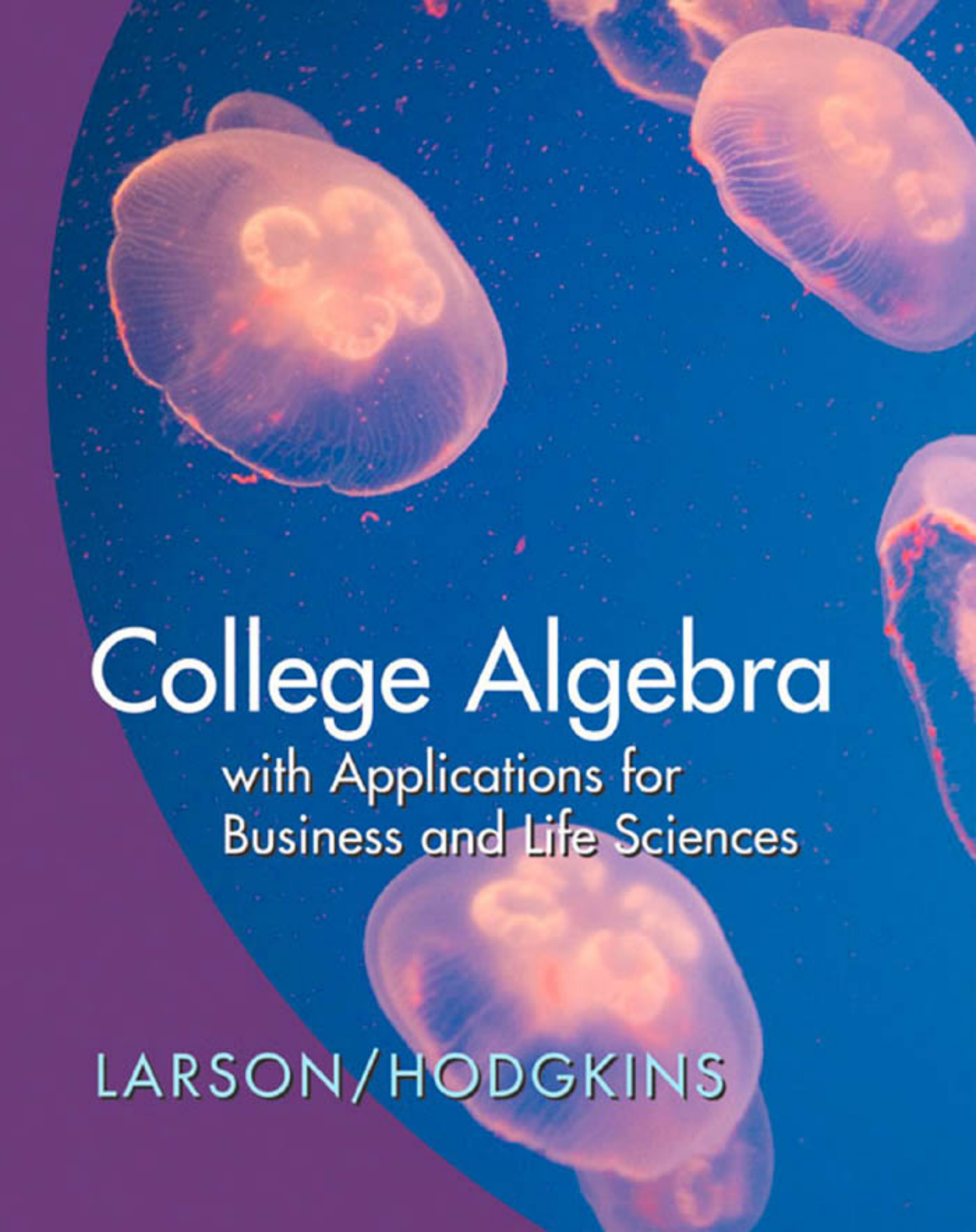


College Algebra

with Applications for Business and Life Sciences

LARSON/HODGKINS

College Algebra

with Applications for Business and the Life Sciences

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Boston New York

Publisher: Richard Stratton
Senior Sponsoring Editor: Cathy Cantin
Senior Development Editor: Maria Morelli
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We have included examples and exercises that use real-life data as well as technology output from a variety of software. This would not have been possible without the help of many people and organizations. Our wholehearted thanks goes to all for their time and effort.

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Printed in the U.S.A.

Library of Congress Control Number: 2007937005

ISBNs
Instructor's Annotated Edition:
ISBN-13: 978-0-547-06999-9
ISBN-10: 0-547-06999-5

For orders, use student text ISBNs:
ISBN-13: 978-0-547-05269-4
ISBN-10: 0-547-05269-3

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From the Desks of Ron Larson and Anne Hodgkins . . .

Do you have students who are taking college algebra as their last college math course, *or* students who are majoring in subjects such as biology and business and plan to go on to applied calculus?

The truth is, many of us are teaching *both* types of students in one classroom.

We realized that we had to change the way we thought about this course in order to reach this diverse audience more effectively and ensure their success. The result: *College Algebra with Applications for Business and the Life Sciences*.

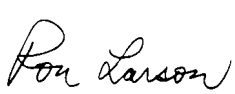
We're excited about this new textbook because it acknowledges where students are when they enter the course . . . and where they should be when they complete it. We review the basic algebra that students have studied previously (in Chapter 0 and in the exercises, notes, study tips, and study sheets throughout the text), *and* we present a solid college algebra course that balances understanding of concepts with the development of strong problem-solving skills. When students have finished with this text, they will be fully prepared to study—and succeed in—applied calculus.

This new textbook program helps students learn the math in the ways we have found most effective for our students, by **practicing their problem-solving skills and reinforcing their understanding in the context of actual problems they may encounter in their lives and their careers.**

And we're also excited about this textbook program because it is being published as part of a whole series of textbooks tailored to the needs of college algebra and applied calculus students majoring in business, biology, and related courses.

College Algebra with Applications for Business and the Life Sciences
Calculus: An Applied Approach, 8e
Brief Calculus: An Applied Approach, 8e
Calculus with Applications for the Life Sciences
College Algebra and Calculus: An Applied Approach

We hope you and your students enjoy *College Algebra with Applications for Business and the Life Sciences*. Feel free to tell us what you think about it. Over the years, we have received many useful comments from both instructors and students, and we value these comments very much.



Ron Larson

Anne V. Hodgkins

Goals for This Text

Establish a Solid Foundation in College Algebra

To help students master algebra concepts, many effective tools are incorporated throughout the text. These features help students evaluate and reinforce their understanding of the math.

- After each worked-out Example, a **Checkpoint** offers the opportunity for immediate practice.
- At the end of each section and before the Section Exercises, a **Concept Check** poses noncomputational questions designed to test students' basic understanding of that section's concepts.
- Each exercise set begins with a **Skills Review** of cumulative exercises that test prerequisite skills from earlier sections.
- The **Mid-Chapter Quiz** offers frequent opportunities for self-assessment so students can discover any topics they might need to study further before they progress too far into the chapter.
- The **Chapter Summary** summarizes skills presented in the chapter and correlates each skill to Review Exercises for extra practice.
- The **Study Strategies** provide invaluable tips for overcoming common study obstacles.
- The **Chapter Test** enables students to identify and strengthen any weaknesses before taking an exam.

Present Real-World Problems to Motivate Interest and Understanding

Applications have been culled from news sources, current events, industry data, world events, and government data. Students see algebra as relevant to the world around them; if they plan to continue their study of math and go on to calculus, the extensive opportunities to create and interpret models from real data provide solid preparation.

Enhance Understanding Using Technology

Students can visualize the math by using powerful technology, such as graphing calculators and spreadsheet software, and so develop a deeper comprehension of mathematical concepts.

- Optional **Technology boxes** feature exercises that offer students opportunities to practice using these tools.
- The  icon in the exercises suggests when a graphing calculator or other technology tool can be used.
- The  icon appears when spreadsheet software, such as Excel, especially relevant to business students, can be incorporated.

Prepare for Success in Applied Calculus and Beyond

Several text features pique students' interest by using everyday examples to extend practice in modeling. Honing these skills will improve students' proficiency and help ensure success in their future careers.

- **Make a Decision** exercises ask students open-ended questions as they apply concepts to real-world problems.
- **Business Capsules** highlight business situations in actual companies and encourage students to include research as part of their problem solving.
- **Extended Applications** are in-depth, applied exercises requiring students to work with large data sets. They often involve work in creating or analyzing models. These exercises are offered online at *college.hmco.com/info/larsonapplied.*

Acknowledgments

We would like to thank our colleagues who have helped us develop this program. Their encouragement, criticisms, and suggestions have been invaluable to us.

Reviewers

Michael Brook, *University of Delaware*
Tim Chappell, *Metropolitan Community College–Penn Valley*
Warrene Ferry, *Jones County Junior College*
David Frank, *University of Minnesota*
Michael Frantz, *University of La Verne*
Linda Herndon, OSB, *Benedictine College*
Ruth E. Hoffman, *Toccoa Falls College*
Eileen Lee, *Framingham State College*
Shahrokh Parvini, *San Diego Mesa College*
Jim Rutherfoord, *Chattahoochee Technical College*
Laurie Varecka, *University of Arizona*

Portions of *College Algebra with Applications for Business and the Life Sciences* have been featured in our previously published *College Algebra: Concepts and Models*. We thank the following reviewers, who have given us many useful insights throughout that book's five editions:

Rosalie Abraham, *Florida Community College at Jacksonville;* Judith A. Ahrens, *Pellissippi State Technical Community College;* Sandra Beken, *Horry-Georgetown Technical College;* Diane Benjamin, *University of Wisconsin–Platteville;* Dona Boccio, *Queensborough Community College;* Kent Craghead, *Colby Community College;* Carol Edwards, *St. Louis Community College at Florissant Valley;* Thomas L. Fitzkee, *Francis Marion University;* Michael Frantz, *University of La Verne;* Nick Geller, *Collin College;* Carolyn H. Goldberg, *Niagara County Community College;* Carl Hughes, *Fayetteville State University;* Buddy A. Johns, *Wichita State University;* Annie Jones, *Calhoun State Community College;* Steven Z. Kahn, *Anne Arundel Community College;* Claire Krukenberg, *Eastern Illinois University;* John Kubicek, *Southwest Missouri State University;* Charles G. Laws, *Cleveland State Community College;* John A. Lewallen, *Southeastern Louisiana University;* Gael Mericle, *Minnesota State University Mankato;* Michael Montano, *Riverside Community College;* Sue Neal, *Wichita State University;* Terrie L. Nichols, *Cuyamaca College;* Mark Omodt, *Anoka-Ramsey Community College;* G. Bryan Stewart, *Tarrant County College;* Jacqueline Stone, *University of Maryland;* David Surowski, *Kansas State University;* Pamela K. Trim, *Southwest Tennessee Community College;* Jamie Whitehead, *Texarkana College*

In addition, we would like to thank the staff of Larson Texts, Inc., who assisted in preparing the manuscript, rendering the art package, and typesetting and proofreading the pages and supplements.

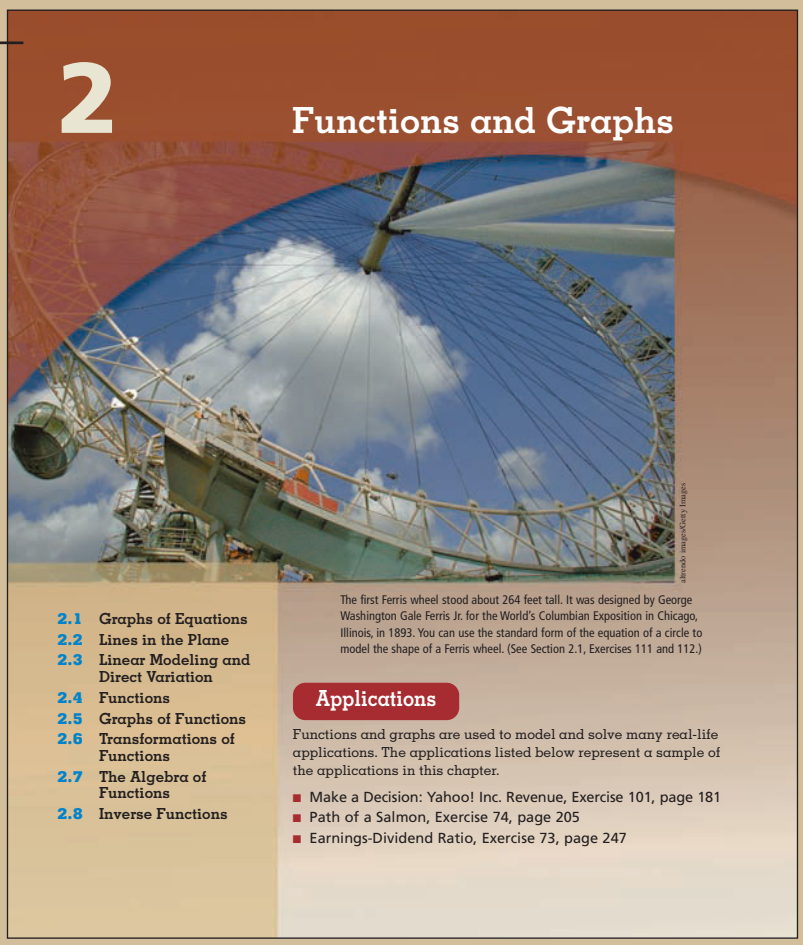
On a personal level, we would like to thank our families, especially Deanna Gilbert Larson and Jay N. Torok, for their love, patience, and support. Also, special thanks goes to R. Scott O'Neil.

Your Textbook Features

Establish a Solid Foundation in College Algebra

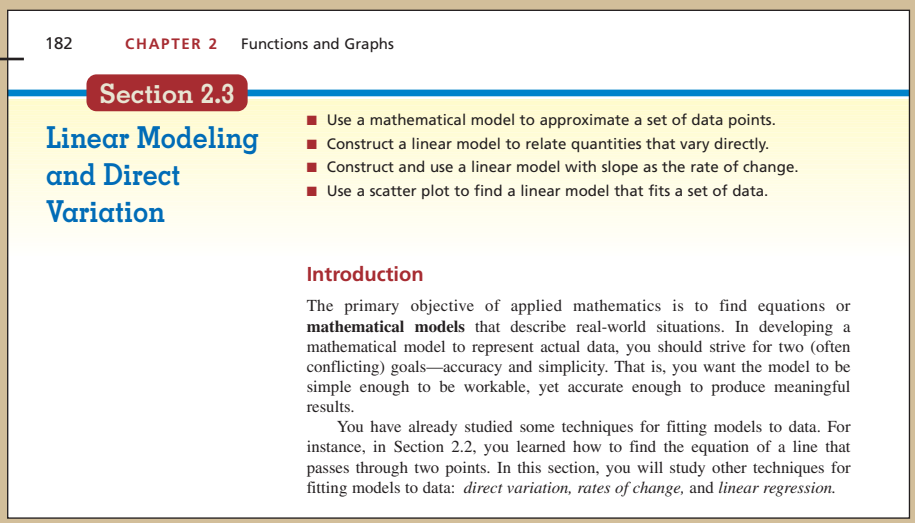
CHAPTER OPENERS

Each opener has an applied example of a core topic from the chapter. The section outline provides a comprehensive overview of the material being presented.



SECTION OBJECTIVES

A bulleted list of learning objectives enables you to preview what will be presented in the upcoming section.



182 CHAPTER 2 Functions and Graphs

Section 2.3

Linear Modeling and Direct Variation

- Use a mathematical model to approximate a set of data points.
- Construct a linear model to relate quantities that vary directly.
- Construct and use a linear model with slope as the rate of change.
- Use a scatter plot to find a linear model that fits a set of data.

Introduction

The primary objective of applied mathematics is to find equations or **mathematical models** that describe real-world situations. In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals—accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results.

You have already studied some techniques for fitting models to data. For instance, in Section 2.2, you learned how to find the equation of a line that passes through two points. In this section, you will study other techniques for fitting models to data: *direct variation, rates of change,* and *linear regression.*

CONCEPT CHECK

1. What can you say about the functions m and n given that $m(n(x)) = x$ for every x in the domain of n and $n(m(x)) = x$ for every x in the domain of m?
2. Given that the functions g and h are inverses of each other and (a, b) is a point on the graph of g, name a point on the graph of h.
3. Explain how to find an inverse function algebraically.
4. The line $y = 2$ intersects the graph of $f(x)$ at two points. Does f have an inverse? Explain.

CONCEPT CHECK

These noncomputational questions appear at the end of each section and are designed to check your understanding of the concepts covered in that section.

DEFINITIONS AND THEOREMS

All definitions and theorems are highlighted for emphasis and easy recognition.

Slope-Intercept Form of the Equation of a Line

The graph of the equation

$$y = mx + b$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

EXAMPLES

There is a wide variety of relevant examples in the text, each titled for easy reference. Many of the solutions are presented graphically, analytically, and/or numerically to provide further insight into mathematical concepts. Examples using real-life situations are identified with the icon Ⓡ.

Applications



Andy Williams/Getty Images

The Capitol building in Washington, D.C. is where each state's Congressional representatives convene. In recent years, no party has had a strong majority, which can make it difficult to pass legislation.

Example 7 **Political Makeup of the U.S. Senate** Ⓡ

Consider three functions R, D, and I that represent the numbers of Republicans, Democrats, and Independents, respectively, in the U.S. Senate from 1967 to 2005. Sketch the graphs of R, D, and I and the sum of R, D, and I in the same coordinate plane. The numbers of senators from each political party are shown below.

Year	R	D	I
1967	36	64	0
1969	42	58	0
1971	44	54	2
1973	42	56	2
1975	37	61	2
1977	38	61	1
1979	41	58	1
1981	53	46	1
1983	54	46	0
1985	53	47	0

Year	R	D	I
1987	45	55	0
1989	45	55	0
1991	44	56	0
1993	43	57	0
1995	52	48	0
1997	55	45	0
1999	55	45	0
2001	50	50	0
2003	51	48	1
2005	55	44	1

SOLUTION The graphs of R, D, and I are shown in Figure 2.66. Note that the sum of R, D, and I is the constant function $R + D + I = 100$. This follows from the fact that the number of senators in the United States is 100 (two from each state).

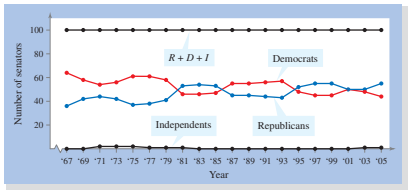


FIGURE 2.66 Numbers of U.S. Senators by Political Party

CHECKPOINT

After each example, a similar problem is presented to encourage immediate practice and to provide further reinforcement of your understanding of the concepts just learned.

✓CHECKPOINT 7

In Example 7, consider the function f given by $f = 100 - (R + D)$. What does f represent in the context of the real-life situation? ■

DISCOVERY

These projects appear before selected topics and allow you to explore concepts on your own. These boxed features are optional, so they can be omitted with no loss of continuity in the coverage of material.

DISCOVERY

The point $(2, 4)$ is on the graph of $f(x) = x^2$. Predict the location of this point if the following transformations are performed.

a. $f(x - 4)$

b. $f(x) + 1$

c. $f(x + 1) - 2$

Use a graphing utility to verify your predictions. Can you find a general description that represents an ordered pair that has been shifted horizontally? vertically?

Enhance Your Understanding Using Technology

TECHNOLOGY BOXES

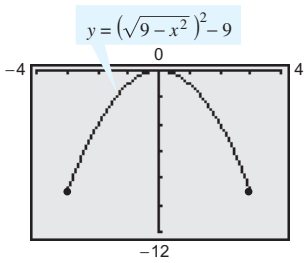
These boxes appear throughout the text and provide guidance on using technology to ease lengthy calculations, present graphical solutions, or discuss where using technology can lead to misleading or wrong solutions.

TECHNOLOGY

In Example 5, the domain of the composite function is $[-3, 3]$. To convince yourself of this, use a graphing utility to graph

$$y = \left(\sqrt{9 - x^2}\right)^2 - 9$$

as shown in the figure below. Notice that the graphing utility does not extend the graph to the left of $x = -3$ or to the right of $x = 3$.



60. Solar Energy Photovoltaic cells convert light energy into electricity. The photovoltaic cell and module domestic shipments S (in peak kilowatts) for the years 1996 to 2005 are shown in the table. *(Source: Energy Information Administration)*

Year	Shipments, S	Year	Shipments, S
1996	13,016	2001	36,310
1997	12,561	2002	45,313
1998	15,069	2003	48,664
1999	21,225	2004	78,346
2000	19,838	2005	134,465

(a) Use a spreadsheet software program to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a spreadsheet software program to find a cubic model and a quartic model for the data.

(c) Use each model to predict the year in which the shipments will be about 1,000,000 peak kilowatts. Then discuss the appropriateness of each model for predicting future values.

TECHNOLOGY EXERCISES

Many exercises in the text can be solved with or without technology. The T icon identifies exercises for which students are specifically instructed to use a graphing calculator or a computer algebra system to solve the problem. Additionally, the S icon denotes exercises best solved by using a spreadsheet.

Prepare for Success in Applied Calculus and Beyond

MAKE A DECISION

Multi-step exercises reinforce students' problem-solving skills and mastery of concepts, as well as taking a real-life application further by testing what they know about a given problem to make a decision within the context of the problem.

101. ***MAKE A DECISION: YAHOO! INC. REVENUE*** In 2000, Yahoo! Inc. had revenues of $1110.2 million. In 2003, their revenues were $1625.1 million. Assume the revenue followed a linear trend. What would the approximate revenue have been in 2005? The actual revenue in 2005 was $5257.7 million. Do you think the yearly revenue followed a linear trend? Explain your reasoning. *(Source: Yahoo! Inc.)*

BUSINESS CAPSULES

Business Capsules appear at the ends of numerous sections. These capsules and their accompanying exercises deal with business situations that are related to the mathematical concepts covered in the chapter.

Business Capsule



AP/Wide World Photos

SunPower Corporation develops and manufactures solar-electric power products. SunPower's new higher efficiency solar cells generate up to 50% more power than other solar technologies. SunPower's technology was developed by Dr. Richard Swanson and his students while he was Professor of Engineering at Stanford University. SunPower's 2006 revenues are projected to increase 300% from its 2005 revenues.

69. **Research Project** Use your campus library, the Internet, or some other reference source to find information about an alternative energy business experiencing strong growth similar to the example above. Write a brief report about the company or small business.

APPLICATIONS INDEX

This list, found on the front and back endsheets, is an index of all the applications presented in the text Examples and Exercises.

EXTENDED APPLICATIONS

These in-depth applied exercises require students to work with large data sets and often involve work in creating or analyzing models. These exercises are offered online at *college.hmco.com/info/larsonapplied*.

Applications

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Tools To Help You Learn and Review

STUDY TIPS

Scattered throughout the text, study tips address special cases, expand on concepts, and help you to avoid common errors.

STUDY TIP

Note in Step 3 of the guidelines for finding inverse functions that it is possible for a function to have no inverse function. For instance, the function given by $f(x) = x^2$ has no inverse function.

SKILLS REVIEW

These exercises at the beginning of each exercise set help students review skills covered in previous sections. The answers are provided at the back of the text to reinforce understanding of the skill sets learned.

Skills Review 2.8 The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.4, 1.1, 1.5, and 2.4.

In Exercises 1–4, find the domain of the function.

1. $f(x) = \sqrt[3]{x+1}$ **2.** $f(x) = \sqrt{x+1}$

3. $g(x) = \dfrac{2}{x^2 - 2x}$ **4.** $h(x) = \dfrac{x}{3x+5}$

In Exercises 5–8, simplify the expression.

5. $2\left(\dfrac{x+5}{2}\right) - 5$ **6.** $7 - 10\left(\dfrac{7-x}{10}\right)$

7. $\sqrt[3]{2\left(\dfrac{x^3}{2} - 2\right) + 4}$ **8.** $\sqrt[5]{(x+2)^5} - 2$

In Exercises 9 and 10, solve for x in terms of y.

9. $y = \dfrac{2x-6}{3}$ **10.** $y = \sqrt[3]{2x-4}$

EXERCISE SETS

These exercises offer opportunities for practice and review. They progress in difficulty from skill-development problems to more challenging problems, to build confidence and understanding.

In Exercises 45–52, find all real values of x such that $f(x) = 0$.

45. $f(x) = 15 - 3x$ **46.** $f(x) = \dfrac{2x-5}{3}$

47. $f(x) = x^2 - 9$ **48.** $f(x) = 2x^2 - 11x + 5$

49. $f(x) = x^3 - x$

50. $f(x) = x^3 - 3x^2 - 4x + 12$

51. $f(x) = \dfrac{3}{x-1} + \dfrac{4}{x-2}$

52. $f(x) = 3 + \dfrac{2}{x-1}$

In Exercises 53–66, find the domain of the function.

53. $g(x) = 1 - 2x^2$ **54.** $f(x) = 5x^2 + 2x - 1$

55. $h(t) = \dfrac{4}{t}$ **56.** $s(y) = \dfrac{3y}{y+5}$

57. $g(y) = \sqrt[3]{y-10}$ **58.** $f(t) = \sqrt[3]{t+4}$

59. $f(x) = \sqrt[4]{1-x^2}$ **60.** $g(x) = \sqrt{x+1}$

61. $g(x) = \dfrac{1}{x} - \dfrac{3}{x+2}$ **62.** $h(x) = \dfrac{10}{x^2 - 2x}$

63. $f(x) = \dfrac{\sqrt{x+1}}{x-2}$ **64.** $f(s) = \dfrac{\sqrt{s-1}}{s-4}$

65. $f(x) = \dfrac{x-4}{\sqrt{x}}$ **66.** $f(x) = \dfrac{x-5}{\sqrt{x^2-9}}$

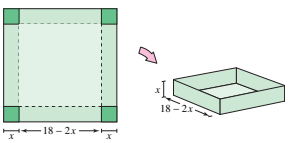
67. Consider $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt[3]{x-2}$. Why are the domains of f and g different?

68. A student says that the domain of

$$f(x) = \frac{\sqrt{x+1}}{x-3}$$

is all real numbers except $x = 3$. Is the student correct? Explain.

69. Volume of a Box An open box is to be made from a square piece of material 18 inches on a side by cutting equal squares from the corners and turning up the sides (see figure).

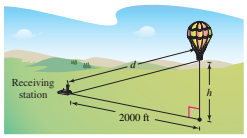


(a) Write the volume V of the box as a function of its height x.

(b) What is the domain of the function?

(c) Determine the volume of a box with a height of 4 inches.

70. Height of a Balloon A balloon carrying a transmitter ascends vertically from a point 2000 feet from the receiving station (see figure). Let d be the distance between the balloon and the receiving station. Write the height h of the balloon as a function of d. What is the domain of this function?



71. Cost, Revenue, and Profit A company produces a product for which the variable cost is $11.75 per unit and the fixed costs are $112,000. The product sells for $21.95 per unit. Let x be the number of units produced and sold.

(a) Add the variable cost and the fixed costs to write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Use the formula

$$P = R - C$$

to write the profit P as a function of the number of units sold.

72. Cost, Revenue, and Profit A company produces a product for which the variable cost is $9.85 per unit and the fixed costs are $85,000. The product sells for $19.95 per unit. Let x be the number of units produced and sold.

(a) Add the variable cost and the fixed costs to write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

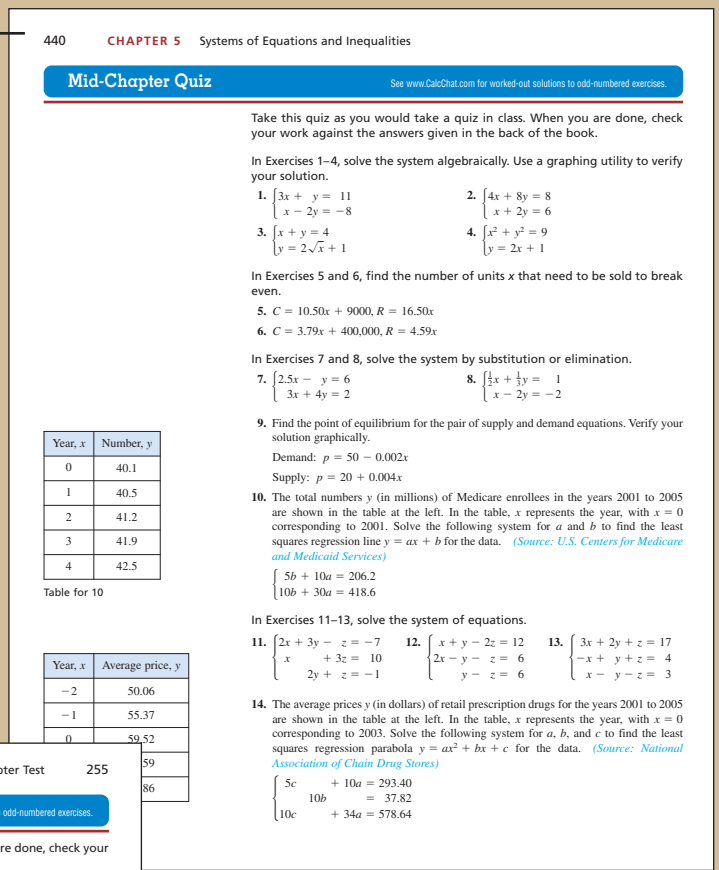
(c) Use the formula

$$P = R - C$$

to write the profit P as a function of the number of units sold.

MID-CHAPTER QUIZ

Appearing in the middle of each chapter, this one-page test allows you to practice skills and concepts learned in the chapter. This opportunity for self-assessment will uncover any potential weak areas that might require further review of the material.



440 CHAPTER 5 Systems of Equations and Inequalities

Mid-Chapter Quiz See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, solve the system algebraically. Use a graphing utility to verify your solution.

1. $\begin{cases} 3x + y = 11 \\ x - 2y = -8 \end{cases}$
2. $\begin{cases} 4x + 8y = 8 \\ x + 2y = 6 \end{cases}$
3. $\begin{cases} x + y = 4 \\ y = 2\sqrt{x} + 1 \end{cases}$
4. $\begin{cases} x^2 + y^2 = 9 \\ y = 2x + 1 \end{cases}$

In Exercises 5 and 6, find the number of units x that need to be sold to break even.

5. $C = 10.50x + 9000, R = 16.50x$
6. $C = 3.79x + 400{,}000, R = 4.59x$

In Exercises 7 and 8, solve the system by substitution or elimination.

7. $\begin{cases} 2.5x - y = 6 \\ 3x + 4y = 2 \end{cases}$
8. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 1 \\ x - 2y = -2 \end{cases}$

9. Find the point of equilibrium for the pair of supply and demand equations. Verify your solution graphically.
 Demand: $p = 50 - 0.002x$
 Supply: $p = 20 + 0.004x$

Year, x	Number, y
0	40.1
1	40.5
2	41.2
3	41.9
4	42.5

Table for 10

10. The total numbers y (in millions) of Medicare enrollees in the years 2001 to 2005 are shown in the table at the left. In the table, x represents the year, with $x = 0$ corresponding to 2001. Solve the following system for a and b to find the least squares regression line $y = ax + b$ for the data. *(Source: U.S. Centers for Medicare and Medicaid Services)*

$$\begin{cases} 5b + 10a = 206.2 \\ 10b + 30a = 418.6 \end{cases}$$

In Exercises 11–13, solve the system of equations.

11. $\begin{cases} 2x + 3y - z = -7 \\ x + 3z = 10 \\ 2y + z = -1 \end{cases}$
12. $\begin{cases} x + y - 2z = 12 \\ 2x - y - z = 6 \\ y - z = 6 \end{cases}$
13. $\begin{cases} 3x + 2y + z = 17 \\ -x + y + z = 4 \\ x - y - z = 3 \end{cases}$

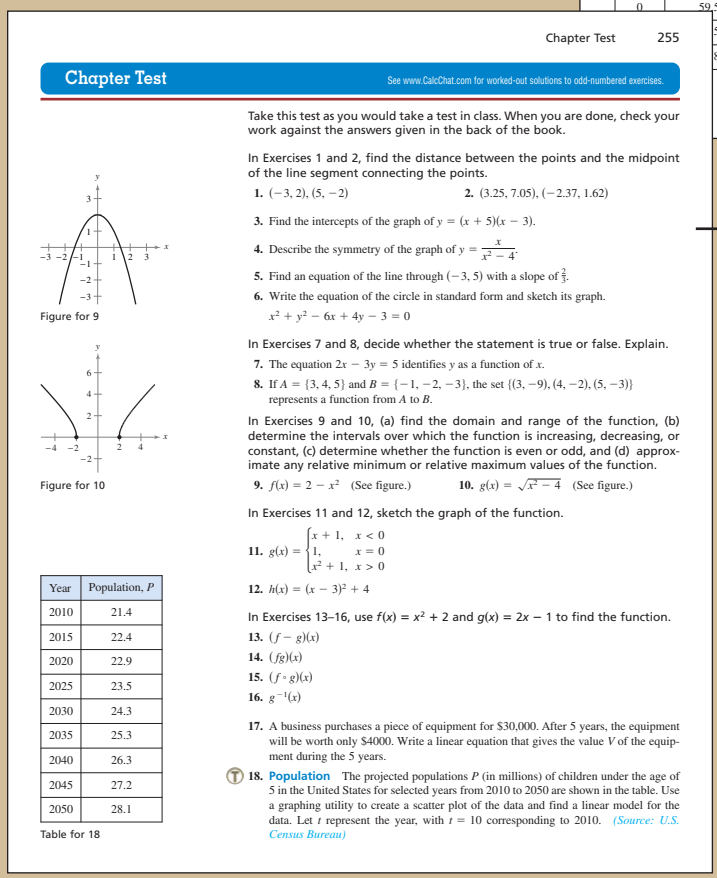
Year, x	Average price, y
−2	50.06
−1	55.37
0	59.52
	59
	86

14. The average prices y (in dollars) of retail prescription drugs for the years 2001 to 2005 are shown in the table at the left. In the table, x represents the year, with $x = 0$ corresponding to 2003. Solve the following system for a, b, and c to find the least squares regression parabola $y = ax^2 + bx + c$ for the data. *(Source: National Association of Chain Drug Stores)*

$$\begin{cases} 5c + 10a = 293.40 \\ 10b = 37.82 \\ 10c + 34a = 578.64 \end{cases}$$

CHAPTER TEST

Appearing at the end of each chapter, this test is designed to simulate an in-class exam. Taking these tests will help you to determine what concepts require further study and review.



Chapter Test 255

Chapter Test See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, find the distance between the points and the midpoint of the line segment connecting the points.

1. $(-3, 2), (5, -2)$
2. $(3.25, 7.05), (-2.37, 1.62)$

3. Find the intercepts of the graph of $y = (x + 5)(x - 3)$.
4. Describe the symmetry of the graph of $y = \frac{x}{x^2 - 4}$.
5. Find an equation of the line through $(-3, 5)$ with a slope of $\frac{2}{3}$.
6. Write the equation of the circle in standard form and sketch its graph.
 $x^2 + y^2 - 6x + 4y - 3 = 0$

Figure for 9

Figure for 10

In Exercises 7 and 8, decide whether the statement is true or false. Explain.

7. The equation $2x - 3y = 5$ identifies y as a function of x.
8. If $A = \{3, 4, 5\}$ and $B = \{-1, -2, -3\}$, the set $\{(3, -9), (4, -2), (5, -3)\}$ represents a function from A to B.

In Exercises 9 and 10, (a) find the domain and range of the function, (b) determine the intervals over which the function is increasing, decreasing, or constant, (c) determine whether the function is even or odd, and (d) approximate any relative minimum or relative maximum values of the function.

9. $f(x) = 2 - x^2$ (See figure.)
10. $g(x) = \sqrt{x^2 - 4}$ (See figure.)

In Exercises 11 and 12, sketch the graph of the function.

11. $g(x) = \begin{cases} x + 1, & x < 0 \\ 1, & x = 0 \\ x^2 + 1, & x > 0 \end{cases}$
12. $h(x) = (x - 3)^2 + 4$

In Exercises 13–16, use $f(x) = x^2 + 2$ and $g(x) = 2x - 1$ to find the function.

13. $(f - g)(x)$
14. $(fg)(x)$
15. $(f \circ g)(x)$
16. $g^{-1}(x)$

17. A business purchases a piece of equipment for $30,000. After 5 years, the equipment will be worth only $4000. Write a linear equation that gives the value V of the equipment during the 5 years.

Year	Population, P
2010	21.4
2015	22.4
2020	22.9
2025	23.5
2030	24.3
2035	25.3
2040	26.3
2045	27.2
2050	28.1

Table for 18

T 18. **Population** The projected populations P (in millions) of children under the age of 5 in the United States for selected years from 2010 to 2050 are shown in the table. Use a graphing utility to create a scatter plot of the data and find a linear model for the data. Let t represent the year, with $t = 10$ corresponding to 2010. *(Source: U.S. Census Bureau)*

CHAPTER SUMMARY AND STUDY STRATEGIES

The *Chapter Summary* reviews the skills covered in the chapter and correlates each skill to the Review Exercises that test the skill. Following each *Chapter Summary* is a short list of *Study Strategies* for addressing topics or situations specific to the chapter.

Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 336.
Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 3.1	Review Exercises
■ Sketch the graph of a quadratic function and identify its vertex and intercepts.	*1–4*
■ Find a quadratic function given its vertex and a point on its graph.	*5, 6*
■ Construct and use a quadratic model to solve an application problem.	*7–12*
Section 3.2	
■ Determine right-hand and left-hand behavior of graphs of polynomial functions.	*13–16*
When n is odd and the leading coefficient is positive, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.	
When n is odd and the leading coefficient is negative, $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.	
When n is even and the leading coefficient is positive, $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.	
When n is even and the leading coefficient is negative, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.	
■ Find the real zeros of a polynomial function.	*17–20*
Section 3.3	
■ Divide one polynomial by a second polynomial using long division.	*21, 22*
■ Simplify a rational expression using long division.	*23, 24*
■ Use synthetic division to divide two polynomials.	*25, 26, 31, 32*
■ Use the Remainder Theorem and synthetic division to evaluate a polynomial.	*27, 28*
■ Use the Factor Theorem to factor a polynomial.	*29, 30*
Section 3.4	
■ Find all possible rational zeros of a function using the Rational Zero Test.	*33, 34*
■ Find all real zeros of a function.	*35–42*
■ real zeros of a polynomial function using the Intermediate	*43, 44*
real zeros of a polynomial function using a graphing utility.	*45, 46*
s for approximating real zeros to solve an application problem.	*47, 48*

valuable study aids to help you master the material in this chapter. The *Student* udes step-by-step solutions to all odd-numbered exercises to help you review dent website at *college.hmco.com/info/larsonapplied* offers algebra help and a *y Guide*. The *Graphing Technology Guide* contains step-by-step commands a wide variety of graphing calculators, including the most recent models.

Section 3.5	Review Exercises
■ Find the complex conjugate of a complex number.	*49–52*
■ Perform operations with complex numbers and write the results in standard form. $(a + bi) + (c + di) = (a + c) + (b + d)i$ $(a + bi) - (c + di) = (a - c) + (b - d)i$ $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$	*53–68*
■ Solve a polynomial equation that has complex solutions.	*69–72*
■ Plot a complex number in the complex plane.	*73, 74*
Section 3.6	
■ Use the Fundamental Theorem of Algebra and the Linear Factorization Theorem to write a polynomial as the product of linear factors.	*75–80*
■ Find a polynomial with real coefficients whose zeros are given.	*81, 82*
■ Factor a polynomial over the rational, real, and complex numbers.	*83, 84*
■ Find all real and complex zeros of a polynomial function.	*85–88*
Section 3.7	
■ Find the domain of a rational function.	*89–92*
■ Find the vertical and horizontal asymptotes of the graph of a rational function.	*89–92*

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$, $a_n \neq 0, b_m \neq 0$.

1. The graph of f has *vertical* asymptotes at the zeros of $q(x)$.
2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $p(x)$ and $q(x)$.
 a. If $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
 b. If $n = m$, the graph of f has the line $y = a_n/b_m$ (ratio of the leading coefficients) as a horizontal asymptote.
 c. If $n > m$, the graph of f has no horizontal asymptote.

■ Sketch the graph of a rational function, including graphs with slant asymptotes.	*93–98*
■ Use a rational function model to solve an application problem.	*99–103*

Study Strategies

■ **Use a Graphing Utility** A graphing calculator or graphing software for a computer can help you in this course in two important ways. As an *exploratory device*, a graphing utility allows you to learn concepts by allowing you to compare graphs of functions. For instance, sketching the graphs of $f(x) = x^3$ and $f(x) = -x^3$ helps confirm that the negative coefficient has the effect of reflecting the graph about the x-axis. As a *problem-solving tool*, a graphing utility frees you from some of the difficulty of sketching complicated graphs by hand. The time you can save can be spent using mathematics to solve real-life problems.

■ **Problem-Solving Strategies** If you get stuck when trying to solve a real-life problem, consider the strategies below.

1. *Draw a Diagram.* If feasible, draw a diagram that represents the problem. Label all known values and unknown values on the diagram.
2. *Solve a Simpler Problem.* Simplify the problem, or write several simple examples of the problem. For instance, if you are asked to find the dimensions that will produce a maximum area, try calculating the areas of several examples.
3. *Rewrite the Problem in Your Own Words.* Rewriting a problem can help you understand it better.
4. *Guess and Check.* Try guessing the answer, then check your guess in the statement of the original problem. By refining your guesses, you may be able to think of a general strategy for solving the problem.

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Supplements for the Instructor

Online Instructor Solutions Manual Found on the instructor website, this manual contains the complete, worked-out solutions for all of the exercises in the text.

HM Testing (powered by Diploma™) (ISBN: 978-0-547-06465-9) HM Testing (powered by Diploma) provides instructors with a wide array of new algorithmic exercises, along with improved functionality and ease of use. Instructors can create, customize, and deliver multiple types of tests—including authoring and editing algorithmic questions.

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Student Solutions Manual (ISBN: 978-0-547-07003-2) This guide contains complete solutions to all odd-numbered exercises in the text.

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Fundamental Concepts of Algebra

0



Jeff Schultz/AlaskaStock.com

The Iditarod Sled Dog Race includes a stop in McGrath, Alaska. Part of the challenge of this event is facing temperatures that reach well below zero. To find the range of a set of temperatures, you must find the distance between two numbers. (See Section 0.1, Exercise 81.)

Applications

The fundamental concepts of algebra have many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- College Costs, Exercise 75, page 28
- Escape Velocity, Example 11, page 35
- Oxygen Level, Exercise 72, page 61

0.1 Real Numbers: Order and Absolute Value
0.2 The Basic Rules of Algebra
0.3 Integer Exponents
0.4 Radicals and Rational Exponents
0.5 Polynomials and Special Products
0.6 Factoring
0.7 Fractional Expressions

Section 0.1

Real Numbers: Order and Absolute Value

- Classify real numbers as natural numbers, integers, rational numbers, or irrational numbers.
- Order real numbers.
- Give a verbal description of numbers represented by an inequality.
- Use inequality notation to describe a set of real numbers.
- Interpret absolute value notation.
- Find the distance between two numbers on the real number line.
- Use absolute value to solve an application problem.

Real Numbers

The formal term that is used in mathematics to refer to a collection of objects is the word **set.** For instance, the set

$$\{1, 2, 3\}$$

contains the three numbers 1, 2, and 3. Note that a pair of braces { } is used to enclose the members of the set. In this text, a pair of braces will always indicate the members of a set. Parentheses () and brackets [] are used to represent other ideas.

The set of numbers that is used in arithmetic is the set of **real numbers.** The term *real* distinguishes real numbers from *imaginary* or *complex* numbers.

A set A is called a **subset** of a set B if every member of A is also a member of B. Here are two examples.

- $\{1, 2, 3\}$ is a subset of $\{1, 2, 3, 4\}$.
- $\{0, 4\}$ is a subset of $\{0, 1, 2, 3, 4\}$.

One of the most commonly used subsets of real numbers is the set of **natural numbers** or **positive integers**

$$\{1, 2, 3, 4, \ldots\}.$$ Set of positive integers

Note that the three dots indicate that the pattern continues. For instance, the set also contains the numbers 5, 6, 7, and so on.

Positive integers can be used to describe many quantities that you encounter in everyday life—for instance, you might be taking four classes this term, or you might be paying $700 a month for rent. But even in everyday life, positive integers cannot describe some concepts accurately. For instance, you could have a zero balance in your checking account, or the temperature could be $-10°$ (10 degrees below zero). To describe such quantities, you need to expand the set of positive integers to include **zero** and the **negative integers.** The expanded set is called the set of **integers,** which can be written as follows.

$$\{\underbrace{\ldots, -3, -2, -1}_{\text{Negative integers}}, \overset{\text{Zero}}{0}, \underbrace{1, 2, 3, \ldots}_{\text{Positive integers}}\}$$

The set of integers is a subset of the set of real numbers. This means that every integer is a real number.

Even with the set of integers, there are still many quantities in everyday life that you cannot describe accurately. The costs of many items are not in whole dollar amounts, but in parts of dollars, such as \$1.19 or \$39.98. You might work $8\frac{1}{2}$ hours, or you might miss the first *half* of a movie. To describe such quantities, the set of integers is expanded to include **fractions.** The expanded set is called the set of **rational numbers.** Formally, a real number is called **rational** if it can be written as the ratio p/q of two integers, where $q \neq 0$. (The symbol $\neq$ means **not equal to.**) For instance,

$$2 = \frac{2}{1}, \quad 0.333\ldots = \frac{1}{3}, \quad 0.125 = \frac{1}{8}, \quad \text{and} \quad 1.126126\ldots = \frac{125}{111}$$

are rational numbers. Real numbers that cannot be written as the ratio of two integers are called **irrational.** For instance, the numbers

$$\sqrt{2} = 1.4142135\ldots \quad \text{and} \quad \pi = 3.1415926\ldots$$

are irrational numbers. The decimal representation of a rational number is either *terminating* or *repeating*. For instance, the decimal representation of

$$\frac{1}{4} = 0.25 \qquad \text{Terminating decimal}$$

is terminating, and the decimal representation of

$$\frac{4}{11} = 0.363636\ldots = 0.\overline{36} \qquad \text{Repeating decimal}$$

is repeating. (The line over "36" indicates which digits repeat.)

The decimal representation of an irrational number neither terminates nor repeats. When you perform calculations using decimal representations of nonterminating decimals, you usually use a decimal approximation that has been **rounded** to a certain number of decimal places. For instance, rounded to four decimal places, the decimal approximations of $\frac{2}{3}$ and π are

$$\frac{2}{3} \approx 0.6667 \quad \text{and} \quad \pi \approx 3.1416.$$

The symbol $\approx$ means **approximately equal to.**

The Venn diagram in Figure 0.1 shows the relationships between the real numbers and several commonly used subsets of the real numbers.

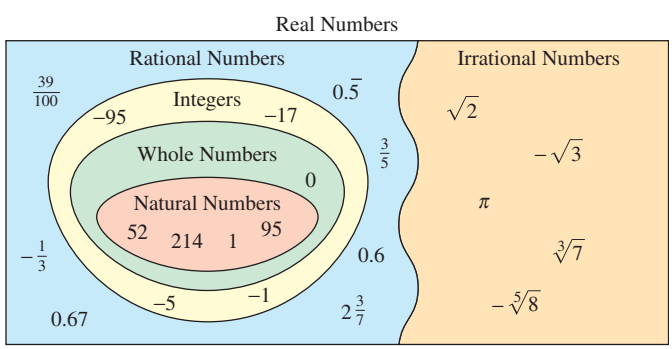


FIGURE 0.1

STUDY TIP

Make sure you understand that not all fractions are rational numbers. For instance, the fraction $\frac{\sqrt{2}}{3}$ is not a rational number.

The Real Number Line and Ordering

The picture that is used to represent the real numbers is the **real number line.** It consists of a horizontal line with a point (the **origin**) labeled as 0 (zero). Points to the left of zero are associated with **negative numbers,** and points to the right of zero are associated with **positive numbers,** as shown in Figure 0.2. The real number zero is neither positive nor negative. So, when you want to talk about real numbers that might be positive *or* zero, you can use the term **nonnegative real numbers.**

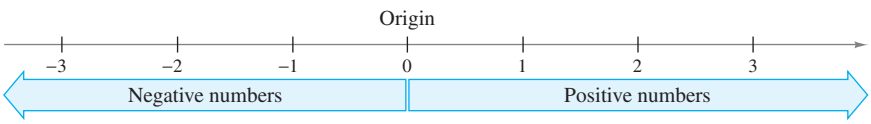


FIGURE 0.2 The Real Number Line

Each point on the real number line corresponds to exactly one real number, and each real number corresponds to exactly one point on the real number line, as shown in Figure 0.3. The number associated with a point on the real number line is the **coordinate** of the point.

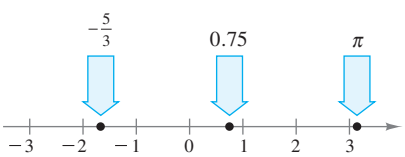


FIGURE 0.3 Every real number corresponds to a point on the real number line.

The real number line provides you with a way of comparing any two real numbers. For instance, if you choose any two (different) numbers on the real number line, one of the numbers must be to the left of the other number. The number to the left is **less than** the number to the right, and the number to the right is **greater than** the number to the left.

Definition of Order on the Real Number Line

If the real number a lies to the left of the real number b on the real number line, a is **less than** b, which is denoted by

$$a < b$$

as shown in Figure 0.4. This relationship can also be described by saying that b is **greater than** a and writing $b > a$. The inequality $a \leq b$ means that a is **less than or equal to** b, and the inequality $b \geq a$ means that b is **greater than or equal to** a.

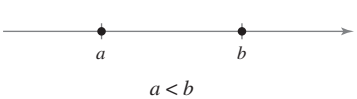
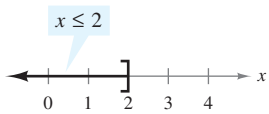
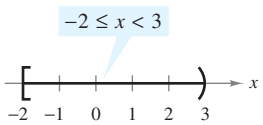


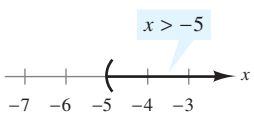
FIGURE 0.4 a is to the left of b.



(a)



(b)



(c)

FIGURE 0.5

The symbols $<$, $>$, $\leq$, and $\geq$ are called **inequality symbols.** Inequalities are useful in denoting subsets of real numbers, as shown in Examples 1 and 2.

Example 1 Interpreting Inequalities

a. The inequality $x \leq 2$ denotes all real numbers that are less than or equal to 2, as shown in Figure 0.5(a).

b. The inequality $-2 \leq x < 3$ means that $x \geq -2$ *and* $x < 3$. This **double inequality** denotes all real numbers between -2 and 3, including -2 but *not* including 3, as shown in Figure 0.5(b).

c. The inequality $x > -5$ denotes all real numbers that are greater than -5, as shown in Figure 0.5(c).

✓CHECKPOINT 1

Give a verbal description of the subset of real numbers represented by $x \geq 7$. ■

In Figure 0.5, notice that a bracket is used to *include* the endpoint of an interval and a parenthesis is used to *exclude* the endpoint.

Example 2 Inequalities and Sets of Real Numbers

a. "c is nonnegative" means that c is greater than or equal to zero, which you can write as $c \geq 0$.

b. "b is at most 5" can be written as $b \leq 5$.

c. "d is negative" can be written as $d < 0$, and "d is greater than -3" can be written as $-3 < d$. Combining these two inequalities produces $-3 < d < 0$.

d. "x is positive" can be written as $0 < x$, and "x is not more than 6" can be written as $x \leq 6$. Combining these two inequalities produces $0 < x \leq 6$.

✓CHECKPOINT 2

Use inequality notation to describe each subset of real numbers.

a. x is at least 5.

b. y is greater than 4, but no more than 11. ■

The following property of real numbers is called the **Law of Trichotomy.** As the "tri" in its name suggests, this law tells you that for any two real numbers a and b, precisely one of *three* relationships is possible.

$$a < b, \quad a = b, \quad \text{or} \quad a > b \qquad \text{Law of Trichotomy}$$

Absolute Value and Distance

The **absolute value** of a real number is its *magnitude,* or its value disregarding its sign. For instance, the absolute value of -3, written $|-3|$, has the value of 3.

STUDY TIP

Be sure you see from the definition that the absolute value of a real number is never negative. For instance, if $a = -5$, then $|-5| = -(-5) = 5$.

Definition of Absolute Value

Let a be a real number. The **absolute value** of a, denoted by $|a|$, is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}.$$

The absolute value of any real number is either positive or zero. Moreover, 0 is the only real number whose absolute value is zero. That is, $|0| = 0$.

Example 3 Finding Absolute Value

a. $|-7| = 7$ **b.** $\left|\frac{1}{2}\right| = \frac{1}{2}$

c. $|-4.8| = 4.8$ **d.** $-|-9| = -(9) = -9$

✓CHECKPOINT 3

Evaluate $|-12|$. ■

Example 4 Comparing Real Numbers

Place the correct symbol ($<$, $>$, or $=$) between the two real numbers.

a. $|-4| \square |4|$ **b.** $|-5| \square 3$ **c.** $-|-1| \square |-1|$

SOLUTION

a. $|-4| = |4|$, because both are equal to 4.

b. $|-5| > 3$, because $|-5| = 5$ and 5 is greater than 3.

c. $-|-1| < |-1|$, because $-|-1| = -1$ and $|-1| = 1$.

✓CHECKPOINT 4

Place the correct symbol ($<$, $>$, or $=$) between the two real numbers.

a. $-|-6| \square -|6|$

b. $-|5| \square |-5|$ ■

Properties of Absolute Value

Let a and b be real numbers. Then the following properties are true.

1. $|a| \geq 0$ **2.** $|-a| = |a|$

3. $|ab| = |a|\,|b|$ **4.** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$

Absolute value can be used to define the distance between two numbers on the real number line. To see how this is done, consider the numbers -3 and 4, as shown in Figure 0.6. To find the distance between these two numbers, subtract either number from the other and then take the absolute value of the difference.

(Distance between -3 and 4) $= |-3 - 4| = |-7| = 7$

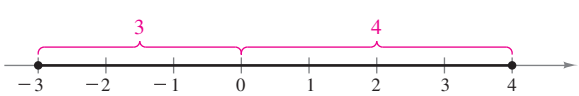


FIGURE 0.6 The distance between -3 and 4 is 7.

Distance Between Two Numbers

Let a and b be real numbers. The **distance between a and b** is given by

$$\text{Distance} = |b - a| = |a - b|.$$

Example 5 Finding the Distance Between Two Numbers

a. The distance between 2 and 7 is $|2 - 7| = |-5| = 5$.

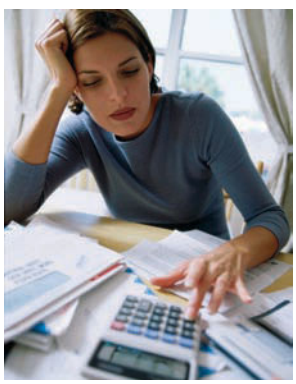
b. The distance between 0 and -4 is $|0 - (-4)| = |4| = 4$.

c. The statement "the distance between x and 2 is at least 3" can be written as $|x - 2| \geq 3$.

✓CHECKPOINT 5

Find the distance between -5 and 3. ■

Application



SuperStock/Jupiter Images

Math plays an important part in keeping your personal finances in order as well as a company's expenses and budget.

Example 6 MAKE A DECISION Budget Variance ®

You monitor monthly expenses for a home health care company. For each type of expense, the company wants the absolute value of the difference between the actual and budgeted amounts to be less than or equal to \$500 *and* less than or equal to 5% of the budgeted amount. By letting a represent the actual expenses and b the budgeted expenses, these restrictions can be written as

$$|a - b| \leq 500 \quad \text{and} \quad |a - b| \leq 0.05b.$$

For travel, office supplies, and wages, the company budgeted \$12,500, \$750, and \$84,600. The actual amounts paid for these expenses were \$12,872.56, \$704.15, and \$85,143.95. Are these amounts within budget restrictions?

SOLUTION One way to determine whether these three expenses are within budget restrictions is to create the table shown.

	Budgeted Expense, b	*Actual Expense, a*	$\lvert a - b \rvert$	$0.05b$
Travel	\$12,500	\$12,872.56	\$372.56	\$625.00
Office supplies	\$750	\$704.15	\$45.85	\$37.50
Wages	\$84,600	\$85,143.95	\$543.95	\$4230.00

From this table, you can see that travel expenses pass both tests, so they are within budget restrictions. Office supply expenses pass the first test but fail the second test, so they are *not* within budget restrictions. Wage expenses fail the first test and pass the second test, so they are *not* within budget restrictions.

✓CHECKPOINT 6

In Example 6, the company budgeted \$28,000 for medical supplies, but actually paid \$30,100. Is this within budget restrictions? ■

CONCEPT CHECK

Is the statement true? If not, explain why.

1. There are no integers in the set of rational numbers.
2. The set of integers is a subset of the set of natural numbers.
3. The expression $x < 5$ describes a subset of the set of rational numbers.
4. When a is negative, $|a| = -a$.

The symbol ® indicates an example that uses or is derived from real-life data.

Exercises 0.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, determine which numbers in the set are (a) natural numbers, (b) integers, (c) rational numbers, and (d) irrational numbers.

1. $\{-9, -\frac{7}{2}, 5, \frac{2}{3}, \sqrt{2}, 0.1\}$

2. $\{\sqrt{5}, -7, -\frac{7}{3}, 0, 3.12, \frac{5}{4}\}$

3. $\{12, -13, 1, \sqrt{4}, \sqrt{6}, \frac{3}{2}\}$

4. $\{3, -1, \frac{1}{3}, \frac{6}{3}, -\frac{1}{2}\sqrt{2}, -7.5\}$

5. $\{\frac{8}{2}, -\frac{8}{3}, \sqrt{10}, -4, 9, 14.2\}$

6. $\{25, -17, \frac{12}{5}, \sqrt{9}, \sqrt{8}, -\sqrt{8}\}$

In Exercises 7–10, use a calculator to find the decimal form of the rational number. If the number is a nonterminating decimal, write the repeating pattern.

7. $\frac{2}{3}$

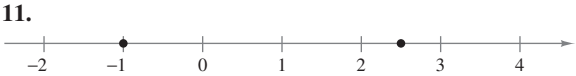
8. $\frac{9}{40}$

9. $\frac{14}{111}$

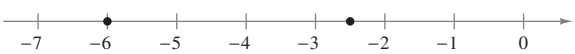
10. $\frac{49}{160}$

In Exercises 11 and 12, approximate the two plotted numbers and place the correct symbol (< or >) between them.

11.



12.



In Exercises 13–18, plot the two real numbers on the real number line and place the appropriate inequality symbol (< or >) between them.

13. $\frac{3}{2}, 7$

14. $-4, -8$

15. $1, -3.5$

16. $\frac{16}{3}, 1$

17. $\frac{5}{6}, \frac{2}{3}$

18. $-\frac{8}{7}, -\frac{3}{7}$

In Exercises 19–22, use a calculator to order the numbers from least to greatest.

19. $\frac{7}{2}, 2\sqrt{3}, 3.45, \frac{204}{60}, \frac{31}{9}$

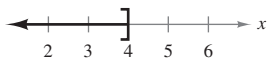
20. $\frac{559}{500}, 1.12, \frac{\sqrt{5}}{2}, \frac{115}{99}, \frac{23}{20}$

21. $\frac{7071}{5000}, \frac{584}{413}, \sqrt{2}, \frac{47}{33}, \frac{127}{90}$

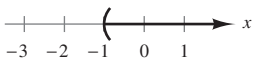
22. $\frac{26}{15}, \sqrt{3}, 1.73\overline{20}, \frac{381}{220}, \sqrt{10} - \sqrt{2}$

In Exercises 23–26, write an inequality that describes the graph.

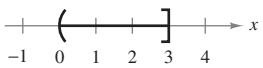
23.



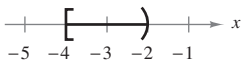
24.



25.



26.



In Exercises 27–36, give a verbal description of the subset of real numbers that is represented by the inequality, and sketch the subset on the real number line.

27. $x < 0$

28. $x < 2$

29. $x \leq 5$

30. $x \geq -2$

31. $x > 3$

32. $x \geq 4$

33. $-2 < x < 2$

34. $0 \leq x \leq 5$

35. $-1 \leq x < 0$

36. $0 < x \leq 6$

In Exercises 37–44, use inequality notation to describe the subset of real numbers.

37. x is positive.

38. t is no more than 20.

39. y is greater than 5 and less than or equal to 12.

40. m is at least -5 and at most 9.

41. The person's age A is at least 35.

42. The yield Y is no more than 42 bushels per acre.

43. The annual rate of inflation r is expected to be at least 3.5%, but no more than 6%.

44. The price p of unleaded gasoline is not expected to go below \$2.13 per gallon during the coming year.

In Exercises 45–54, evaluate the expression.

45. $|-10|$

46. $|0|$

47. $-3 - |-3|$

48. $|-1| - |-2|$

49. $-3|-3|$

50. $-5|-5|$

51. $\frac{-5}{|-5|}$

52. $\frac{|-4|}{-4}$

53. $|3 - \pi|$

54. $|4 - \pi|$

In Exercises 55–60, place the correct symbol (<, >, or =) between the two real numbers.

55. $|-7|$ ☐ $|7|$

56. -5 ☐ $-|5|$

57. $|-3|$ ☐ $-|-3|$

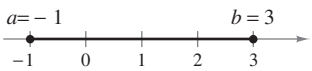
58. $-|-6|$ ☐ $|-6|$

59. $-|-2|$ ☐ $-|2|$

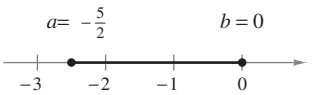
60. $-(-2)$ ☐ -2

In Exercises 61–70, find the distance between a and b.

61. $a = -1$, $b = 3$



62. $a = -\frac{5}{2}$, $b = 0$



63. $a = -4, b = -\frac{3}{2}$

64. $a = \frac{1}{4}, b = \frac{11}{4}$

65. $a = -\frac{7}{2}, b = 0$

66. $a = \frac{3}{4}, b = \frac{9}{4}$

67. $a = 126, b = 75$

68. $a = -126, b = -75$

69. $a = \frac{16}{5}, b = \frac{112}{75}$

70. $a = 9.34, b = -5.65$

In Exercises 71–78, use absolute value notation to describe the sentence.

71. The distance between z and $\frac{3}{2}$ is greater than 1.

72. The distance between x and 5 is no more than 3.

73. The distance between x and -10 is at least 6.

74. The distance between z and 0 is less than 8.

75. y is at least six units from 0.

76. x is less than eight units from 0.

77. x is more than five units from m.

78. y is at most two units from a.

79. Travel While traveling on the Pennsylvania Turnpike, you pass milepost 57 near Pittsburgh, then milepost 236 near Gettysburg. How far do you travel during that time period?

80. Travel While traveling on the Pennsylvania Turnpike, you pass milepost 326 near Valley Forge, then milepost 351 near Philadelphia. How far do you travel during that time period?

Temperature In Exercises 81 and 82, the record January temperatures (in degrees Fahrenheit) for a city are given. Find the distance between the numbers to determine the range of temperatures for January.

81. McGrath, Alaska: lowest: $-75°$F
highest: 54°F

82. Flagstaff, Arizona: lowest: $-22°$F
highest: 66°F

MAKE A DECISION: BUDGET VARIANCE In Exercises 83–88, the accounting department of an Internet start-up company is checking to see whether various actual expenses differ from the budgeted expenses by more than $500 or by more than 5%. Complete the missing parts of the table. Then determine whether the actual expense passes the "budget variance test."

	Budgeted Expense, b	*Actual Expense, a*	$\lvert a - b \rvert$	$0.05b$
83.	$30,000	$29,123.45		
84.	$125,500	$126,347.85		
85.	$12,000	$11,735.68		
86.	$8300	$8632.59		
87.	$40,800	$39,862.17		
88.	$2625	$2196.89		

MAKE A DECISION: QUALITY CONTROL In Exercises 89–94, the quality control inspector for a tire factory is testing the rim diameters of various tires. A tire is rejected if its rim diameter varies too much from its expected measure. The diameter should not differ by more than 0.02 inch or by more than 0.12% of the expected diameter measure. Complete the missing parts of the table. Then determine whether the tire is passed or rejected according to the inspector's guide lines.

	Expected Diameter, b	*Actual Diameter, a*	$\lvert a - b \rvert$	$0.0012b$
89.	14 in.	13.998 in.		
90.	15 in.	15.012 in.		
91.	16 in.	15.973 in.		
92.	17 in.	16.992 in.		
93.	18 in.	18.027 in.		
94.	19 in.	18.993 in.		

95. Think About It Consider $|u + v|$ and $|u| + |v|$.

(a) Are the values of the expressions always equal? If not, under what conditions are they unequal?

(b) If the two expressions are not equal for certain values of u and v, is one of the expressions always greater than the other? Explain.

96. Think About It Is there a difference between saying that a real number is positive and saying that a real number is nonnegative? Explain.

97. Describe the differences among the sets of natural numbers, integers, rational numbers, and irrational numbers.

98. Think About it Can it ever be true that $|a| = -a$ for a real number a? Explain.

Section 0.2

The Basic Rules of Algebra

- Identify the terms of an algebraic expression.
- Evaluate an algebraic expression.
- Identify basic rules of algebra.
- Perform operations on real numbers.
- Use a calculator to evaluate an algebraic expression.

Algebraic Expressions

One of the basic characteristics of algebra is the use of letters (or combinations of letters) to represent numbers. The letters used to represent numbers are called **variables,** and combinations of letters and numbers are called **algebraic expressions.** Some examples of algebraic expressions are

$$5x, \quad 2x - 3, \quad \frac{4}{x^2 + 2}, \quad \text{and} \quad 7x + y.$$

Algebraic Expression

A collection of letters (called **variables**) and real numbers (called **constants**) that are combined using the operations of addition, subtraction, multiplication, and division is an **algebraic expression.** (Other operations can also be used to form an algebraic expression.)

The **terms** of an algebraic expression are those parts that are separated by addition. For example, the algebraic expression $x^2 - 5x + 8$ has three terms: x^2, $-5x$, and 8. Note that $-5x$, rather than $5x$, is a term, because

$$x^2 - 5x + 8 = x^2 + (-5x) + 8.$$

The terms x^2 and $-5x$ are the **variable terms** of the expression, and 8 is the **constant term** of the expression. The numerical factor of a variable term is the **coefficient** of the variable term. For instance, the coefficient of the variable term $-5x$ is -5, and the coefficient of the variable term x^2 is 1.

Example 1 Identifying the Terms of an Algebraic Expression

Algebraic Expression	*Terms*
a. $4x - 3$	$4x, -3$
b. $2x + 4y - 5$	$2x, 4y, -5$

✓CHECKPOINT 1

Identify the terms of each algebraic expression.

a. $8 - 15x$

b. $4x^2 - 3y - 7$ ■

STUDY TIP

When you evaluate an expression with grouping symbols (such as parentheses), be careful to use the correct order of operations.

Example 2 Symbols of Grouping

a. $7 - 3(4 - 2) = 7 - 3(2) = 7 - 6 = 1$

b. $(4 - 5) - (3 - 6) = (-1) - (-3) = -1 + 3 = 2$

✓CHECKPOINT 2

Simplify the expression $5(7 - 3) + 9$. ■

The **Substitution Principle** states, "If $a = b$, then a can be replaced by b in any expression involving a." You use this principle to **evaluate** an algebraic expression by substituting numerical values for each of the variables in the expression. In the first evaluation shown below, 3 is substituted for x in the expression $-3x + 5$.

Expression	*Value of Variable*	*Substitution*	*Value of Expression*
$-3x + 5$	$x = 3$	$-3(3) + 5$	$-9 + 5 = -4$
$3x^2 + 2x - 1$	$x = -1$	$3(-1)^2 + 2(-1) - 1$	$3 - 2 - 1 = 0$
$-2x(x + 4)$	$x = -2$	$-2(-2)(-2 + 4)$	$-2(-2)(2) = 8$
$\frac{1}{x - 2}$	$x = 2$	$\frac{1}{2 - 2}$	Undefined

TECHNOLOGY

To evaluate the expression $3 + 4x$ for the values 2 and 5, use the *last entry* feature of a graphing utility.

1. Evaluate $3 + 4 \cdot 2$.
2. Press [2nd] [ENTRY] (recalls previous expression to the home screen).
3. Cursor to 2, replace 2 with 5, and press [ENTER].

For specific keystrokes for the *last entry* feature, go to the text website at *college.hmco.com/info/larsonapplied.*

Example 3 Evaluating Algebraic Expressions

Evaluate each algebraic expression when $x = -2$ and $y = 3$.

a. $4y - 2x$ **b.** $5 + x^2$ **c.** $5 - y^2$

SOLUTION

a. When $x = -2$ and $y = 3$, the expression $4y - 2x$ has a value of

$$4(3) - 2(-2) = 12 + 4 = 16.$$

b. When $x = -2$, the expression $5 + x^2$ has a value of

$$5 + (-2)^2 = 5 + 4 = 9.$$

c. When $y = 3$, the expression $5 - y^2$ has a value of

$$5 - (3)^2 = 5 - 9 = -4.$$

✓CHECKPOINT 3

Evaluate $3y + x^2$ when $x = 4$ and $y = -2$. ■

Basic Rules of Algebra

The four basic arithmetic operations are **addition, multiplication, subtraction,** and **division,** denoted by the symbols $+$, $\times$ or $\cdot$, $-$, and $\div$, respectively. Of these, addition and multiplication are considered to be the two primary arithmetic operations. Subtraction and division are defined as the inverse operations of addition and multiplication, as follows.

The symbol (T) indicates when to use graphing technology or a symbolic computer algebra system to solve a problem or an exercise. The solutions of other exercises may also be facilitated by use of appropriate technology.

Subtraction: Add the opposite

$$a - b = a + (-b)$$

Division: Multiply by the reciprocal

If $b \neq 0$, then $a \div b = a\left(\frac{1}{b}\right) = \frac{a}{b}$.

In these definitions, $-b$ is called the **additive inverse** (or opposite) of b, and $1/b$ is called the **multiplicative inverse** (or reciprocal) of b. In place of $a \div b$, you can use the fraction symbol a/b. In this fractional form, a is called the **numerator** of the fraction and b is called the **denominator.**

The **basic rules of algebra,** listed below, are true for variables and algebraic expressions as well as for real numbers.

Basic Rules of Algebra

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property	*Example*
Commutative Property of Addition	
$a + b = b + a$	$4x + x^2 = x^2 + 4x$
Commutative Property of Multiplication	
$ab = ba$	$(4 - x)x^2 = x^2(4 - x)$
Associative Property of Addition	
$(a + b) + c = a + (b + c)$	$(-x + 5) + 2x^2 = -x + (5 + 2x^2)$
Associative Property of Multiplication	
$(ab)c = a(bc)$	$(2x \cdot 3y)(8) = (2x)(3y \cdot 8)$
Distributive Property	
$a(b + c) = ab + ac$	$3x(5 + 2x) = 3x \cdot 5 + 3x \cdot 2x$
$(a + b)c = ac + bc$	$(y + 8)y = y \cdot y + 8 \cdot y$
Additive Identity Property	
$a + 0 = a$	$5y^2 + 0 = 5y^2$
Multiplicative Identity Property	
$a \cdot 1 = a$	$(4x^2)(1) = 4x^2$
Additive Inverse Property	
$a + (-a) = 0$	$5x^3 + (-5x^3) = 0$
Multiplicative Inverse Property	
$a \cdot \frac{1}{a} = 1, \quad a \neq 0$	$(x^2 + 4)\left(\frac{1}{x^2 + 4}\right) = 1$

Because subtraction is defined as "adding the opposite," the Distributive Property is also true for subtraction. For instance, the "subtraction form" of $a(b + c) = ab + ac$ is $a(b - c) = a[b + (-c)] = ab + a(-c) = ab - ac$.

Example 4 Identifying the Basic Rules of Algebra

Identify the rule of algebra illustrated by each statement.

a. $(4x^2)5 = 5(4x^2)$

b. $(2y^3 + y) - (2y^3 + y) = 0$

c. $(4 + x^2) + 3x^2 = 4 + (x^2 + 3x^2)$

d. $(x - 5)7 + (x - 5)x = (x - 5)(7 + x)$

e. $2x \cdot \frac{1}{2x} = 1, \quad x \neq 0$

SOLUTION

a. This equation illustrates the Commutative Property of Multiplication.

b. This equation illustrates the Additive Inverse Property.

c. This equation illustrates the Associative Property of Addition.

d. This equation illustrates the Distributive Property in reverse order.

$$ab + ac = a(b + c) \qquad \text{Distributive Property}$$

$$(x - 5)7 + (x - 5)x = (x - 5)(7 + x)$$

e. This equation illustrates the Multiplicative Inverse Property. Note that it is important that x be a nonzero number. If x were allowed to be zero, you would be in trouble because the reciprocal of zero is undefined.

✓CHECKPOINT 4

Identify the rule of algebra illustrated by each statement.

a. $3x^2 \cdot 1 = 3x^2$

b. $x^2 + 5 = 5 + x^2$ ■

The following three lists summarize the basic properties of negation, zero, and fractions. When you encounter such lists, you should not only *memorize* a verbal description of each property, but you should also try to gain an *intuitive feeling* for the validity of each.

Properties of Negation

Let a and b be real numbers, variables, or algebraic expressions.

Property	*Example*
1. $(-1)a = -a$	$(-1)7 = -7$
2. $-(-a) = a$	$-(-6) = 6$
3. $(-a)b = -(ab) = a(-b)$	$(-5)3 = -(5 \cdot 3) = 5(-3)$
4. $(-a)(-b) = ab$	$(-2)(-6) = 2 \cdot 6$
5. $-(a + b) = (-a) + (-b)$	$-(3 + 8) = (-3) + (-8)$

Be sure you see the difference between the opposite of a number and a negative number. If a is negative, then its opposite, $-a$, is positive. For instance, if $a = -5$, then $-a = -(-5) = 5$.

Properties of Zero

Let a and b be real numbers, variables, or algebraic expressions. Then the following properties are true.

1. $a + 0 = a$ and $a - 0 = a$

2. $a \cdot 0 = 0$

3. $\dfrac{0}{a} = 0, \quad a \neq 0$

4. $\dfrac{a}{0}$ is undefined.

5. Zero-Factor Property: If $ab = 0$, then $a = 0$ or $b = 0$.

The "or" in the Zero-Factor Property includes the possibility that both factors are zero. This is called an **inclusive or,** and it is the way the word "or" is always used in mathematics.

Properties of Fractions

Let a, b, c, and d be real numbers, variables, or algebraic expressions such that $b \neq 0$ and $d \neq 0$. Then the following properties are true.

1. *Equivalent fractions:* $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$.

2. *Rules of signs:* $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ and $\dfrac{-a}{-b} = \dfrac{a}{b}$

3. *Generate equivalent fractions:* $\dfrac{a}{b} = \dfrac{ac}{bc}, \quad c \neq 0$

4. *Add or subtract with like denominators:* $\dfrac{a}{b} \pm \dfrac{c}{b} = \dfrac{a \pm c}{b}$

5. *Add or subtract with unlike denominators:* $\dfrac{a}{b} \pm \dfrac{c}{d} = \dfrac{ad \pm bc}{bd}$

6. *Multiply fractions:* $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$

7. *Divide fractions:* $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}, \quad c \neq 0$

In Property 1 (equivalent fractions) the phrase "if and only if" implies two statements. One statement is: If $a/b = c/d$, then $ad = bc$. The other statement is: If $ad = bc$, where $b \neq 0$ and $d \neq 0$, then $a/b = c/d$.

Example 5 **Properties of Zero and Properties of Fractions**

a. $x - \frac{0}{5} = x - 0 = x$ Properties 3 and 1 of zero

b. $\frac{x}{5} = \frac{3 \cdot x}{3 \cdot 5} = \frac{3x}{15}$ Generate equivalent fractions.

c. $\frac{x}{3} + \frac{2x}{5} = \frac{x \cdot 5 + 3 \cdot 2x}{15} = \frac{11x}{15}$ Add fractions with unlike denominators.

d. $\frac{7}{x} \div \frac{3}{2} = \frac{7}{x} \cdot \frac{2}{3} = \frac{14}{3x}$ Divide fractions.

✓CHECKPOINT 5

Simplify the expression $\frac{x}{4} + \frac{2x}{3}$. ■

If a, b, and c are integers such that $ab = c$, then a and b are **factors** or **divisors** of c. For example, 2 and 3 are factors of 6 because $2 \cdot 3 = 6$. A **prime number** is a positive integer that has exactly two factors: itself and 1. For example, 2, 3, 5, 7, and 11 are prime numbers, whereas 1, 4, 6, 8, 9, and 10 are not. The numbers 4, 6, 8, 9, and 10 are **composite** because they can be written as the products of two or more prime numbers. The number 1 is neither prime nor composite. The **Fundamental Theorem of Arithmetic** states that every positive integer greater than 1 is a prime number or can be written as the product of prime numbers in precisely one way (disregarding order). For instance, the *prime factorization* of 24 is

$$24 = 2 \cdot 2 \cdot 2 \cdot 3.$$

When you are adding or subtracting fractions that have unlike denominators, you can use Property 4 of fractions by rewriting both of the fractions so that they have the same denominator. This is called the **least common denominator** method.

STUDY TIP

To find the LCD, first factor the denominators.

$15 = 3 \cdot 5$

$9 = 3^2$

$5 = 5$

The LCD is the product of the prime factors, with each factor given the highest power of its occurrence in any denominator. So, the LCD is

$3^2 \cdot 5 = 45.$

Example 6 **Adding and Subtracting Fractions**

Evaluate $\frac{2}{15} - \frac{5}{9} + \frac{4}{5}$.

SOLUTION Begin by factoring the denominators to find the least common denominator (LCD). Use the LCD, 45, to rewrite the fractions and simplify.

$$\frac{2}{15} - \frac{5}{9} + \frac{4}{5} = \frac{2 \cdot 3}{15 \cdot 3} - \frac{5 \cdot 5}{9 \cdot 5} + \frac{4 \cdot 9}{5 \cdot 9}$$

$$= \frac{6 - 25 + 36}{45}$$

$$= \frac{17}{45}$$

✓CHECKPOINT 6

Evaluate $\frac{3}{4} + \frac{2}{3} - \frac{1}{2}$. ■

Equations

An **equation** is a statement of equality between two expressions. So, the statement

$$a + b = c + d$$

means that the expressions $a + b$ and $c + d$ represent the same number. For instance, because $1 + 4$ and $3 + 2$ both represent the number 5, you can write $1 + 4 = 3 + 2$. Three important properties of equality follow.

Properties of Equality

Let a, b, and c be real numbers, variables, or algebraic expressions.

1. Reflexive: $a = a$

2. Symmetric: If $a = b$,then $b = a$.

3. Transitive: If $a = b$and $b = c$,then $a = c$.

In algebra, you often rewrite expressions by making substitutions that are permitted under the Substitution Principle. Two important consequences of the Substitution Principle are the following rules.

1. If $a = b$, then $a + c = b + c$. **2.** If $a = b$, then $ac = bc$.

The first rule allows you to add the same number to each side of an equation. The second allows you to multiply each side of an equation by the same number. The converses of these two rules are also true and are listed below.

1. If $a + c = b + c$, then $a = b$. **2.** If $ac = bc$ and $c \neq 0$, then $a = b$.

So, you can also subtract the same number from each side of an equation as well as divide each side of an equation by the same nonzero number.

Calculators and Rounding

The table below shows keystrokes for several similar functions on a standard scientific calculator and a graphing calculator. These keystrokes may not be the same as those for your calculator. Consult your user's guide for specific keystrokes.

Graphing Calculator	*Scientific Calculator*
ENTER	=
(−)	+/−
^	y^x
x^{-1}	1/x

For example, you can evaluate 13^3 on a graphing calculator or a scientific calculator as follows.

Graphing Calculator	*Scientific Calculator*
13 ^ 3 ENTER	13 y^x 3 =

Example 7 Using a Calculator

Scientific Calculator

	Expression	*Keystrokes*	*Display*
a.	$7 - (5 \cdot 3)$	7 (−) 5 (×) 3 (=)	−8
b.	$-12^2 - 100$	12 (x^2) (+/−) (−) 100 (=)	−244
c.	$24 \div 2^3$	24 (÷) 2 (y^x) 3 (=)	3
d.	$3(10 - 4^2) \div 2$	3 (×) (10 (−) 4 (x^2)) (÷) 2 (=)	−9
e.	37% of 40	.37 (×) 40 (=)	14.8

Graphing Calculator

	Expression	*Keystrokes*	*Display*
a.	$7 - (5 \cdot 3)$	7 (−) 5 (×) 3 (ENTER)	−8
b.	$-12^2 - 100$	(−) 12 (x^2) (−) 100 (ENTER)	−244
c.	$24 \div 2^3$	24 (÷) 2 (^) 3 (ENTER)	3
d.	$3(10 - 4^2) \div 2$	3 (10 (−) 4 (x^2)) (÷) 2 (ENTER)	−9
e.	37% of 40	.37 (×) 40 (ENTER)	14.8

TECHNOLOGY

Be sure you see the difference between the change sign key (+/−) or ((−)) and the subtraction key (−), as used in Example 7(b).

✓CHECKPOINT 7

Write the keystrokes you can use to evaluate

$$6(8^3 - 481)$$

on a graphing calculator or a scientific calculator. ■

When rounding decimals, look at the *decision digit* (the digit at the right of the last digit you want to keep). Round up when the decision digit is 5 or greater, and round down when it is 4 or less.

Example 8 Rounding Decimal Numbers

	Number	*Rounded to Three Decimal Places*	
a.	$\sqrt{2} = 1.4142135\ldots$	1.414	Round down because $2 < 4$.
b.	$\pi = 3.1415926\ldots$	3.142	Round up because $5 = 5$.
c.	$\frac{7}{9} = 0.7777777\ldots$	0.778	Round up because $7 > 5$.

✓CHECKPOINT 8

Use a calculator to evaluate

$$4\left(\frac{2}{3} + \frac{4}{5}\right).$$

Then round the result to two decimal places. ■

CONCEPT CHECK

1. Write an algebraic expression that contains a variable term, a constant term, and a coefficient. Identify the parts of your expression.
2. Is $(a - b) + c = a - (b + c)$ when a, b, and c are nonzero real numbers? Explain.
3. Is the expression $-x$ always negative? Explain.
4. Explain how to divide a/b by c/d when b, c, and d are nonzero real numbers.

Skills Review 0.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.1.

In Exercises 1–4, place the correct inequality symbol ($<$ or $>$) between the two real numbers.

1. $-4 \quad -2$ **2.** $0 \quad -3$ **3.** $\sqrt{3} \quad 1.73$ **4.** $-\pi \quad -3$

In Exercises 5–8, find the distance between the two real numbers.

5. 4, 6 **6.** $-2, 2$ **7.** $0, -5$ **8.** $-1, 3$

In Exercises 9 and 10, evaluate the expression.

9. $|-7| + |7|$ **10.** $-|8 - 10|$

Exercises 0.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, identify the terms of the algebraic expression.

1. $7x + 4$ **2.** $-5 + 3x$

3. $x^2 - 4x + 8$ **4.** $4x^3 + x - 5$

5. $2x^2 - 9x + 13$ **6.** $3x^4 + 2x^3 - 1$

In Exercises 7–10, simplify the expression.

7. $(8 - 17) + 3$ **8.** $-3(5 - 2)$

9. $(4 - 7)(-2)$ **10.** $-5(-2 - 6)$

In Exercises 11–16, evaluate the expression for each value of x. (If not possible, state the reason.)

	Expression	*Values*	
11.	$4x - 6$	(a) $x = -1$	(b) $x = 0$
12.	$5 - 3x$	(a) $x = -3$	(b) $x = 2$
13.	$x^2 - 3x + 4$	(a) $x = -2$	(b) $x = 2$
14.	$-x^3 + 2x - 1$	(a) $x = 0$	(b) $x = 2$
15.	$\dfrac{x}{x - 2}$	(a) $x = -2$	(b) $x = 2$
16.	$\dfrac{x + 3}{x - 3}$	(a) $x = 3$	(b) $x = -3$

In Exercises 17–22, evaluate the expression when $x = 3$, $y = -2$, and $z = 4$.

17. $x + 3y + z$ **18.** $6z + 5x - 3y$

19. $x^2 - 5y + 4z$ **20.** $z^2 + 6y - x$

21. $\dfrac{x - y}{5z}$ **22.** $\dfrac{4z - 2y}{20x}$

In Exercises 23–38, identify the rule(s) of algebra illustrated by the statement.

23. $3 + 4 = 4 + 3$ **24.** $x + 9 = 9 + x$

25. $-15 + 15 = 0$ **26.** $(x + 2) - (x + 2) = 0$

27. $2(x + 3) = 2x + 6$

28. $(5 + 11) \cdot 6 = 5 \cdot 6 + 11 \cdot 6$

29. $2\left(\frac{1}{2}\right) = 1$

30. $\dfrac{1}{h + 6}(h + 6) = 1, \quad h \neq -6$

31. $h + 0 = h$ **32.** $(z - 2) + 0 = z - 2$

33. $57 \cdot 1 = 57$ **34.** $1 \cdot (1 + x) = 1 + x$

35. $6 + (7 + 8) = (6 + 7) + 8$

36. $x + (y + 10) = (x + y) + 10$

37. $x(3y) = (x \cdot 3)y = (3x)y$

38. $\frac{1}{7}(7 \cdot 12) = \left(\frac{1}{7} \cdot 7\right)12 = 1 \cdot 12 = 12$

In Exercises 39–42, write the prime factorization of the integer.

39. 48 **40.** 24

41. 240 **42.** 150

In Exercises 43–50, perform the indicated operation(s). (Write fractional answers in simplest form.)

43. $2\left(\dfrac{77}{-11}\right)$ **44.** $\dfrac{27 - 35}{4}$

45. $\frac{5}{8} + \frac{1}{4} - \frac{5}{6}$ **46.** $\frac{10}{11} + \frac{6}{33} - \frac{13}{66}$

47. $\frac{2}{5} \cdot \frac{7}{8}$ **48.** $\left(-\frac{2}{3}\right) \cdot \frac{5}{8} \cdot \frac{3}{4}$

49. $\frac{2}{3} \div 8$ **50.** $\left(\frac{3}{5} \div 3\right) - \left(6 \cdot \frac{4}{8}\right)$

In Exercises 51–54, use a calculator to evaluate the expression. (Round to two decimal places.)

51. $3\left(-\frac{5}{12}+\frac{3}{8}\right)$

52. $2\left(-7+\frac{1}{6}\right)$

53. $\dfrac{11.46-5.37}{3.91}$

54. $\dfrac{-8.31+4.83}{7.65}$

In Exercises 55–58, use a calculator to solve.

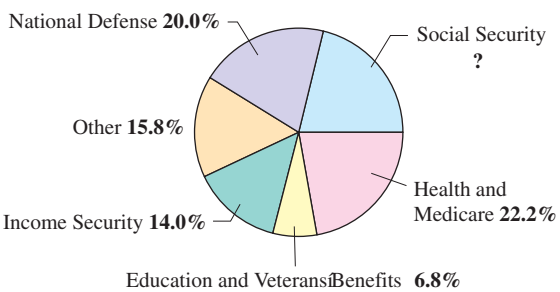
55. 35% of 68

56. 35% of 820

57. 125% of 37

58. 147% of 22

59. Federal Government Expenses The circle graph shows the types of expenses for the federal government in 2005. *(Source: Office of Management and Budget)*



(a) What percent of the total expenses was the amount spent on Social Security?

(b) The total of the 2005 expenses was $2,472,200,000,000. Find the amount spent for each category in the circle graph. (Round to the nearest billion dollars.)

60. Research Study The percent of people in a research study that have a particular health risk is 39.5%. The total number of people in the study is 12,857. How many people have the health risk?

61. Clinical Trial The percent of patients in a clinical trial of a cancer treatment showing a decrease in tumor size is 49.2%. There are 3445 patients in the trial. How many patients show a decrease in tumor size?

62. Calculator Keystrokes Write the algebraic expression that corresponds to each set of keystrokes.

(a) 5 × (2.7 − 9.4) = Scientific

5 (2.7 − 9.4) ENTER Graphing

(b) 2 × (4 +/− + 2) = Scientific

2 ((−) 4 + 2) ENTER Graphing

63. Calculator Keystrokes Write the keystrokes used to evaluate each algebraic expression on either a scientific or a graphing calculator.

(a) $5(18-2^3)\div 10$ (b) $-6^2-[7+(-2)^3]$

The symbol (S) indicates an exercise in which you are instructed to use a spreadsheet.

(S) In Exercises 64 and 65, a breakdown of pet spending for one year in the United States is given. Find the percent of total pet spending for each subcategory. Then use a spreadsheet software program to make a labeled circle graph for the percent data. *(Source: American Pet Products Manufacturers Association)*

64. Total pet spending (2005):	**$36.3 billion**
Food:	$14.7 billion
Vet care:	$8.7 billion
Supplies/OTC medicine:	$8.7 billion
Live animal purchases:	$1.7 billion
Grooming and boarding:	$2.5 billion
65. Total pet spending (2006):	**$38.4 billion**
Food:	$15.2 billion
Vet care:	$9.4 billion
Supplies/OTC medicine:	$9.3 billion
Live animal purchases:	$1.8 billion
Grooming and boarding:	$2.7 billion

Business Capsule



Tim Sloan/AFP/Getty Images

PetSmart, the largest U.S. pet store chain with 909 stores, has grown by offering pet lodging services in some stores. PetsHotels provide amenities such as supervised play areas with toys and slides, hypoallergenic lambskin blankets, TV, healthy pet snacks, and special fee services such as grooming, training, and phoning pet parents. These services are twice as profitable as retail sales, and they tend to attract greater sales as well. PetSmart's sales were 29% higher in stores with established PetsHotels than in those without them. In 2006, PetSmart had a goal to expand from 62 to 435 PetsHotels.

66. Research Project Use your campus library, the Internet, or some other reference source to find information about "special services" companies experiencing strong growth as in the example above. Write a brief report about one of these companies.

Section 0.3

Integer Exponents

- Use properties of exponents.
- Use scientific notation to represent real numbers.
- Use a calculator to raise a number to a power.
- Use interest formulas to solve an application problem.

Properties of Exponents

Repeated multiplication of a real number by itself can be written in **exponential form.** Here are some examples.

Repeated Multiplication	*Exponential Form*
$7 \cdot 7$	7^2
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

STUDY TIP

It is important to recognize the difference between exponential forms such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. Similarly, in $(5x)^3$, the parentheses indicate that the exponent applies to the 5 as well as to the x, whereas in $5x^3 = 5(x^3)$, the exponent applies only to the x.

Exponential Notation

Let a be a real number, a variable, or an algebraic expression, and let n be a positive integer. Then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** The expression a^n is read as "a to the nth **power**" or simply "a to the n."

When multiplying exponential expressions with the same base, *add* exponents.

$$a^m \cdot a^n = a^{m+n}$$ Add exponents when multiplying.

For instance, to multiply 2^2 and 2^3, you can write

$$2^2 \cdot 2^3 = \overbrace{(2 \cdot 2)}^{\text{Two factors}} \cdot \overbrace{(2 \cdot 2 \cdot 2)}^{\text{Three factors}} = \overbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}^{\text{Five factors}}$$
$$= 2^{2+3} = 2^5.$$

On the other hand, when dividing exponential expressions, *subtract* exponents. That is,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0.$$ Subtract exponents when dividing.

These and other properties of exponents are summarized in the list on the following page.

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (Assume all denominators and bases are nonzero.)

	Property	*Example*	
1.	$a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6$	Product of Powers
2.	$\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$	Quotient of Powers
3.	$(ab)^m = a^m b^m$	$(5x)^3 = 5^3 x^3 = 125x^3$	Power of a Product
4.	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$	Power of a Quotient
5.	$(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12}$	Power of a Power
6.	$a^{-n} = \dfrac{1}{a^n}$	$y^{-4} = \dfrac{1}{y^4}$	Definition of negative exponent
7.	$a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$	Definition of zero exponent
8.	$\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n, \quad a \neq 0, b \neq 0$	$\left(\dfrac{3}{2}\right)^{-3} = \left(\dfrac{2}{3}\right)^3$	
9.	$\lvert a^2 \rvert = \lvert a \rvert^2 = a^2$	$\lvert 2^2 \rvert = \lvert 2 \rvert^2 = 2^2$	

Notice that these properties of exponents apply for *all* integers m and n, not just positive integers. For instance, by the Quotient of Powers Property,

$$\frac{3^4}{3^{-5}} = 3^{4-(-5)} = 3^{4+5} = 3^9.$$

DISCOVERY

Example 1 **Using Properties of Exponents**

a. $3^4 \cdot 3^{-1} = 3^{4-1} = 3^3 = 27$

b. $\dfrac{5^6}{5^4} = 5^{6-4} = 5^2 = 25$

c. $5\left(\dfrac{2}{5}\right)^3 = 5 \cdot \dfrac{2^3}{5^3} = 5 \cdot 5^{-3} \cdot 2^3 = 5^{-2} \cdot 2^3 = \dfrac{2^3}{5^2} = \dfrac{8}{25}$

d. $(-5 \cdot 2^3)^2 = (-5)^2 \cdot (2^3)^2 = 25 \cdot 2^6 = 25 \cdot 64 = 1600$

e. $(-3ab^4)(4ab^{-3}) = -3(4)(a)(a)(b^4)(b^{-3}) = -12a^2b$

f. $3a(-4a^2)^0 = 3a(1) = 3a, \quad a \neq 0$

g. $\left(\dfrac{5x^3}{y}\right)^2 = \dfrac{5^2(x^3)^2}{y^2} = \dfrac{25x^6}{y^2}$

✓CHECKPOINT 1

Evaluate the expression $4^2 \cdot 4^3$. ■

The next example shows how expressions involving negative exponents can be rewritten using positive exponents.

STUDY TIP

Rarely in algebra is there only one way to solve a problem. Don't be concerned if the steps you use to solve a problem are not exactly the same as the steps presented here. The important thing is to use steps that you understand *and* that, of course, are justified by the rules of algebra. For instance, you might prefer the following steps to simplify Example 2(d).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^{2} = \frac{y^2}{9x^4}$$

Example 2 Rewriting with Positive Exponents

a. $x^{-1} = \frac{1}{x}$ Definition of negative exponent

b. $\frac{1}{3x^{-2}} = \frac{1(x^2)}{3} = \frac{x^2}{3}$ The exponent -2 applies only to x.

c. $\frac{12a^3b^{-4}}{4a^{-2}b} = \frac{12a^3 \cdot a^2}{4b \cdot b^4}$ Definition of negative exponent

$= \frac{3a^5}{b^5}$ Product of Powers Property

d. $\left(\frac{3x^2}{y}\right)^{-2} = \frac{3^{-2}(x^2)^{-2}}{y^{-2}}$ Power of a Quotient and Power of a Product Properties

$= \frac{3^{-2}x^{-4}}{y^{-2}}$ Power of a Power Property

$= \frac{y^2}{3^2x^4}$ Definition of negative exponent

$= \frac{y^2}{9x^4}$ Simplify.

✓CHECKPOINT 2

Rewrite $\left(\frac{3}{x^{-2}z^4}\right)^{-3}$ with positive exponents and simplify. ■

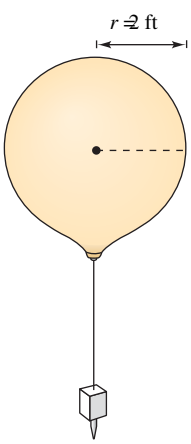


FIGURE 0.7

Example 3 Ratio of Volume to Surface Area

The volume V and surface area S of a sphere are given by

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad S = 4\pi r^2$$

where r is the radius of the sphere. A spherical weather balloon has a radius of 2 feet, as shown in Figure 0.7. Find the ratio of the volume to the surface area.

SOLUTION To find the ratio, write the quotient V/S and simplify.

$$\frac{V}{S} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{\frac{4}{3}\pi 2^3}{4\pi 2^2} = \frac{1}{3}(2) = \frac{2}{3}$$

✓CHECKPOINT 3

Evaluate $\frac{\frac{5}{7}x^7}{25x^5}$ when $x = 7$. ■

Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, a drop of water contains more than 33 billion billion molecules—that is, 33 followed by 18 zeros.

$$33{,}000{,}000{,}000{,}000{,}000{,}000$$

It is convenient to write such numbers in **scientific notation.** This notation has the form $c \times 10^n$, where $1 \leq c < 10$ and n is an integer. So, the number of molecules in a drop of water can be written in scientific notation as

$$3.3 \times 10{,}000{,}000{,}000{,}000{,}000{,}000 = 3.3 \times 10^{19}.$$

The *positive* exponent 19 indicates that the number is *large* (10 or more) and that the decimal point has been moved 19 places. A *negative* exponent in scientific notation indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.0000000000000000000000000009.$$

28 decimal places

Example 4 Converting to Scientific Notation

a. $0.0000572 = 5.72 \times 10^{-5}$ Number is less than 1.

b. $149{,}400{,}000 = 1.494 \times 10^{8}$ Number is greater than 10.

c. $32.675 = 3.2675 \times 10^{1}$ Number is greater than 10.

✓CHECKPOINT 4

Write 0.00345 in scientific notation. ■

Example 5 Converting to Decimal Notation

a. $3.125 \times 10^{2} = 312.5$ Number is greater than 10.

b. $3.73 \times 10^{-6} = 0.00000373$ Number is less than 1.

c. $7.91 \times 10^{5} = 791{,}000$ Number is greater than 10.

✓CHECKPOINT 5

Write 4.28×10^{5} in decimal notation. ■

Most calculators automatically use scientific notation when showing large (or small) numbers that exceed the display range. Try multiplying $86{,}500{,}000 \times 6000$. If your calculator follows standard conventions, its display should be

5.19 11 or 5.19E11 .

This means that $c = 5.19$ and the exponent of 10 is $n = 11$, which implies that the number is 5.19×10^{11}. To *enter* numbers in scientific notation, your calculator should have an exponential entry key labeled EXP or EE.

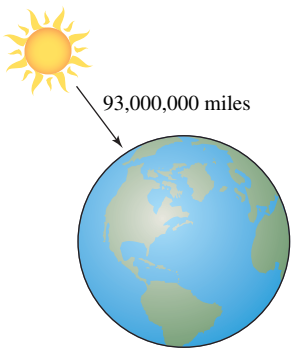


FIGURE 0.8

✓CHECKPOINT 6

Evaluate the expression

$$\frac{4.6 \times 10^5}{2.3 \times 10^2}.$$ ■

Example 6 The Speed of Light

The distance between Earth and the sun is approximately 93 million miles, as shown in Figure 0.8. How long does it take for light to travel from the sun to Earth? Use the fact that light travels at a rate of approximately 186,000 miles per second.

SOLUTION Using the formula distance = (rate)(time), you find the time as follows.

$$\begin{aligned}\text{Time} = \frac{\text{distance}}{\text{rate}} &= \frac{93 \text{ million miles}}{186{,}000 \text{ miles per second}} \\ &= \frac{9.3 \times 10^7 \text{ miles}}{1.86 \times 10^5 \text{ miles/second}} \\ &= 5 \times 10^2 \text{ seconds} \\ &\approx 8.33 \text{ minutes}\end{aligned}$$

Note that to convert 500 seconds to 8.33 minutes, you divide by 60, because there are 60 seconds in one minute.

Powers and Calculators

One of the most useful features of a calculator is its ability to evaluate exponential expressions. Consult your user's guide for specific keystrokes.

Example 7 Using a Calculator to Raise a Number to a Power

Scientific Calculator

	Expression	*Keystrokes*	*Display*
a.	$13^4 + 5$	13 [yˣ] 4 [+] 5 [=]	28566
b.	$3^{-2} + 4^{-1}$	3 [yˣ] 2 [+/−] [+] 4 [yˣ] 1 [+/−] [=]	0.361111111
c.	$\frac{3^5 + 1}{3^5 - 1}$	[(] 3 [yˣ] 5 [+] 1 [)] [÷] [(] 3 [yˣ] 5 [−] 1 [)] [=]	1.008264463

Graphing Calculator

	Expression	*Keystrokes*	*Display*
a.	$13^4 + 5$	13 [^] 4 [+] 5 [ENTER]	28566
b.	$3^{-2} + 4^{-1}$	3 [^] [(−)] 2 [+] 4 [^] [(−)] 1 [ENTER]	.3611111111
c.	$\frac{3^5 + 1}{3^5 - 1}$	[(] 3 [^] 5 [+] 1 [)] [÷] [(] 3 [^] 5 [−] 1 [)] [ENTER]	1.008264463

TECHNOLOGY

Make sure you include parentheses as needed when entering expressions in your calculator. Notice the use of parentheses in Example 7(c).

✓CHECKPOINT 7

Use a calculator to evaluate

$$\frac{4^4 - 6}{2^5 + 18}.$$ ■

Applications

The formulas shown below can be used to find the balance in a savings account.

Balance in an Account

The balance A in an account that earns an annual interest rate r (in decimal form) for t years is given by one of the following.

$A = P(1 + rt)$ Simple interest

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compound interest

In both formulas, P is the principal (or the initial deposit). In the formula for compound interest, n is the number of compoundings *per year*. Make sure you convert all units of time t to years. For instance, 6 months $= \frac{1}{2}$ year. So, $t = \frac{1}{2}$.

Example 8 *MAKE A DECISION* Finding the Balance in an Account

You are trying to decide how to invest $5000 for 10 years. Which savings plan will earn more money?

a. 4% simple annual interest

b. 3.5% interest compounded quarterly

SOLUTION

a. The balance after 10 years is

$$\begin{aligned} A &= P(1 + rt) \\ &= 5000[1 + 0.04(10)] \\ &= \$7000. \end{aligned}$$

b. The balance after 10 years is

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} \\ &= 5000\left(1 + \frac{0.035}{4}\right)^{(4)(10)} \\ &\approx \$7084.54. \end{aligned}$$

Savings plan (a) will earn $7000 - 5000 = \$2000$ and savings plan (b) will earn $7084.54 - 5000 = \$2084.54$. So, plan (b) will earn more money.

✓CHECKPOINT 8

In Example 8, how much money would you earn in a savings plan with 3.4% annual interest compounded monthly? ■

In addition to finding the balance in an account, the compound interest formula can also be used to determine the rate of inflation. To apply the formula, you must know the cost of an item for two different years, as demonstrated in Example 9.

Example 9 Finding the Rate of Inflation

In 1984, the cost of a first-class postage stamp was \$0.20. By 2007, the cost increased to \$0.41, as shown in Figure 0.9. Find the average annual rate of inflation for first-class postage over this 23-year period. *(Source: U.S. Postal Service)*

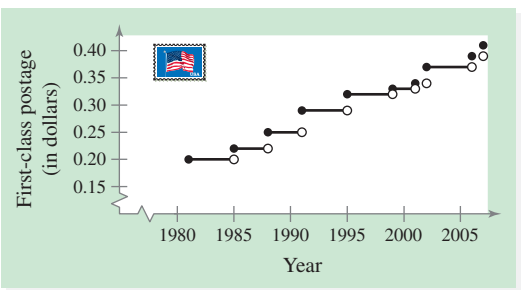


FIGURE 0.9

SOLUTION To find the average annual rate of inflation, use the formula for compound interest with *annual* compounding. So, you need to find the value of r that will make the following equation true.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$0.41 = 0.20(1 + r)^{23}$$

You can begin by guessing that the average annual rate of inflation was 5%. Entering $r = 0.05$ in the formula, you find that $0.20(1 + 0.05)^{23} \approx 0.6143$. Because this result is more than 0.41, try some smaller values of r. Finally, you can discover that

$$0.20(1 + 0.032)^{23} \approx 0.41.$$

So, the average annual rate of inflation for first-class postage from 1984 to 2007 was about 3.2%.

✓CHECKPOINT 9

The fee for a medical school application was \$85. Three years later, the application fee is \$95. What is the average annual rate of inflation over this three-year period? ■

CONCEPT CHECK

1. Explain how to simplify the expression $a^{0.5}(a^{1.5})$.
2. Because $-2^3 = -8$ and $(-2)^3 = -8$, a student concludes that $-a^n = (-a)^n$, where n is an integer. Do you agree? Can you give an example where $-a^n \neq (-a)^n$?
3. A student claims "Any number with a zero exponent is equal to 1." Is the student correct? Explain.
4. Is 0.12×10^5 written in scientific notation? Explain.

Skills Review 0.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.2.

In Exercises 1–10, perform the indicated operation(s) and simplify.

1. $\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)$
2. $\left(\frac{1}{4}\right)(5)(4)$
3. $3\left(\frac{2}{7}\right) + 11\left(\frac{2}{7}\right)$
4. $11\left(\frac{1}{4}\right) + \frac{5}{4}$
5. $\frac{1}{2} \div 2$
6. $\frac{1}{3} \div \frac{1}{3}$
7. $\frac{1}{7} + \frac{1}{3} - \frac{1}{21}$
8. $\frac{1}{3} + \frac{1}{2} - \frac{5}{6}$
9. $\frac{1}{12} - \frac{1}{3} + \frac{1}{8}$
10. $\left(\frac{1}{2} - \frac{1}{3}\right) \div \frac{1}{6}$

Exercises 0.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–20, evaluate the expression. Write fractional answers in simplest form.

1. $2^2 \cdot 2^4$
2. $3 \cdot 3^5$
3. $\frac{2^6}{2^3}$
4. $\frac{5^7}{5^5}$
5. $(3^3)^2$
6. $(2^5)^3$
7. -3^4
8. $(-3)^4$
9. $8 \cdot 2^{-2} \cdot 4^{-1}$
10. $6 \cdot 2^{-3} \cdot 3^{-1}$
11. $\left(\frac{1}{2}\right)^{-3}$
12. $\left(\frac{2}{3}\right)^{-3}$
13. $5^{-1} - 2^{-1}$
14. $4^{-1} - 2^{-2}$
15. $(2^3 \cdot 3^2)^2$
16. $(-3 \cdot 4^2)^3$
17. $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$
18. $\left(\frac{-5}{4}\right)^3\left(\frac{4}{5}\right)^2$
19. 3^0
20. $(-2)^0$

In Exercises 21–24, evaluate the expression for the indicated value of x.

	Expression	*Value*
21.	$\frac{x^4}{2x^2}$	$x = -6$
22.	$4x^{-3}$	$x = 2$
23.	$7x^{-2}$	$x = 4$
24.	$8x^0 - (8x)^0$	$x = -7$

In Exercises 25–44, simplify the expression.

25. $(-5z)^3$
26. $(-2w)^5$
27. $(8x^4)(2x^3)$
28. $5x^4(x^2)$
29. $10(x^2)^2$
30. $(4x^4)^3$
31. $(-z)^3(3z^4)$
32. $(6y^2)(2y^3)^3$
33. $\frac{25y^8}{10y^4}$
34. $\frac{10x^9}{4x^6}$
35. $\left(\frac{4}{y}\right)^3\left(\frac{3}{y}\right)^4$
36. $\left(\frac{5}{z}\right)^2\left(\frac{2}{z}\right)^3$
37. $\frac{15(x+3)^3}{9(x+3)^2}$
38. $\frac{24(x-2)^2}{8(x-2)^4}$
39. $\frac{7x^2}{x^3}$
40. $\frac{5z^5}{z^7}$
41. $\frac{x^2 \cdot x^n}{x^3 \cdot x^n}$
42. $\frac{x^n \cdot x^{2n}}{x^{3n}}$
43. $3^n \cdot 3^{2n}$
44. $2^m \cdot 2^{3m}$

In Exercises 45–54, rewrite the expression with positive exponents and simplify.

45. $(2x^5)^0, \quad x \neq 0$
46. $(x+5)^0, \quad x \neq -5$
47. $(y+2)^{-2}(y+2)^{-1}$
48. $(x+y)^{-5}(x+y)^9$
49. $(4y^{-2})(8y^4)$
50. $(-2x^2)^3(4x^3)^{-1}$
51. $\left(\frac{x}{10}\right)^{-1}$
52. $\left(\frac{y}{5}\right)^{-2}$
53. $\left(\frac{x^{-3}y^4}{5}\right)^{-3}$
54. $\left(\frac{2z^2}{y}\right)^{-2}$

In Exercises 55–60, write the number in scientific notation.

55. **Land Area of Earth:** 57,300,000 square miles
56. **Water Area of Earth:** 139,500,000 square miles
57. **Light Year:** 9,460,000,000,000 kilometers
58. **Mass of a Bacterium:** 0.0000000000000003 gram
59. **Thickness of a Soap Bubble:** 0.0000001 meter
60. **One Micron (millionth of a meter):** 0.00003937 inch

In Exercises 61–64, write the number in decimal notation.

61. Number of Air Sacs in the Lungs: 3.5×10^8

62. Temperature of the Core of the Sun: 1.5×10^7 degrees Celsius

63. Charge of an Electron: 1.602×10^{-19} coulomb

64. Width of a Human Hair: 9.0×10^{-5} meter

In Exercises 65 and 66, evaluate each expression without using a calculator.

65. (a) $(1.2 \times 10^7)(5 \times 10^{-3})$ (b) $\dfrac{6.0 \times 10^8}{3.0 \times 10^{-3}}$

66. (a) $(9.8 \times 10^{-2})(3 \times 10^7)$ (b) $\dfrac{9.0 \times 10^5}{4.5 \times 10^{-2}}$

In Exercises 67–70, use a calculator to evaluate each expression. Write your answer in scientific notation. (Round to three decimal places.)

67. (a) $0.000345(8{,}900{,}000{,}000)$

(b) $\dfrac{67{,}000{,}000 + 93{,}000{,}000}{0.0052}$

68. (a) $0.000045(9{,}200{,}000)$

(b) $\dfrac{0.0000928 - 0.0000021}{0.0061}$

69. (a) $(9.3 \times 10^6)^3(6.1 \times 10^{-4})$ (b) $\dfrac{(2.414 \times 10^4)^6}{(1.68 \times 10^5)^5}$

70. (a) $(1.2 \times 10^2)^2(5.3 \times 10^{-5})$ (b) $\dfrac{(3.28 \times 10^{-6})^{10}}{(5.34 \times 10^{-3})^{22}}$

In Exercises 71 and 72, write each number in scientific notation. Perform the operation and write your answer in scientific notation.

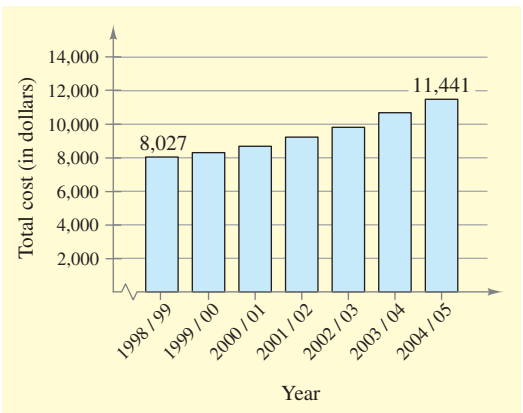
71. (a) $48{,}000{,}000{,}000(250{,}000{,}000)$ (b) $\dfrac{0.000000012}{0.0000064}$

72. (a) $0.00000034(0.00000006)$ (b) $\dfrac{18{,}000{,}000{,}000}{2{,}400{,}000}$

73. Balance in an Account You deposit \$10,000 in an account with an annual interest rate of 6.75% for 12 years. Determine the balance in the account when the interest is compounded (a) daily ($n = 365$), (b) weekly, (c) monthly, and (d) quarterly. How is the balance affected by the type of compounding?

74. Balance in an Account You deposit \$2000 in an account with an annual interest rate of 7.5% for 15 years. Determine the balance in the account when the interest is compounded (a) daily ($n = 365$), (b) weekly, (c) monthly, and (d) quarterly. How is the balance affected by the type of compounding?

75. College Costs The bar graph shows the average yearly costs of attending a public four-year college in the United States for the academic years 1998/1999 to 2004/2005. Find the average rate of inflation over this seven-year period. *(Source: U.S. National Center for Education Statistics)*



76. College Costs The average yearly cost of tuition, fees, and room and board at private four-year colleges in the United States was \$19,929 for the academic year 1998/1999 and \$26,489 for the academic year 2004/2005. Find the average yearly rate of inflation over this seven-year period. *(Source: U.S. National Center for Education Statistics)*

77. Becoming a Millionaire The compound interest formula can be rewritten as

$$P = \frac{A}{(1 + r/n)^{nt}}.$$

Find the principal amount P that would have had to have been invested on the day you were born at 7.5% annual interest compounded quarterly to make you a millionaire on your 21st birthday.

78. Electron Microscopes Electron microscopes provide greater magnification than traditional light microscopes by using focused beams of electrons instead of visible light. It is the extremely short wavelengths of the electron beams that make electron microscopes so powerful. The wavelength λ (in meters) of any object in motion is given by $\lambda = \dfrac{6.626 \times 10^{-34}}{mv}$, where m is the mass (in kilograms) of the object and v is its velocity (in meters per second). Find the wavelength of an electron with a mass of 9.11×10^{-31} kilogram and a velocity of 5.9×10^6 meters per second.

(Submitted by Brian McIntyre, Senior Laboratory Engineer for the Optics Electron Microscopy Facility at the University of Rochester.)

Section 0.4

Radicals and Rational Exponents

- Simplify a radical.
- Rationalize a denominator.
- Use properties of rational exponents.
- Combine radicals.
- Use a calculator to evaluate a radical.
- Use a radical expression to solve an application problem.

Radicals and Properties of Radicals

A **square root** of a number is defined as one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of 25. In a similar way, a **cube root** of a number is one of its three equal factors. Here are some examples.

Number	*Equal Factors*	*Root*
$25 = (-5)^2$	$(-5)(-5)$	-5 (square root)
$-64 = (-4)^3$	$(-4)(-4)(-4)$	-4 (cube root)
$81 = 3^4$	$3 \cdot 3 \cdot 3 \cdot 3$	3 (fourth root)

Definition of *n*th Root of a Number

Let a and b be real numbers and let n be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a*.** If $n = 2$, the root is a **square root,** and if $n = 3$, the root is a **cube root.**

From this definition, you can see that some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The following definition distinguishes between these two roots.

Principal *n*th Root of a Number

Let a be a real number that has at least one real nth root. **The principal *n*th root of *a*** is the nth root that has the same sign as a, and it is denoted by the **radical symbol**

$$\sqrt[n]{a}.$$ Principal nth root

The positive integer n is the **index** (the plural of index is *indexes* or *indices*) of the radical, and the number a is the **radicand.** If $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$.

Example 1 Evaluating Expressions Involving Radicals

a. The principal square root of 121 is $\sqrt{121} = 11$ because $11^2 = 121$.

b. The principal cube root of $\frac{125}{64}$ is $\sqrt[3]{\frac{125}{64}} = \frac{5}{4}$ because $\left(\frac{5}{4}\right)^3 = \frac{5^3}{4^3} = \frac{125}{64}$.

c. The principal fifth root of -32 is $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

d. $-\sqrt{49} = -7$ because $7^2 = 49$.

e. $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce -81.

✓CHECKPOINT 1

Evaluate $-\sqrt[3]{-8}$. ■

From Example 1, you can make the following generalizations about nth roots of a real number.

1. If a is a positive real number and n is a positive *even* integer, then a has exactly two real nth roots, which are denoted by $\sqrt[n]{a}$ and $-\sqrt[n]{a}$.
2. If a is any real number and n is an *odd* integer, then a has only one (real) nth root. It is the principal nth root and is denoted by $\sqrt[n]{a}$.
3. If a is negative and n is an *even* integer, then a has no (real) nth root.

Integers such as 1, 4, 9, 16, 49, and 81 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers. Then the following properties are true.

Property	*Example*
1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \quad b \neq 0$	$\frac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\frac{27}{9}} = \sqrt[4]{3}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5. $\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6. For n even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{(-12)^2} = \lvert -12 \rvert = 12$
For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

A common special case of Property 6 is

$$\sqrt{a^2} = |a|.$$

Simplifying Radicals

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called *rationalizing the denominator*).
3. The index of the radical has been reduced as far as possible.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

Example 2 Simplifying Even Roots

a. $\sqrt[4]{48} = \sqrt[4]{16 \cdot 3}$ Find largest fourth-power factor.

$= \sqrt[4]{2^4 \cdot 3}$ Rewrite.

$= 2\sqrt[4]{3}$ Find fourth root.

b. $\sqrt{75x^3} = \sqrt{25x^2 \cdot 3x}$ Find largest square factor.

$= \sqrt{(5x)^2 \cdot 3x}$ Rewrite.

$= 5x\sqrt{3x}, \quad x \geq 0$ Find root of perfect square.

c. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

✓CHECKPOINT 2

Simplify $\sqrt{18x^5}$. ■

In Example 2(c), note that the absolute value symbol is included in the answer because $\sqrt[4]{x^4} = |x|$.

Example 3 Simplifying Odd Roots

a. $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3}$ Find largest cube factor.

$= \sqrt[3]{2^3 \cdot 3}$ Rewrite.

$= 2\sqrt[3]{3}$ Find root of perfect cube.

b. $\sqrt[5]{32a^{11}} = \sqrt[5]{32a^{10} \cdot a}$ Find largest fifth-power factor.

$= \sqrt[5]{(2a^2)^5 \cdot a}$ Rewrite.

$= 2a^2\sqrt[5]{a}$ Find fifth root.

c. $\sqrt[3]{-40x^6} = \sqrt[3]{(-8x^6) \cdot 5}$ Find largest cube factor.

$= \sqrt[3]{(-2x^2)^3 \cdot 5}$ Rewrite.

$= -2x^2\sqrt[3]{5}$ Find root of perfect cube.

✓CHECKPOINT 3

Simplify $\sqrt[3]{54x^4}$. ■

Some fractions have radicals in the denominator. To **rationalize a denominator** of the form $a + b\sqrt{m}$, multiply the numerator and denominator by the **conjugate** $a - b\sqrt{m}$.

$a + b\sqrt{m}$ and $a - b\sqrt{m}$ Conjugates

When $a = 0$, the rationalizing factor of $\sqrt{m}$ is itself, $\sqrt{m}$.

Example 4 Rationalizing Single-Term Denominators

a. To rationalize the denominator of the following fraction, multiply *both* the numerator and the denominator by $\sqrt{3}$ to obtain

$$\frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{2\sqrt{3^2}} = \frac{5\sqrt{3}}{2(3)} = \frac{5\sqrt{3}}{6}.$$

b. To rationalize the denominator of the following fraction, multiply *both* the numerator and the denominator by $\sqrt[3]{5^2}$. Note how this eliminates the radical from the denominator by producing a perfect *cube* in the radicand.

$$\frac{2}{\sqrt[3]{5}} = \frac{2}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \frac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}} = \frac{2\sqrt[3]{25}}{5}$$

✓CHECKPOINT 4

Simplify $\frac{1}{\sqrt[3]{4}}$ by rationalizing the denominator. ■

Example 5 Rationalizing a Denominator with Two Terms

$$\frac{2}{3 + \sqrt{7}} = \frac{2}{3 + \sqrt{7}} \cdot \frac{3 - \sqrt{7}}{3 - \sqrt{7}}$$ Multiply numerator and denominator by conjugate.

$$= \frac{2(3 - \sqrt{7})}{3^2 - (\sqrt{7})^2}$$ Multiply fractions.

$$= \frac{2(3 - \sqrt{7})}{9 - 7}$$ Simplify.

$$= \frac{2(3 - \sqrt{7})}{2}$$ Divide out like factors.

$$= 3 - \sqrt{7}$$ Simplify.

✓CHECKPOINT 5

Simplify $\frac{6}{3 - \sqrt{3}}$ by rationalizing the denominator. ■

Don't confuse an expression such as $\sqrt{2} + \sqrt{7}$ with $\sqrt{2 + 7}$. In general, $\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$.

Rational Exponents

The definition on the following page shows how radicals are used to define **rational exponents.** Until now, work with exponents has been restricted to integer exponents.

STUDY TIP

The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken. It doesn't matter which operation is performed first, provided the nth root exists. Here is an example.

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$$

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined to be $a^{1/n} = \sqrt[n]{a}$.

If m is a positive integer that has no common factor with n, then $a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m$ and $a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$.

The properties of exponents that were listed in Section 0.3 also apply to rational exponents (provided the roots indicated by the denominators exist). Some of those properties are relisted here, with different examples.

Properties of Exponents

Let r and s be rational numbers, and let a and b be real numbers, variables, or algebraic expressions. If the roots indicated by the rational exponents exist, then the following properties are true.

	Property	*Example*
1.	$a^r a^s = a^{r+s}$	$4^{1/2}(4^{1/3}) = 4^{5/6}$
2.	$\dfrac{a^r}{a^s} = a^{r-s}, \quad a \neq 0$	$\dfrac{x^2}{x^{1/2}} = x^{2-(1/2)} = x^{3/2}, x \neq 0$
3.	$(ab)^r = a^r b^r$	$(2x)^{1/2} = 2^{1/2}(x^{1/2})$
4.	$\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}, \quad b \neq 0$	$\left(\dfrac{x}{3}\right)^{1/3} = \dfrac{x^{1/3}}{3^{1/3}}$
5.	$(a^r)^s = a^{rs}$	$(x^3)^{1/3} = x$
6.	$a^{-r} = \dfrac{1}{a^r}, \quad a \neq 0$	$4^{-1/2} = \dfrac{1}{4^{1/2}} = \dfrac{1}{2}$
7.	$\left(\dfrac{a}{b}\right)^{-r} = \left(\dfrac{b}{a}\right)^r, \quad a \neq 0, \quad b \neq 0$	$\left(\dfrac{x}{4}\right)^{-1/2} = \left(\dfrac{4}{x}\right)^{1/2} = \dfrac{2}{x^{1/2}}$

STUDY TIP

Rational exponents can be tricky. Remember, the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. For instance, $(-8)^{5/6}$ is not defined because $\sqrt[6]{-8}$ is not a real number. And yet, $(-8)^{2/3}$ is defined because $\sqrt[3]{-8} = -2$.

Rational exponents are particularly useful for evaluating roots of numbers on a calculator, for reducing the index of a radical, and for simplifying (and factoring) algebraic expressions. Examples 6 and 7 demonstrate some of these uses.

Example 6 **Simplifying with Rational Exponents**

a. $(27)^{1/3} = \sqrt[3]{27} = 3$

b. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

c. $(-5x^{2/3})(3x^{-1/3}) = -15x^{(2/3)-(1/3)} = -15x^{1/3}, \quad x \neq 0$

✓CHECKPOINT 6

Simplify $(3^{1/2})(3^{3/2})$. ■

Example 7 Reducing the Index of a Radical

a. $\sqrt[6]{a^4} = a^{4/6} = a^{2/3} = \sqrt[3]{a^2}$

b.
$$\sqrt[3]{\sqrt{125}} = (125^{1/2})^{1/3} \quad \text{Rewrite with rational exponents.}$$
$$= (125)^{1/6} \quad \text{Multiply exponents.}$$
$$= (5^3)^{1/6} \quad \text{Rewrite base as perfect cube.}$$
$$= 5^{3/6} \quad \text{Multiply exponents.}$$
$$= 5^{1/2} \quad \text{Reduce exponent.}$$
$$= \sqrt{5} \quad \text{Rewrite as radical.}$$

✓CHECKPOINT 7

Use rational exponents to reduce the index of the radical $\sqrt[3]{2^6}$. ■

Radical expressions can be combined (added or subtracted) if they are **like radicals**—that is, if they have the same index and radicand. For instance, $2\sqrt{3x}$ and $\frac{1}{2}\sqrt{3x}$ are like radicals, but $\sqrt[3]{3x}$ and $2\sqrt{3x}$ are not like radicals.

Example 8 Simplifying and Combining Like Radicals

a.
$$2\sqrt{48} + 3\sqrt{27} = 2\sqrt{16 \cdot 3} + 3\sqrt{9 \cdot 3} \quad \text{Find square factors.}$$
$$= 8\sqrt{3} + 9\sqrt{3} \quad \text{Find square roots.}$$
$$= 17\sqrt{3} \quad \text{Combine like terms.}$$

b.
$$\sqrt[3]{16x} - \sqrt[3]{54x} = \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot 2x} \quad \text{Find cube factors.}$$
$$= 2\sqrt[3]{2x} - 3\sqrt[3]{2x} \quad \text{Find cube roots.}$$
$$= -\sqrt[3]{2x} \quad \text{Combine like terms.}$$

✓CHECKPOINT 8

Simplify the expression $\sqrt{25x} + \sqrt{x}$. ■

Radicals and Calculators

You can use a calculator to evaluate radicals by using the square root key $\boxed{\sqrt{\ }}$, the cube root key $\boxed{\sqrt[3]{\ }}$, or the xth root key $\boxed{\sqrt[x]{\ }}$. You can also use the exponential key $\boxed{\wedge}$ or $\boxed{y^x}$. To use these keys, first convert the radical to exponential form.

Example 9 Evaluating a Cube Root with a Calculator

Two ways to evaluate $\sqrt[3]{25}$ using a calculator are shown below.

$\boxed{25}\ \boxed{y^x}\ \boxed{(}\ \boxed{1}\ \boxed{\div}\ \boxed{3}\ \boxed{)}\ \boxed{=}$ Exponential key

$\boxed{\sqrt[3]{\ }}\ \boxed{25}\ \boxed{)}\ \boxed{\text{ENTER}}$ Cube root key

Most calculators will display 2.924017738. So,

$$\sqrt[3]{25} \approx 2.924.$$

✓CHECKPOINT 9

Use a calculator to approximate the value of $\sqrt[3]{18}$. ■

Example 10 Evaluating Radicals with a Calculator

a. Use the following keystroke sequence to evaluate $\sqrt[3]{-4}$.

4 [+/−] [y^x] [(] 1 [÷] 3 [)] [=] Scientific

[(−)] 4 [^] [(] 1 [÷] 3 [)] [ENTER] Graphing

The calculator display is -1.587401052, which implies that

$$\sqrt[3]{-4} \approx -1.587.$$

b. Use the following keystroke sequence to evaluate $(1.4)^{-2/5}$.

1.4 [y^x] [(] 2 [÷] 5 [+/−] [)] [=] Scientific

1.4 [^] [(] [(−)] 2 [÷] 5 [)] [ENTER] Graphing

The calculator display is 0.874075175, which implies that

$$(1.4)^{-2/5} \approx 0.874.$$

✓CHECKPOINT 10

Use a calculator to approximate the value of $2.2^{-1.2}$. Round to three decimal places. ■

Application

Example 11 Escape Velocity

A rocket, launched vertically from Earth, has an initial velocity of 10,000 meters per second. All of the fuel is used for launching. The *escape velocity*, or the minimum initial velocity necessary for the rocket to escape the gravitational field of Earth, is

$$\sqrt{\frac{2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{6.37 \times 10^{6}}} \text{ meters per second.}$$

Will the rocket escape Earth's gravitational field?

SOLUTION The escape velocity is

$$\sqrt{\frac{2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{6.37 \times 10^{6}}} \approx 11{,}190.7 \text{ meters per second.}$$

The initial velocity of 10,000 meters per second is less than the escape velocity of 11,190.7 meters per second. So, the rocket will not escape Earth's gravitational field.



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✓CHECKPOINT 11

Will an object traveling at 10,000 meters per second exceed the escape velocity of Venus, which is

$$\sqrt{\frac{2(6.67 \times 10^{-11})(4.87 \times 10^{24})}{6.05 \times 10^{6}}}$$

meters per second? ■

CONCEPT CHECK

Let m and n be positive real numbers greater than 1.

1. Are the expressions $\left(\frac{m^2}{n}\right)^3$ and $\frac{m^5}{n^4}$ equivalent? Explain.
2. How many real cube roots does $-n$ have? Explain.
3. Is $3m\sqrt[4]{16mn^5}$ in simplest form? If not, simplify the expression.
4. Explain how to rationalize the denominator of $\frac{3}{\sqrt[5]{2}}$.

Skills Review 0.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.3.

In Exercises 1–10, simplify the expression.

1. $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2$

2. $3(-4)^2$

3. $(-2x)^3$

4. $(-2x^3)(-3x^4)$

5. $(7x^5)(4x)$

6. $(5x^4)(25x^2)^{-1}$

7. $\frac{12z^6}{4z^2}$

8. $\left(\frac{2x}{5}\right)^2\left(\frac{2x}{5}\right)^{-4}$

9. $\left(\frac{3y^2}{x}\right)^0, \quad x \neq 0, \ y \neq 0$

10. $[(x+2)^2(x+2)^3]^2$

Exercises 0.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, fill in the missing form.

	Radical Form	*Rational Exponent Form*
1.	$\sqrt{9} = 3$	
2.	$\sqrt[3]{125} = 5$	
3.		$32^{1/5} = 2$
4.		$-(144^{1/2}) = -12$
5.		$196^{1/2} = 14$
6.		$614.125^{1/3} = 8.5$
7.	$\sqrt[3]{-216} = -6$	
8.	$\sqrt[5]{-243} = -3$	
9.	$\left(\sqrt[4]{81}\right)^3 = 27$	
10.	$\sqrt[4]{81^3} = 27$	
11.		$125^{2/3} = 25$
12.		$16^{5/4} = 32$

In Exercises 13–30, evaluate the expression.

13. $\sqrt{9}$

14. $\sqrt[3]{64}$

15. $-\sqrt[3]{-27}$

16. $\sqrt[3]{0}$

17. $\frac{14}{\sqrt{49}}$

18. $\frac{\sqrt[4]{81}}{3}$

19. $\left(\sqrt[3]{-125}\right)^3$

20. $\sqrt[4]{562^4}$

21. $16^{1/2}$

22. $27^{1/3}$

23. $36^{3/2}$

24. $25^{3/2}$

25. $\sqrt{2} \cdot \sqrt{3}$

26. $\sqrt{2} \cdot \sqrt{5}$

27. $\left(\frac{16}{81}\right)^{-3/4}$

28. $\left(\frac{9}{4}\right)^{-1/2}$

29. $\left(-\frac{1}{64}\right)^{-1/3}$

30. $\left(-\frac{125}{27}\right)^{-1/3}$

In Exercises 31–36, simplify the expression.

31. $\sqrt[3]{16x^5}$

32. $\sqrt[4]{(3x^2)^4}$

33. $\sqrt{75x^{-2}y^4}$

34. $\sqrt[5]{96x^5}$

35. $\sqrt[5]{64y^{-5}}$

36. $\sqrt{8x^4y^3z^{-2}}$

In Exercises 37–40, evaluate the expression when $x = 2$, $y = 3$, and $z = 5$.

37. $\sqrt{2xy^4z^2}$

38. $\sqrt{3x^2yz^6}$

39. $\sqrt[4]{16x^{-4}y^8z^4}$

40. $\sqrt[5]{243x^5y^{-5}z^{15}}$

In Exercises 41–48, rewrite the expression by rationalizing the denominator. Simplify your answer.

41. $\frac{1}{\sqrt{5}}$

42. $\frac{5}{\sqrt{10}}$

43. $\frac{8}{\sqrt[3]{2}}$

44. $\frac{5}{\sqrt[3]{(5x)^2}}$

45. $\frac{2x}{5 - \sqrt{3}}$

46. $\frac{5x}{\sqrt{14} - 2}$

47. $\frac{3}{\sqrt{5} + \sqrt{6}}$

48. $\frac{5}{2\sqrt{10} - 5}$

In Exercises 49–60, simplify the expression.

49. $5^{1/2} \cdot 5^{3/2}$

50. $4^{1/3} \cdot 4^{5/3}$

51. $\frac{2^{3/2}}{2}$

52. $\frac{5^{1/2}}{5}$

53. $\frac{x^2}{x^{1/2}}$

54. $\frac{x \cdot x^{1/2}}{x^{3/2}}$

55. $\sqrt[3]{5} \cdot \sqrt[3]{5^2}$

56. $\sqrt[5]{3^7} \cdot \sqrt[5]{3^3}$

57. $(x^6x^3)^{1/3}$

58. $(x^3x^{12})^{1/5}$

59. $(16x^8y^4)^{3/4}$

60. $(27x^6y^9)^{2/3}$

In Exercises 61–66, use rational exponents to reduce the index of the radical.

61. $\sqrt{\sqrt{32}}$

62. $\sqrt{\sqrt{x^4}}$

63. $\sqrt[4]{3^2}$

64. $\sqrt[4]{(3x^2)^4}$

65. $\sqrt[9]{x^3}$

66. $\sqrt[6]{(x+2)^4}$

In Exercises 67–72, simplify the expression.

67. $5\sqrt{x} - 3\sqrt{x}$

68. $3\sqrt{x+1} + 10\sqrt{x+1}$

69. $5\sqrt{50} + 3\sqrt{8}$

70. $2\sqrt{27} - \sqrt{75}$

71. $2\sqrt{4y} - 2\sqrt{9y}$

72. $2\sqrt{108} + \sqrt{147}$

In Exercises 73–80, use a calculator to approximate the number. (Round to three decimal places.)

73. $\sqrt[3]{45}$

74. $\sqrt{57}$

75. $5.7^{2/5}$

76. $24.7^{1.1}$

77. $0.26^{-0.8}$

78. $3.75^{-1/2}$

79. $\dfrac{3-\sqrt{5}}{2}$

80. $\dfrac{-4+\sqrt{12}}{4}$

81. **Calculator** Write the keystrokes you can use to evaluate $\dfrac{4-\sqrt{7}}{3}$ in one step on your calculator.

82. **Calculator** Write the keystrokes you can use to evaluate $\sqrt[3]{(-5)^5}$ in one step on your calculator.

In Exercises 83–88, complete the statement with <, =, or >.

83. $\sqrt{5} + \sqrt{3} \quad \sqrt{5+3}$

84. $\sqrt{3} - \sqrt{2} \quad \sqrt{3-2}$

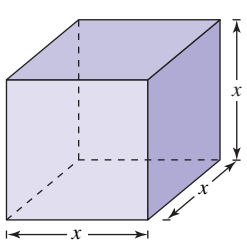
85. $5 \quad \sqrt{3^2+2^2}$

86. $5 \quad \sqrt{3^2+4^2}$

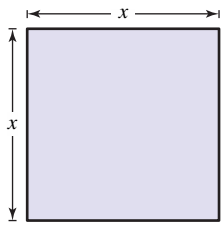
87. $\sqrt{3}\cdot\sqrt[4]{3} \quad \sqrt[8]{3}$

88. $\sqrt{\dfrac{3}{11}} \quad \dfrac{\sqrt{3}}{\sqrt{11}}$

89. **Geometry** Find the dimensions of a cube that has a volume of 15,625 cubic inches (see figure).



90. **Geometry** Find the dimensions of a square classroom that has 1100 square feet of floor space (see figure).

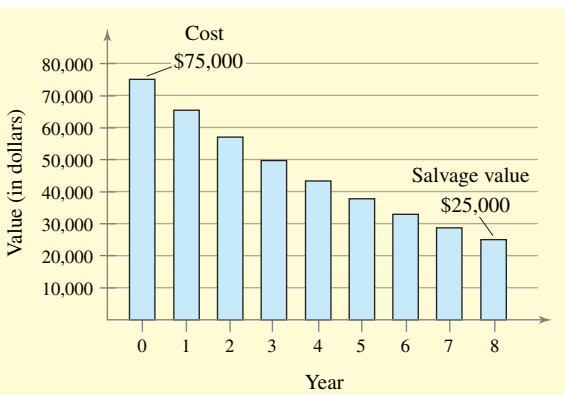


Declining Balances Depreciation In Exercises 91 and 92, find the annual depreciation rate r by using the declining balances formula

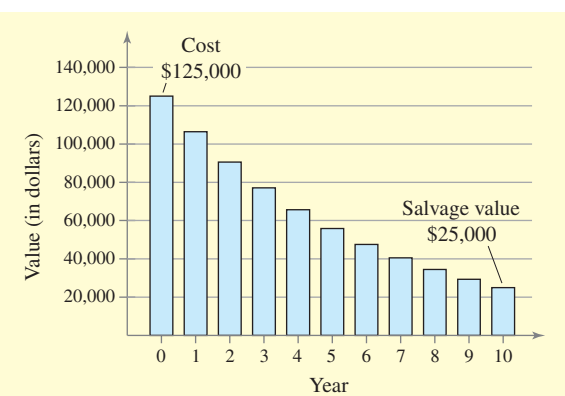
$$r = 1 - \left(\frac{S}{C}\right)^{1/n}$$

where n is the useful life of the item (in years), S is the salvage value (in dollars), and C is the original cost (in dollars).

91. A truck whose original cost is \$75,000 is depreciated over an eight-year period, as shown in the bar graph.



92. A printing press whose original cost is \$125,000 is depreciated over a 10-year period, as shown in the bar graph.



93. Escape Velocity The escape velocity (in meters per second) on the moon is

$$\sqrt{\frac{2(6.67 \times 10^{-11})(7.36 \times 10^{22})}{1.74 \times 10^{6}}}.$$

If all the fuel is consumed during launching, will a rocket with an initial velocity of 2000 meters per second escape the gravitational field of the moon?

94. Escape Velocity The escape velocity (in meters per second) on Mars is

$$\sqrt{\frac{2(6.67 \times 10^{-11})(6.42 \times 10^{23})}{3.37 \times 10^{6}}}.$$

Will an object traveling at 6000 meters per second escape the gravitational field of Mars?

95. Period of a Pendulum The period T (in seconds) of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{32}}$$

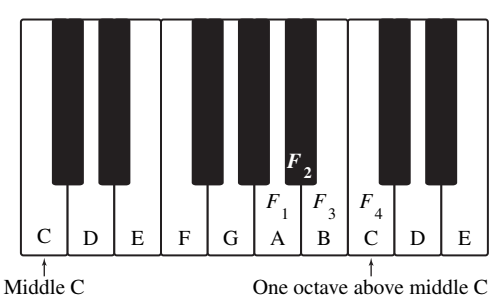
where L is the length (in feet) of the pendulum. Find the period of a pendulum whose length is 4 feet.

96. Period of a Pendulum Use the formula given in Exercise 95 to find the period of a pendulum whose length is 2.5 feet.

97. Erosion A stream of water moving at the rate of v feet per second can carry particles of size $0.03\sqrt{v}$ inches. Find the size of the largest particle that can be carried by a stream flowing at the rate of $\frac{1}{2}$ foot per second.

98. Erosion A stream of water moving at the rate of v feet per second can carry particles of size $0.03\sqrt{v}$ inches. Find the size of the largest particle that can be carried by a stream flowing at the rate of $\frac{7}{9}$ foot per second.

Notes on a Musical Scale In Exercises 99–102, find the frequency of the indicated note on a piano (see figure). The musical note A above middle C has a frequency of 440 vibrations per second. If we denote this frequency by F_1, then the frequency of the next higher note is given by $F_2 = F_1 \cdot 2^{1/12}$. Similarly, the frequency of the next note is given by $F_3 = F_2 \cdot 2^{1/12}$.



99. Find the frequency of the musical note B above middle C.

100. Find the frequency of the musical note C that is one octave above middle C.

101. *MAKE A DECISION* Which note would you expect to have a higher frequency? Explain your reasoning.

(a) Musical note E one octave above middle C

(b) Musical note D one octave above middle C

102. *MAKE A DECISION* Assume the pattern shown on the piano continues. Which note would you expect to have a higher frequency?

(a) Musical note D one octave above middle C

(b) Musical note G one octave above middle C

Estimating Speed A formula used to help determine the speed of a car from its skid marks is $S = \sqrt{30Df}$, where S is the least possible speed (in miles per hour) of the car before its brakes are applied, D is the length of the car's skid marks (in feet) and f is the drag factor of the road surface. In Exercises 103 and 104, find the least possible speed of the car for the given conditions.

103. Skid marks:60 feet, drag factor:0.90

104. Skid marks:100 feet, drag factor:0.75

Wind Chill A wind chill temperature is a measure of how cold it *feels* outside. The wind chill temperature W (in degrees Fahrenheit) is given by

$$W = 35.75 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$$

where T is the actual temperature (in degrees Fahrenheit) and v is the wind speed (in miles per hour). In Exercises 105 and 106, find the wind chill temperature for the given conditions. *(Source: NOAA's National Weather Service)*

105. Actual temperature: 30°F,wind speed:20 mph

106. Actual temperature: 10°F,wind speed:10 mph

107. Calculator Experiment Enter any positive real number in your calculator and repeatedly take the square root. What real number does the display appear to be approaching?

108. Calculator Experiment Square the real number $2/\sqrt{5}$ and note that the radical is eliminated from the denominator. Is this equivalent to rationalizing the denominator? Why or why not?

109. Think About It How can you show that $a^0 = 1, a \neq 0$? (*Hint:* Use the property of exponents $a^m/a^n = a^{m-n}$.)

110. Explain why $\sqrt{4x^2} \neq 2x$ for every real number x.

111. Explain why $\sqrt{2} + \sqrt{3} \neq \sqrt{5}$.

Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. After you are done, check your work against the answers given in the back of the book.



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When occupancy rates are not maximized, renters can sometimes negotiate for a lower rent. But when the market is overwhelmed by renters, rates are driven up.

In Exercises 1 and 2, place the correct symbol (<, >, or =) between the two real numbers.

1. $-|-7| \quad |-7|$ **2.** $-(-3) \quad |-3|$

In Exercises 3 and 4, use inequality notation to describe the subset of real numbers.

3. x is positive or x is equal to zero.

4. The apartment occupancy rate r will be at least 95% during the coming year.

5. Describe the subset of real numbers that is represented by the inequality $-2 \le x < 3$, and sketch the subset on the real number line.

6. Identify the terms of the algebraic expression $3x^2 - 7x + 2$.

In Exercises 7–10, perform the indicated operation(s). Write fractional answers in simplest form.

7. $-4 - (-7)$ **8.** $\dfrac{31 - 5}{-2}$

9. $\dfrac{2}{3} \cdot \dfrac{5}{4} \cdot \dfrac{3}{7}$ **10.** $\dfrac{11}{15} \div \dfrac{3}{5}$

In Exercises 11–13, rewrite the expression with positive exponents and simplify.

11. $(-x)^3(2x^4)$ **12.** $\dfrac{5y^7}{15y^3}$

13. $\left(\dfrac{x^{-2}y^2}{3}\right)^{-3}$

14. You deposit \$5000 in an account with an annual interest rate of 6.5%, compounded quarterly. Find the balance in the account after 10 years.

In Exercises 15 and 16, evaluate the expression.

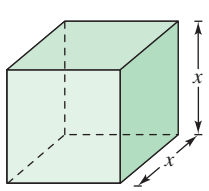
15. $\dfrac{-\sqrt[4]{81}}{3}$ **16.** $\left(\sqrt[3]{-64}\right)^3$

In Exercises 17–19, simplify the expression.

17. $3^{1/2} \cdot 3^{3/2}$ **18.** $\sqrt[3]{81} - 4\sqrt[3]{3}$

19. $\sqrt[10]{12^5}$

20. Find the dimensions of a cube that has a volume of 10,648 cubic centimeters.



Section 0.5

Polynomials and Special Products

- Write a polynomial in standard form.
- Add, subtract, and multiply polynomials.
- Use special products to multiply polynomials.
- Use polynomials to solve an application problem.

Polynomials

One of the simplest and most common types of algebraic expressions is a **polynomial.** Here are some examples.

$$2x + 5, \quad 3x^4 - 7x^2 + 2x + 4, \quad 5x^2y^2 - xy + 3$$

The first two are *polynomials in x* and the third is a *polynomial in x and y*. The terms of a polynomial in x have the form ax^k, where a is the **coefficient** and k is the **degree** of the term. Because a polynomial is defined as an algebraic sum, the coefficients take on the signs between the terms. For instance, the polynomial

$$2x^3 - 5x^2 + 1 = 2x^3 + (-5)x^2 + (0)x + 1$$

has coefficients 2, −5, 0, and 1.

Definition of a Polynomial in *x*

Let $a_n, \ldots, a_2, a_1, a_0$ be real numbers and let n be a *nonnegative integer*. A **polynomial in *x*** is an expression of the form

$$a_nx^n + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree** n, and the number a_n is the **leading coefficient.** The number a_0 is the **constant term.** The constant term is considered to have a degree of zero.

Note in the definition of a polynomial in x that the polynomial is written in descending powers of x. This is called the **standard form** of a polynomial.

Example 1 Rewriting a Polynomial in Standard Form

	Polynomial	*Standard Form*	*Degree*
a.	$4x^2 - 5x^3 - 2 + 3x$	$-5x^3 + 4x^2 + 3x - 2$	3
b.	$4 - 9x^2$	$-9x^2 + 4$	2
c.	8	$8 (8 = 8x^0)$	0

✓CHECKPOINT 1

Rewrite the polynomial $7 - 9x^2 + 3x$ in standard form and state its degree. ■

Polynomials with one, two, and three terms are called **monomials, binomials,** and **trinomials,** respectively.

A polynomial that has all zero coefficients is called the **zero polynomial,** denoted by 0. This particular polynomial is not considered to have a degree.

Example 2 Identifying a Polynomial and Its Degree

a. $-2x^3 + x^2 + 3x - 2$ is a polynomial of degree 3.

b. $\sqrt{x^2 - 3x}$ is not a polynomial because the radical sign indicates a noninteger power of x.

c. $x^2 + 5x^{-1}$ is not a polynomial because of the negative exponent.

✓CHECKPOINT 2

Determine whether the expression $\dfrac{2x + 5}{x}$ is a polynomial. If it is, state the degree. ■

For a polynomial in more than one variable, the *degree of a term* is the sum of the powers of the variables in the term. The *degree of the polynomial* is the highest degree of all its terms. For instance, the polynomial

$$5x^3y - x^2y^2 + 2xy - 5$$

has two terms of degree 4, one term of degree 2, and one term of degree 0. The degree of the polynomial is 4.

Operations with Polynomials

You can **add** and **subtract** polynomials in much the same way that you add and subtract real numbers—you simply add or subtract the *like terms* (terms having the same variables to the same powers) by adding their coefficients. For instance, $-3x^2$ and $5x^2$ are like terms and their sum is given by

$$-3x^2 + 5x^2 = (-3 + 5)x^2 = 2x^2.$$

Example 3 Sums and Differences of Polynomials

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

$= (5x^3 + x^3) + (2x^2 - 7x^2) - x + (8 - 3)$ Group like terms.

$= 6x^3 - 5x^2 - x + 5$ Combine like terms.

b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

$= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x$ Distribute sign.

$= (7x^4 - 3x^4) + (4x^2 - x^2) + (-4x - 3x) + 2$ Group like terms.

$= 4x^4 + 3x^2 - 7x + 2$ Combine like terms.

✓CHECKPOINT 3

Find the sum $(2x^2 + x + 3) + (4x + 1)$ and write the resulting polynomial in standard form. ■

A common mistake is to fail to change the sign of *each* term inside parentheses preceded by a minus sign. For instance, note the following.

$-(3x^4 - 4x^2 + 3x) = -3x^4 + 4x^2 - 3x$ Correct

~~$-(3x^4 - 4x^2 + 3x) = -3x^4 - 4x^2 + 3x$~~ Common mistake

To find the **product** of two polynomials, you can use the left and right Distributive Properties. For example, if you treat $(5x + 7)$ as a single quantity, you can multiply $(3x - 2)$ by $(5x + 7)$ as follows.

$$\begin{aligned}(3x - 2)(5x + 7) &= 3x(5x + 7) - 2(5x + 7)\\ &= (3x)(5x) + (3x)(7) - (2)(5x) - (2)(7)\\ &= 15x^2 + 21x - 10x - 14\\ &= 15x^2 + 11x - 14\end{aligned}$$

Product of **F**irst terms — Product of **O**uter terms — Product of **I**nner terms — Product of **L**ast terms

You can use the four special products shown in the boxes above to write the product of two binomials in the FOIL form in just one step. This is called the **FOIL Method.**

Example 4 Using the FOIL Method

Use the FOIL Method to find the product of $2x - 4$ and $x + 5$.

SOLUTION

$$\begin{aligned}(2x - 4)(x + 5) &= \overset{\text{F}}{2x^2} + \overset{\text{O}}{10x} - \overset{\text{I}}{4x} - \overset{\text{L}}{20}\\ &= 2x^2 + 6x - 20\end{aligned}$$

✓CHECKPOINT 4

Find the product of $3x + 1$ and $x - 1$. ■

When multiplying two polynomials, be sure to multiply *each* term of one polynomial by *each* term of the other. The following vertical pattern is a convenient way to multiply two polynomials.

Example 5 Using a Vertical Format to Multiply Polynomials

Multiply $(x^2 - 2x + 2)$ by $(x^2 + 2x + 2)$.

SOLUTION

$$\begin{array}{rl}
x^2 - 2x + 2 & \text{Standard form}\\
x^2 + 2x + 2 & \text{Standard form}\\
\hline
x^4 - 2x^3 + 2x^2 & \Leftarrow\ x^2(x^2 - 2x + 2)\\
2x^3 - 4x^2 + 4x & \Leftarrow\ 2x(x^2 - 2x + 2)\\
2x^2 - 4x + 4 & \Leftarrow\ 2(x^2 - 2x + 2)\\
\hline
x^4 + 0x^3 + 0x^2 + 0x + 4 = x^4 + 4 & \text{Combine like terms.}
\end{array}$$

So, $(x^2 - 2x + 2)(x^2 + 2x + 2) = x^4 + 4$.

✓CHECKPOINT 5

Multiply $(x^2 + x + 4)$ by $(x^2 - 3x + 1)$. ■

Special Products

Special Products

Let u and v be real numbers, variables, or algebraic expressions.

Special Product	*Example*
Sum and Difference of Two Terms	
$(u + v)(u - v) = u^2 - v^2$	$(x + 4)(x - 4) = x^2 - 16$
Square of a Binomial	
$(u + v)^2 = u^2 + 2uv + v^2$	$(x + 3)^2 = x^2 + 6x + 9$
$(u - v)^2 = u^2 - 2uv + v^2$	$(3x - 2)^2 = 9x^2 - 12x + 4$
Cube of a Binomial	
$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$	$(x + 2)^3 = x^3 + 6x^2 + 12x + 8$
$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$	$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$

Example 6 Sum and Difference of Two Terms

$$(5x + 9)(5x - 9) = (5x)^2 - 9^2 = 25x^2 - 81$$

✓CHECKPOINT 6

Find the product $(3 - x)(3 + x)$. ■

Example 7 Square of a Binomial

$$\begin{aligned}(6x - 5)^2 &= (6x)^2 - 2(6x)(5) + 5^2 \\ &= 36x^2 - 60x + 25\end{aligned}$$

✓CHECKPOINT 7

Find the product $(x - 4)^2$. ■

Example 8 Cube of a Binomial

$$\begin{aligned}(3x + 2)^3 &= (3x)^3 + 3(3x)^2(2) + 3(3x)(2)^2 + 2^3 \\ &= 27x^3 + 54x^2 + 36x + 8\end{aligned}$$

✓CHECKPOINT 8

Find the product $(x - 3)^3$. ■

Example 9 The Product of Two Trinomials

$$\begin{aligned}(x + y - 2)(x + y + 2) &= [(x + y) - 2][(x + y) + 2] \\ &= (x + y)^2 - 2^2 \\ &= x^2 + 2xy + y^2 - 4\end{aligned}$$

✓CHECKPOINT 9

Find the product $(x + 5 - y)(x + 5 + y)$. ■

Applications

Example 10 A Savings Plan



AP/Wide World Photos

Many families set up savings accounts to help pay their children's college expenses.

At the same time each year for five consecutive years, you deposit money in an account that earns 7% interest, compounded annually. The deposit amounts are \$1500, \$1800, \$2400, \$2600, and \$3000. After the last deposit, is there enough money to pay a \$12,000 tuition bill?

SOLUTION Using the formula for compound interest, for *each* deposit you have

$$\text{Balance} = P\left(1 + \frac{r}{n}\right)^{nt} = P(1 + 0.07)^t = P(1.07)^t.$$

For the first deposit, $P = 1500$ and $t = 4$. For the second deposit, $P = 1800$ and $t = 3$, and so on. The balances for the five deposits are as follows.

Date	*Deposit*	*Time in Account*	*Balance in Account*
First Year	\$1500	4 years	$1500(1.07)^4$
Second Year	\$1800	3 years	$1800(1.07)^3$
Third Year	\$2400	2 years	$2400(1.07)^2$
Fourth Year	\$2600	1 year	$2600(1.07)$
Fifth Year	\$3000	0 years	3000

By adding these five balances, you can find the total balance in the account to be

$$1500(1.07)^4 + 1800(1.07)^3 + 2400(1.07)^2 + 2600(1.07) + 3000.$$

Note that this expression is in polynomial form. By evaluating the expression, you can find the balance to be \$12,701.03, as shown in Figure 0.10.

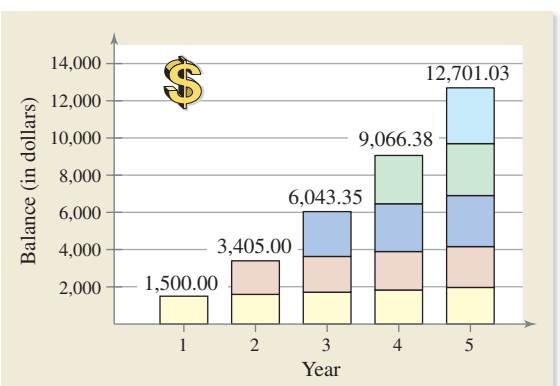


FIGURE 0.10

After the fifth deposit, there is enough money in the account to pay the college tuition bill.

✓CHECKPOINT 10

In Example 10, suppose the account earns 5% interest. What is the balance of the account after the last deposit? ■

Example 11 Geometry: Volume of a Box

An open box is made by cutting squares from the corners of a piece of metal that measures 16 inches by 20 inches and turning up the sides, as shown in Figure 0.11. The sides of the cut-out squares are all x inches long, so the box is x inches tall. Write an expression for the volume of the box. Then find the volume when $x = 1$, $x = 2$, and $x = 3$ inches.

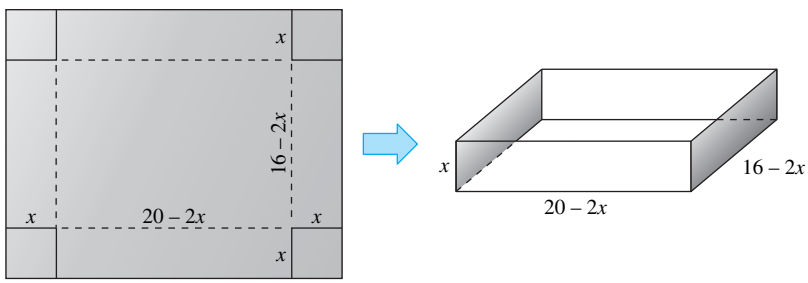


FIGURE 0.11

SOLUTION

Verbal Model: Volume = Length · Width · Height

Labels:
Height = x (inches)
Width = $16 - 2x$ (inches)
Length = $20 - 2x$ (inches)

Equation:
$$\begin{aligned} \text{Volume} &= (20 - 2x)(16 - 2x)(x) \\ &= (320 - 72x + 4x^2)(x) \\ &= 320x - 72x^2 + 4x^3 \end{aligned}$$

When $x = 1$ inch, the volume of the box is

$$\text{Volume} = 320(1) - 72(1)^2 + 4(1)^3 = 252 \text{ cubic inches.}$$

When $x = 2$ inches, the volume of the box is

$$\text{Volume} = 320(2) - 72(2)^2 + 4(2)^3 = 384 \text{ cubic inches.}$$

When $x = 3$ inches, the volume of the box is

$$\text{Volume} = 320(3) - 72(3)^2 + 4(3)^3 = 420 \text{ cubic inches.}$$

✓CHECKPOINT 11

In Example 11, suppose the original piece of metal is 10 inches by 12 inches. Write an expression for the volume of the box. Then find the volume when $x = 2$ and $x = 3$. ■

CONCEPT CHECK

1. Is $2 - 3x + x^3 - x^5$ written in standard form? Explain.
2. How many terms are in the sum of $x^3 - 4x^2 + 3$ and $2x^2 - x$?
3. A student claims that $(x - 3)(x + 4) = x^2 - 12$. Is the student correct? Explain.
4. Describe how you would show that $\sqrt{a^2 + b^2} \neq a + b$, where $a, b \neq 0$, using an algebraic argument. Then give a numerical example.

Skills Review 0.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 0.4.

In Exercises 1–10, perform the indicated operation(s).

1. $(7x^2)(6x)$

2. $(10z^3)(-2z^{-1})$

3. $(-3x^2)^3$

4. $-3(x^2)^3$

5. $\dfrac{27z^5}{12z^2}$

6. $\sqrt{24} \cdot \sqrt{2}$

7. $\left(\dfrac{2x}{3}\right)^{-2}$

8. $16^{3/4}$

9. $\dfrac{4}{\sqrt{8}}$

10. $\sqrt[3]{-27x^3}$

Exercises 0.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the degree and leading coefficient of the polynomial.

1. $2x^2 - x + 1$

2. $-3x^4 + 2x^2 - 5$

3. $x^5 - 1$

4. 3

5. $3x^5 - 6x^4 + x - 2$

6. $-3x$

In Exercises 7–12, determine whether the algebraic expression is a polynomial. If it is, write the polynomial in standard form and state its degree.

7. $2x - 3x^3 + 8$

8. $2x^3 + x - 3x^{-1}$

9. $\dfrac{3x + 4}{x}$

10. $\dfrac{2x^2 + 5x - 3}{3}$

11. $w^2 - w^4 + 2w^3$

12. $\sqrt{y^2 - y^4}$

In Exercises 13–16, evaluate the polynomial for each value of x.

13. $4x + 5$
(a) $x = -2$ (b) $x = -1$ (c) $x = 0$ (d) $x = 3$

14. $-x^2 + 3$
(a) $x = -3$ (b) $x = -2$ (c) $x = 0$ (d) $x = 1$

15. $-2x^2 + 3x + 4$
(a) $x = -2$ (b) $x = -1$ (c) $x = 0$ (d) $x = 1$

16. $x^3 - 4x^2 + x$
(a) $x = -1$ (b) $x = 0$ (c) $x = 1$ (d) $x = 2$

In Exercises 17–28, perform the indicated operation(s) and write the resulting polynomial in standard form.

17. $(6x + 5) - (8x + 15)$

18. $(3x^2 + 1) - (2x^2 - 2x + 3)$

19. $-(x^3 + 5) + (3x^3 - 4x)$

20. $-(5x^2 - 1) + (-3x^2 + 5)$

21. $(15x^2 - 6) - (-8x^3 - 14x^2 - 17)$

22. $(15x^4 - 18x - 19) - (13x^4 - 5x + 15)$

23. $3x(x^2 - 2x + 1)$

24. $z^2(2z^2 + 3z + 1)$

25. $-4x(3 - x^3)$

26. $-5y(2y - y^2)$

27. $-3x(-x)(3x - 7)$

28. $(2 - x^2)(-2x)(4x)$

In Exercises 29–54, find the product.

29. $(x + 3)(x + 4)$

30. $(x - 5)(x + 10)$

31. $(3x - 5)(2x + 1)$

32. $(7x - 2)(4x - 3)$

33. $(x + 5)(x - 5)$

34. $(3x + 2)(3x - 2)$

35. $(x + 6)^2$

36. $(3x - 2)^2$

37. $(2x - 5y)^2$

38. $(5 - 8x)^2$

39. $[(x - 3) + y]^2$

40. $[(x + 1) - y]^2$

41. $(x + 1)^3$

42. $(x - 2)^3$

43. $(2x - y)^3$

44. $(3x + 2y)^3$

45. $(3y^2 - 1)(3y^2 + 1)$

46. $(3x^2 - 4y^2)(3x^2 + 4y^2)$

47. $(m - 3 + n)(m - 3 - n)$

48. $(x + y + 1)(x + y - 1)$

49. $\left(\sqrt{x} + \sqrt{y}\right)\left(\sqrt{x} - \sqrt{y}\right)$

50. $\left(5 + \sqrt{x}\right)\left(5 - \sqrt{x}\right)$

51. $(x^2 - x + 1)(x^2 + x + 1)$

52. $(x^2 + 3x - 2)(x^2 - 3x - 2)$

53. $5x(x + 1) - 3x(x + 1)$

54. $(2x - 1)(x + 3) + 3(x + 3)$

55. Error Analysis A student claims that

$(x - 3)^2 = x^2 + 9$.

Describe and correct the student's error.

56. Error Analysis A student claims that

$(x - 3)(x + 3) = (x - 3)^2$.

Describe and correct the student's error.

57. Compound Interest After 3 years, an investment of \$1000 earning an interest rate r compounded annually will be worth $1000(1 + r)^3$ dollars. Write this expression as a polynomial in standard form.

58. Compound Interest After 2 years, an investment of \$800 earning an interest rate r compounded annually will be worth $800(1 + r)^2$ dollars. Write this expression as a polynomial in standard form.

59. Savings Plan At the same time each year for five consecutive years, you deposit money in an account that earns annually compounded interest. The deposits are \$1500, \$1700, \$900, \$2200, and \$3000. Is there enough money in the account after the last deposit to pay a \$10,000 college tuition bill at an interest rate of 6%? 5%? 4%?

60. Savings Plan You have an investment that pays an annual dividend. Each January for six consecutive years, you reinvest this dividend in an account that earns 6.25% interest, compounded annually. The dividends are shown in the table. Is there enough money in the account after the sixth deposit for a \$7500 down payment on a car?

Year	Dividend
1	\$920
2	\$1000
3	\$780
4	\$1310
5	\$1020
6	\$1200

61. Federal Student Aid The total amount (in millions of dollars) of federal student aid disbursed in the years 1998 through 2005 can be approximated by

$453.11x^2 - 5546.7x + 55{,}833$

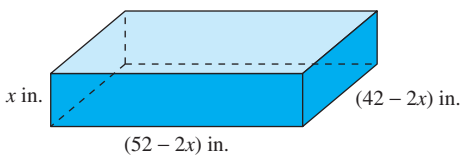
where x represents the year, with $x = 8$ corresponding to 1998. Evaluate the polynomial when $x = 14$ and $x = 15$. Then describe your results in everyday terms. *(Source: U.S. Department of Education)*

62. Federal Pell Grants The amount (in dollars) of the average Pell Grant awarded in the years 1998 through 2005 can be approximated by

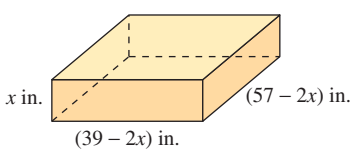
$-4.874x^3 + 155.85x^2 - 1507.9x + 6443$

where x represents the year, with $x = 8$ corresponding to 1998. Evaluate the polynomial when $x = 14$ and $x = 15$. Then describe your results in everyday terms. *(Source: U.S. Department of Education)*

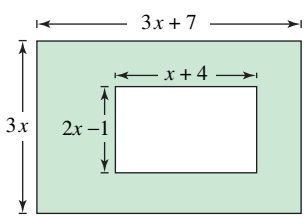
63. Geometry A box has a length of $(52 - 2x)$ inches, a width of $(42 - 2x)$ inches, and a height of x inches. Find the volume when $x = 3$, $x = 7$, and $x = 9$ inches. Which x-value gives the greatest volume?



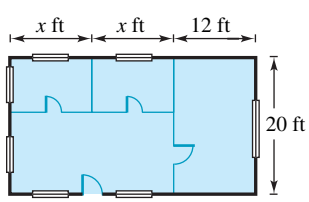
64. Geometry A box has a length of $(57 - 2x)$ inches, a width of $(39 - 2x)$ inches, and a height of x inches. Find the volume when $x = 4$, $x = 6$, and $x = 10$ inches. Which x-value gives the greatest volume?



65. Geometry Find the area of the shaded region in the figure. Write your answer as a polynomial in standard form.



66. Geometry Find a polynomial that represents the total number of square feet in the floor plan.



67. Extended Application To work an extended application involving the population of the United States from 1990 to 2005, visit this text's website at *college.hmco.com.* *(Data Source: U.S. Census Bureau)*

Section 0.6

Factoring

- Factor a polynomial by removing common factors.
- Factor a polynomial in a special form.
- Factor a trinomial as the product of two binomials.
- Factor a polynomial by grouping.

Common Factors

The process of writing a polynomial as a product is called **factoring.** It is an important tool for solving equations and reducing fractional expressions.

A polynomial that cannot be factored using integer coefficients is called **prime** or **irreducible over the integers.** For instance, the polynomial $x^2 - 3$ is irreducible over the integers. [Over the *real numbers*, this polynomial can be factored as

$$x^2 - 3 = \left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right).]$$

A polynomial is **completely factored** when each of its factors is prime. For instance,

$$x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4)$$ Completely factored

is completely factored, but

$$x^3 - x^2 - 4x + 4 = (x - 1)(x^2 - 4)$$ Not completely factored

is not completely factored. Its complete factorization is

$$x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

The simplest type of factoring involves a polynomial that can be written as the product of a monomial and another polynomial. To factor such a polynomial, you can use the Distributive Property in the *reverse* direction.

$$ab + ac = a(b + c)$$ a is a common factor.

Example 1 Removing Common Factors

Factor each expression.

a. $6x^3 - 4x$ **b.** $(x - 2)(2x) + (x - 2)(3)$

SOLUTION

a. Each term of this polynomial has $2x$ as a common factor.

$$6x^3 - 4x = 2x(3x^2) - 2x(2) = 2x(3x^2 - 2)$$

b. The binomial factor $(x - 2)$ is common to both terms.

$$(x - 2)(2x) + (x - 2)(3) = (x - 2)(2x + 3)$$

✓CHECKPOINT 1

Factor the expression $(x + 1)^2 + 2x(x + 1)$. ■

Factoring Special Polynomial Forms

Factoring Special Polynomial Forms

Factored Form	*Example*
Difference of Two Squares	
$u^2 - v^2 = (u + v)(u - v)$	$9x^2 - 4 = (3x + 2)(3x - 2)$
Perfect Square Trinomial	
$u^2 + 2uv + v^2 = (u + v)^2$	$x^2 + 6x + 9 = (x + 3)^2$
$u^2 - 2uv + v^2 = (u - v)^2$	$x^2 - 6x + 9 = (x - 3)^2$
Sum or Difference of Two Cubes	
$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$
$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	$27x^3 - 1 = (3x - 1)(9x^2 + 3x + 1)$

STUDY TIP

In Example 2, note that the first step in factoring a polynomial is to check for common factors. Once the common factor is removed, it is often possible to recognize patterns that were not obvious at first glance.

Example 2 Removing a Common Factor First

Factor the expression $3 - 12x^2$.

SOLUTION

$$\begin{aligned} 3 - 12x^2 &= 3(1 - 4x^2) && \text{3 is a common factor.} \\ &= 3[1^2 - (2x)^2] && \text{Difference of two squares} \\ &= 3(1 + 2x)(1 - 2x) && \text{Completely factored} \end{aligned}$$

✓CHECKPOINT 2

Factor the expression $x^3 - x$. ■

Example 3 Factoring the Difference of Two Squares

a. $$\begin{aligned} (x + 2)^2 - y^2 &= [(x + 2) + y][(x + 2) - y] \\ &= (x + 2 + y)(x + 2 - y) \\ &= (x + y + 2)(x - y + 2) \end{aligned}$$

b. You can factor $16x^4 - 81$ by applying the difference of two squares formula twice.

$$\begin{aligned} 16x^4 - 81 &= (4x^2)^2 - 9^2 \\ &= (4x^2 + 9)(4x^2 - 9) && \text{First application} \\ &= (4x^2 + 9)[(2x)^2 - 3^2] \\ &= (4x^2 + 9)(2x + 3)(2x - 3) && \text{Second application} \end{aligned}$$

✓CHECKPOINT 3

Factor the expression $100 - 4y^2$. ■

A perfect square trinomial is the square of a binomial, and it has the following form. Note that the first and last terms of a perfect square trinomial are squares and the middle term is twice the product of u and v.

$$u^2 + 2uv + v^2 = (u + v)^2 \quad \text{or} \quad u^2 - 2uv + v^2 = (u - v)^2$$

Same sign Same sign

Example 4 Factoring Perfect Square Trinomials

a. $16x^2 + 8x + 1 = (4x)^2 + 2(4x)(1) + 1^2 = (4x + 1)^2$

b. $x^2 - 10x + 25 = x^2 - 2(x)(5) + 5^2 = (x - 5)^2$

✓CHECKPOINT 4

Factor the expression $x^2 - 12x + 36$. ■

The next two formulas show that sums and differences of cubes factor easily. Pay special attention to the signs of the terms.

Like signs Like signs

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2) \qquad u^3 - v^3 = (u - v)(u^2 + uv + v^2)$$

Unlike signs Unlike signs

Example 5 Factoring the Sum and Difference of Cubes

Factor each expression.

a. $x^3 - 27$ **b.** $3x^3 + 192$

SOLUTION

a. $x^3 - 27 = x^3 - 3^3$ Rewrite 27 as 3^3.

$= (x - 3)(x^2 + 3x + 9)$ Factor.

b. $3x^3 + 192 = 3(x^3 + 64)$ 3 is a common factor.

$= 3(x^3 + 4^3)$ Rewrite 64 as 4^3.

$= 3(x + 4)(x^2 - 4x + 16)$ Factor.

✓CHECKPOINT 5

Factor the expression $y^3 - 1$. ■

Trinomials with Binomial Factors

To factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.

Factors of a

$$ax^2 + bx + c = (\square x + \square)(\square x + \square)$$

Factors of c

The goal is to find a combination of factors of a and c such that the outer and inner products add up to the middle term bx. For instance, for the trinomial

$$6x^2 + 17x + 5$$

you can write

$$(2x + 5)(3x + 1) = \overset{\text{F}}{\downarrow}\,6x^2 + \overset{\text{O}}{\downarrow}\,2x + \overset{\text{I}}{\downarrow}\,15x + \overset{\text{L}}{\downarrow}\,5$$

$$= 6x^2 + \underset{\text{O + I}}{\downarrow}\,17x + 5.$$

Note that the outer (O) and inner (I) products add up to $17x$.

Example 6 Factoring a Trinomial: Leading Coefficient Is 1

Factor the trinomial $x^2 - 7x + 12$.

SOLUTION For this trinomial, you have $a = 1$, $b = -7$, and $c = 12$. Because b is negative and c is positive, both factors of 12 must be negative. That is, $12 = (-2)(-6)$, $12 = (-1)(-12)$, or $12 = (-3)(-4)$. So, the possible factorizations of $x^2 - 7x + 12$ are

$$(x - 2)(x - 6), \quad (x - 1)(x - 12), \quad \text{and} \quad (x - 3)(x - 4).$$

Testing the middle term, you can find the correct factorization to be

$$x^2 - 7x + 12 = (x - 3)(x - 4).$$

✓CHECKPOINT 6

Factor the trinomial $x^2 + x - 6$. ■

Example 7 Factoring a Trinomial: Leading Coefficient Is Not 1

Factor the trinomial $2x^2 + x - 15$.

SOLUTION For this trinomial, you have $a = 2$ and $c = -15$, which means that the factors of -15 must have unlike signs. The eight possible factorizations are as follows.

$$(2x - 1)(x + 15) \qquad (2x + 1)(x - 15)$$
$$(2x - 3)(x + 5) \qquad (2x + 3)(x - 5)$$
$$(2x - 5)(x + 3) \qquad (2x + 5)(x - 3)$$
$$(2x - 15)(x + 1) \qquad (2x + 15)(x - 1)$$

Testing the middle term, you can find the correct factorization to be

$$2x^2 + x - 15 = (2x - 5)(x + 3).$$

✓CHECKPOINT 7

Factor the trinomial $2x^2 - 5x + 3$. ■

Factoring by Grouping

Sometimes polynomials with more than three terms can be **factored by grouping.**

Example 8 Factoring by Grouping

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms.} \\ &= x^2(x - 2) - 3(x - 2) && \text{Factor groups.} \\ &= (x - 2)(x^2 - 3) && \text{Distributive Property} \end{aligned}$$

✓CHECKPOINT 8

Factor the polynomial $x^3 + x^2 + 5x + 5$. ■

When factoring by grouping, sometimes several different groupings will work. For instance, a different grouping could have been used in Example 8.

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 3x) - (2x^2 - 6) \\ &= x(x^2 - 3) - 2(x^2 - 3) \\ &= (x^2 - 3)(x - 2) \end{aligned}$$

As you can see, you obtain the same result as in Example 8.

Factoring by grouping can save you some of the trial and error involved in factoring a trinomial. To factor a trinomial of the form $ax^2 + bx + c$ by grouping, rewrite the middle term as the sum of two factors of the product ac that add up to b. This technique is illustrated in Example 9.

Example 9 Factoring a Trinomial by Grouping

Use factoring by grouping to factor $2x^2 + 5x - 3$.

SOLUTION In the trinomial $2x^2 + 5x - 3$, $a = 2$ and $c = -3$, so the product ac is -6. Notice that -6 factors as $(6)(-1)$, and $6 - 1 = 5 = b$. So, you can rewrite the middle term as $5x = 6x - x$. This produces the following.

$$\begin{aligned} 2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 && \text{Rewrite middle term.} \\ &= (2x^2 + 6x) - (x + 3) && \text{Group terms.} \\ &= 2x(x + 3) - (x + 3) && \text{Factor groups.} \\ &= (x + 3)(2x - 1) && \text{Distributive Property} \end{aligned}$$

The trinomial factors as $2x^2 + 5x - 3 = (x + 3)(2x - 1)$.

✓CHECKPOINT 9

Use factoring by grouping to factor $2x^2 + 5x - 12$. ■

CONCEPT CHECK

1. What is the common factor in the polynomial $3x^3 - 27x$?
2. Is $x^4 + 3x^3 - 8x - 24 = (x + 3)(x^3 - 8)$ factored completely? Explain.
3. Describe how you would show that $a^2 + b^2 \neq (a + b)^2$, where $a, b \neq 0$.
4. Can you factor $x^3 - 2x^2 - 8x + 24$ by grouping? Explain.

Skills Review 0.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.5.

In Exercises 1–10, find the product.

1. $3x(5x - 2)$
2. $-2y(y + 1)$
3. $(2x + 3)^2$
4. $(3x - 8)^2$
5. $(2x - 3)(x + 8)$
6. $(4 - 5z)(1 + z)$
7. $(2y + 1)(2y - 1)$
8. $(x + a)(x - a)$
9. $(x + 4)^3$
10. $(2x - 3)^3$

Exercises 0.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, factor out the common factor.

1. $3x + 6$
2. $6y - 30$
3. $3x^3 - 6x$
4. $4x^3 - 6x^2 + 12x$
5. $(x - 1)^2 + 6(x - 1)$
6. $3x(x + 2) - 4(x + 2)$

In Exercises 7–12, factor the difference of two squares.

7. $x^2 - 36$
8. $x^2 - \frac{1}{9}$
9. $16x^2 - 9y^2$
10. $x^2 - 49y^2$
11. $(x - 1)^2 - 4$
12. $25 - (z + 5)^2$

In Exercises 13–18, factor the perfect square trinomial.

13. $x^2 - 4x + 4$
14. $x^2 + 10x + 25$
15. $4y^2 + 12y + 9$
16. $9x^2 - 12x + 4$
17. $y^2 - \frac{2}{3}y + \frac{1}{9}$
18. $z^2 + z + \frac{1}{4}$

In Exercises 19–24, factor the sum or difference of cubes.

19. $x^3 - 8$
20. $x^3 - 27$
21. $y^3 + 125$
22. $y^3 + 1000$
23. $8t^3 - 1$
24. $27x^3 + 8$

In Exercises 25–38, factor the trinomial.

25. $x^2 + x - 2$
26. $x^2 + 6x + 8$
27. $w^2 - 5w + 6$
28. $z^2 - z - 6$
29. $y^2 + y - 20$
30. $z^2 - 4z - 21$
31. $x^2 - 30x + 200$
32. $x^2 - 5x - 150$
33. $3x^2 - 5x + 2$
34. $2x^2 - x - 1$
35. $9x^2 - 3x - 2$
36. $12y^2 + 7y + 1$
37. $6x^2 + 37x + 6$
38. $5u^2 + 13u - 6$

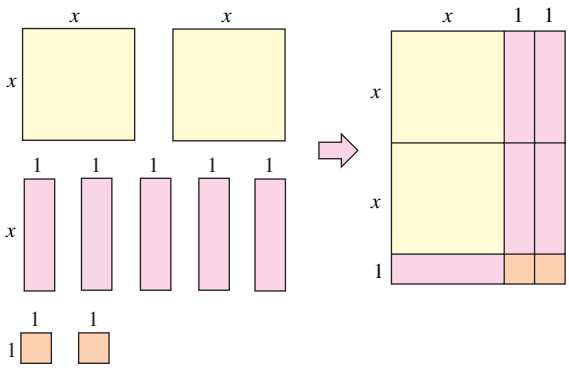
In Exercises 39–44, factor by grouping.

39. $x^3 - x^2 + 2x - 2$
40. $x^3 + 5x^2 - 5x - 25$
41. $2x^3 - x^2 - 6x + 3$
42. $5x^3 - 10x^2 + 3x - 6$
43. $6 + 2y - 3y^3 - y^4$
44. $z^5 + 2z^3 + z^2 + 2$

In Exercises 45–68, completely factor the expression.

45. $4x^2 - 8x$
46. $12x^3 - 48x$
47. $y^3 - 9y$
48. $x^3 - 4x^2$
49. $3x^2 - 48$
50. $7y^2 - 63$
51. $x^2 - 2x + 1$
52. $9x^2 - 6x + 1$
53. $1 - 4x + 4x^2$
54. $16 + 6x - x^2$
55. $2y^3 - 7y^2 - 15y$
56. $3x^4 + x^3 - 10x^2$
57. $-2x^2 - 4x + 2x^3$
58. $13x + 6 + 5x^2$
59. $3x^3 + x^2 + 15x + 5$
60. $5 - x + 5x^2 - x^3$
61. $x^4 - 4x^3 + x^2 - 4x$
62. $3u - 2u^2 + 6 - u^3$
63. $25 - (x + 5)^2$
64. $(t - 1)^2 - 49$
65. $(x^2 + 1)^2 - 4x^2$
66. $(x^2 + 8)^2 - 36x^2$
67. $2t^3 - 16$
68. $3x^3 + 81$

Geometric Modeling In Exercises 69–72, make a "geometric factoring model" to represent the given factorization. For instance, a factoring model for $2x^2 + 5x + 2 = (2x + 1)(x + 2)$ is shown below.



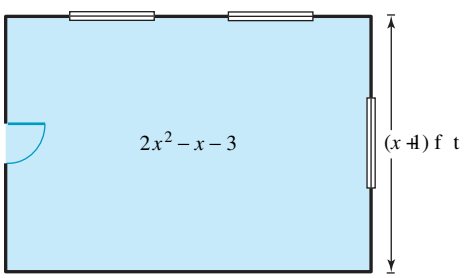
69. $x^2 + 3x + 2 = (x + 2)(x + 1)$

70. $x^2 + 4x + 3 = (x + 3)(x + 1)$

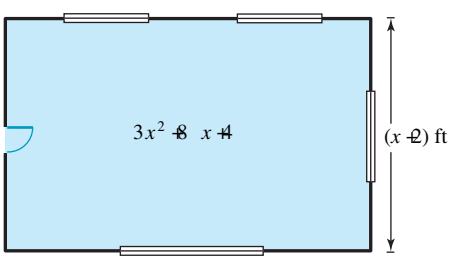
71. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

72. $3x^2 + 7x + 2 = (3x + 1)(x + 2)$

73. Geometry The room shown in the figure has a floor space of $(2x^2 - x - 3)$ square feet. If the width of the room is $(x + 1)$ feet, what is the length?



74. Geometry The room shown in the figure has a floor space of $(3x^2 + 8x + 4)$ square feet. If the width of the room is $(x + 2)$ feet, what is the length?



75. ***MAKE A DECISION*** Factor each trinomial. State whether you used factoring by grouping or factoring by trial and error.

(a) $x^2 + 11x + 24$ (b) $3x^2 + 7x - 20$

76. Find all integers b such that $x^2 + bx + 24$ can be factored. Describe how you found these values of b.

77. Find all integers $c > 0$ such that $x^2 + 8x + c$ can be factored. Describe how you found these values of c.

78. Think About It A student claims that

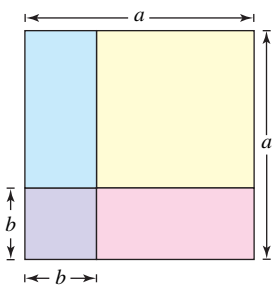
$x^3 - 8 = (x - 2)^3$.

Describe and correct the student's error.

79. Think About It Describe two different ways to factor $2x^2 - 7x - 15$.

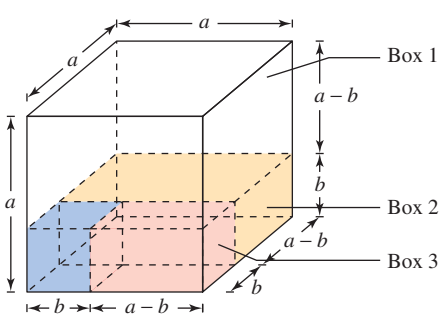
80. Geometric Modeling The figure shows a large square with an area of a^2 that contains a smaller square with an area of b^2. If the smaller square is removed, the remaining figure has an area of $a^2 - b^2$. Rearrange the parts of the remaining figure to illustrate the factoring formula

$a^2 - b^2 = (a - b)(a + b)$.



81. Geometric Modeling The figure shows a large cube with a volume of a^3 that contains a smaller cube with a volume of b^3. If the smaller cube is removed, the remaining solid has a volume of $a^3 - b^3$ and consists of the three rectangular boxes labeled Box 1, Box 2, and Box 3. Explain how you can use the figure to obtain the factoring formula

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.



Section 0.7

Fractional Expressions

- Find the domain of an algebraic expression.
- Simplify a rational expression.
- Perform operations with rational expressions.
- Simplify a complex fraction.

Domain of an Expression

The set of all real numbers for which an algebraic expression is defined is called the **domain** of the expression. For instance, the domain of

$$\frac{1}{x}$$

is all real numbers other than $x = 0$. Two algebraic expressions are **equivalent** if they have the same domain and yield the same values for all numbers in their domain. For instance, the expressions

$$[(x + 1) + (x + 2)] \text{ and } 2x + 3$$

are equivalent.

STUDY TIP

The domain of an algebraic expression does not include any value that creates *division by zero* or *the square root of a negative number*.

Example 1 Finding the Domain of an Algebraic Expression

a. The domain of the polynomial

$$2x^3 + 3x + 4$$

is the set of all real numbers. In fact, the domain of any polynomial is the set of all real numbers (unless the domain is specifically restricted).

b. The domain of the polynomial

$$x^2 + 5x + 2, \quad x > 0$$

is the set of positive real numbers, because the polynomial is specifically restricted to that set.

c. The domain of the radical expression

$$\sqrt{x}$$

is the set of nonnegative real numbers, because the square root of a negative number is not a real number.

d. The domain of the expression

$$\frac{x + 2}{x - 3}$$

is the set of all real numbers except $x = 3$, because the value $x = 3$ results in division by zero, which is undefined.

✓CHECKPOINT 1

Find the domain of $\dfrac{1}{x - 5}$. ■

Simplifying Rational Expressions

The quotient of two algebraic expressions is a **fractional expression.** Moreover, the quotient of two *polynomials* such as

$$\frac{1}{x}, \quad \frac{2x-1}{x+1}, \quad \text{or} \quad \frac{x^2-1}{x^2+1}$$

is a **rational expression.** Recall that a fraction is in simplest form if its numerator and denominator have no factors in common aside from ± 1. To write a fraction in simplest form, divide out common factors.

$$\frac{a \cdot \cancel{c}^{1}}{b \cdot \cancel{c}_{1}} = \frac{a}{b}, \quad b \neq 0, \quad c \neq 0$$

The key to success in simplifying rational expressions lies in your ability to *factor* polynomials. For example,

$$\frac{18x^2-18}{6x-6} = \frac{3(\cancel{6}^{1})(x+1)(\cancel{x-1}^{1})}{\cancel{6}_{1}(\cancel{x-1}_{1})} = 3(x+1), \quad x \neq 1.$$

Note that the original expression is undefined when $x = 1$ (because division by zero is undefined). Because this is not obvious in the simplified expression, you must add the domain restriction $x \neq 1$ to the simplified expression to make it *equivalent* to the original.

Example 2 Simplifying a Rational Expression

$$\frac{x^2+4x-12}{3x-6} = \frac{(x+6)(\cancel{x-2}^{1})}{3(\cancel{x-2}_{1})} \qquad \text{Factor completely.}$$

$$= \frac{x+6}{3}, \quad x \neq 2 \qquad \text{Divide out common factors.}$$

✓CHECKPOINT 2

Write the expression $\dfrac{2x^2-2}{3x-3}$ in simplest form. ■

In Example 2, do not make the mistake of trying to simplify further by dividing out *terms.*

$$\frac{x+6}{3} = \frac{x+\cancel{6}^{2}}{\cancel{3}_{1}} = x+2$$

Remember that to simplify fractions, you divide out *factors*, not terms.

When simplifying rational expressions, be sure to factor each polynomial completely before concluding that the numerator and denominator have no factors in common. Moreover, changing the sign of a factor may allow further simplification, as demonstrated in part (b) of the next example.

Example 3 Simplifying Rational Expressions

a. $\dfrac{x^3 - 4x}{x^2 + x - 2} = \dfrac{x(x+2)(x-2)}{(x+2)(x-1)}$ — Factor completely.

$= \dfrac{x(x-2)}{x-1}, \quad x \neq -2$ — Divide out common factors.

b. $\dfrac{12 + x - x^2}{2x^2 - 9x + 4} = \dfrac{(4-x)(3+x)}{(2x-1)(x-4)}$ — Factor completely.

$= \dfrac{-(x-4)(3+x)}{(2x-1)(x-4)}$ — $4 - x = -(x - 4)$

$= -\dfrac{3+x}{2x-1}, \quad x \neq 4$ — Divide out common factors.

✓CHECKPOINT 3

Write the expression $\dfrac{3 - 2x - x^2}{2x^2 - 2}$ in simplest form. ■

To multiply or divide rational expressions, use the properties of fractions (see Section 0.2). Recall that to divide fractions you invert the divisor and multiply.

Example 4 Multiplying Rational Expressions

$\dfrac{6x^2 - 6x}{x^2 + 2x - 3} \cdot \dfrac{x^2 + x - 6}{2x}$ — Original product

$= \dfrac{6x(x-1)(x+3)(x-2)}{(x-1)(x+3)(2x)}$ — Factor and multiply.

$= \dfrac{3(2x)(x-1)(x+3)(x-2)}{(x-1)(x+3)(2x)}$ — Divide out common factors.

$= 3(x-2), \quad x \neq -3, x \neq 0, x \neq 1$ — Simplify.

✓CHECKPOINT 4

Multiply and simplify:

$$\frac{3}{x-2} \cdot \frac{x-2}{3x+3}.$$ ■

Example 5 Dividing Rational Expressions

$\dfrac{2x}{3x - 12} \div \dfrac{x^2 - 2x}{x^2 - 6x + 8} = \dfrac{2x}{3x - 12} \cdot \dfrac{x^2 - 6x + 8}{x^2 - 2x}$ — Invert and multiply.

$= \dfrac{(2x)(x-2)(x-4)}{(3)(x-4)(x)(x-2)}$ — Factor and multiply.

$= \dfrac{(2x)(x-2)(x-4)}{(3)(x-4)(x)(x-2)}$ — Divide out common factors.

$= \dfrac{2}{3}, \quad x \neq 0, x \neq 2, x \neq 4$ — Simplify.

✓CHECKPOINT 5

Divide and simplify:

$$\frac{4x + 4y}{5} \div \frac{x + y}{2}.$$ ■

To add or subtract rational expressions, use the least common denominator (LCD) method or the following basic property of fractions that was covered on page 14.

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad b \neq 0, d \neq 0$$

This property is efficient for adding or subtracting *two* fractions that have no common factors in their denominators.

Example 6 Adding Rational Expressions

$$\frac{x}{x-3} + \frac{2}{3x+4} = \frac{x(3x+4) + 2(x-3)}{(x-3)(3x+4)} \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$= \frac{3x^2 + 4x + 2x - 6}{(x-3)(3x+4)} \qquad \text{Distributive Property}$$

$$= \frac{3x^2 + 6x - 6}{(x-3)(3x+4)} \qquad \text{Combine like terms.}$$

$$= \frac{3(x^2 + 2x - 2)}{(x-3)(3x+4)} \qquad \text{Factor.}$$

✓CHECKPOINT 6

Subtract: $\frac{4}{x} - \frac{2x}{3}$. ■

For fractions with a repeated factor in their denominators, the LCD method works well. Recall that the least common denominator of two or more fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator.

Example 7 Combining Rational Expressions: The LCD Method

Perform the indicated operations and simplify.

$$\frac{3}{x-1} - \frac{2}{x} + \frac{x+3}{x^2-1}$$

SOLUTION Using the factored denominators $(x-1)$, x, and $(x+1)(x-1)$, you can see that the least common denominator is $x(x+1)(x-1)$.

$$\frac{3}{x-1} - \frac{2}{x} + \frac{x+3}{x^2-1}$$

$$= \frac{3(x)(x+1)}{x(x+1)(x-1)} - \frac{2(x+1)(x-1)}{x(x+1)(x-1)} + \frac{(x+3)(x)}{x(x+1)(x-1)}$$

$$= \frac{3(x)(x+1) - 2(x+1)(x-1) + (x+3)(x)}{x(x+1)(x-1)}$$

$$= \frac{3x^2 + 3x - 2x^2 + 2 + x^2 + 3x}{x(x+1)(x-1)}$$

$$= \frac{2x^2 + 6x + 2}{x(x+1)(x-1)} = \frac{2(x^2 + 3x + 1)}{x(x+1)(x-1)}$$

✓CHECKPOINT 7

Perform the indicated operations and simplify:

$$\frac{5}{x} + \frac{4}{x-1} - \frac{4}{x(x-1)}.$$ ■

Complex Fractions

Fractional expressions with separate fractions in the numerator or denominator are called **complex fractions.** Here are two examples.

$$\frac{\left(\frac{1}{x}\right)}{x^2+1} \quad \text{and} \quad \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2+1}\right)}$$

A complex fraction can be simplified by combining the fractions in its numerator into a single fraction and then combining the fractions in its denominator into a single fraction. Then invert the denominator and multiply.

Example 8 Simplifying a Complex Fraction

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left[\frac{2-3(x)}{x}\right]}{\left[\frac{1(x-1)-1}{x-1}\right]} \qquad \text{Combine fractions.}$$

$$= \frac{\left(\frac{2-3x}{x}\right)}{\left(\frac{x-2}{x-1}\right)} \qquad \text{Simplify.}$$

$$= \frac{2-3x}{x} \cdot \frac{x-1}{x-2} \qquad \text{Invert and multiply.}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

✓CHECKPOINT 8

Simplify the complex fraction $\dfrac{\left(\frac{x}{3}-1\right)}{x-3}$. ■

Another way to simplify in Example 8 is to multiply its numerator and denominator by the LCD of all fractions in its numerator and denominator.

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} \cdot \frac{x(x-1)}{x(x-1)} \qquad \text{LCD is } x(x-1).$$

$$= \frac{\left(\frac{2-3x}{\cancel{x}}\right) \cdot \cancel{x}(x-1)}{\left(\frac{x-2}{\cancel{x-1}}\right) \cdot x\cancel{(x-1)}} = \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

CONCEPT CHECK

1. Is $x \geq 0$ the domain of $\sqrt{x-2}$? Explain.
2. Explain why $\dfrac{x}{x^2-4} + \dfrac{5}{x+2} \neq \dfrac{x+5}{x^2+x-2}$.
3. In the expression $(3x-2) \div (x+1)$, explain why $x \neq -1$.
4. What is a complex fraction? Give an example.

Skills Review 0.7

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.6.

In Exercises 1–10, completely factor the polynomial.

1. $5x^2 - 15x^3$

2. $16x^2 - 9$

3. $9x^2 - 6x + 1$

4. $9 + 12y + 4y^2$

5. $z^2 + 4z + 3$

6. $x^2 - 15x + 50$

7. $3 + 8x - 3x^2$

8. $3x^2 - 46x + 15$

9. $s^3 + s^2 - 4s - 4$

10. $y^3 + 64$

Exercises 0.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, determine if each value of x is in the domain of the expression.

1. $\dfrac{x+2}{5x+2}$ (a) $x = -\dfrac{2}{5}$ (b) $x = 2$

2. $\dfrac{2x+3}{x-4}$ (a) $x = -\dfrac{3}{2}$ (b) $x = 4$

3. $\sqrt{2x+4}$ (a) $x = -2$ (b) $x = 2$

4. $\sqrt{3x-9}$ (a) $x = -3$ (b) $x = 3$

In Exercises 5–12, find the domain of the expression.

5. $3x^2 - 4x + 7$

6. $6x^2 + 7x - 9,\ x > 0$

7. $\dfrac{1}{x-2}$

8. $\dfrac{x+1}{2x+1}$

9. $\dfrac{x-1}{x^2-4x}$

10. $\dfrac{4x+3}{x^2-36}$

11. $\sqrt{x+1}$

12. $\dfrac{1}{\sqrt{x+1}}$

In Exercises 13–18, find the missing factor and state any domain restrictions necessary to make the two fractions equivalent.

13. $\dfrac{5}{2x} = \dfrac{5(\quad)}{6x^2}$

14. $\dfrac{3}{4} = \dfrac{3(\quad)}{4(x+1)}$

15. $\dfrac{x+1}{x} = \dfrac{(x+1)(\quad)}{x(x-2)}$

16. $\dfrac{3y-4}{y+1} = \dfrac{(3y-4)(\quad)}{y^2-1}$

17. $\dfrac{3x}{x-3} = \dfrac{3x(\quad)}{x^2-x-6}$

18. $\dfrac{1-z}{z^2} = \dfrac{(1-z)(\quad)}{z^3+z^2}$

In Exercises 19–34, write the rational expression in simplest form.

19. $\dfrac{15x^2}{10x}$

20. $\dfrac{24y^3}{56y^7}$

21. $\dfrac{2x}{4x+4}$

22. $\dfrac{9x^2+9x}{2x+2}$

23. $\dfrac{x-5}{10-2x}$

24. $\dfrac{3-x}{8x-24}$

25. $\dfrac{x^2-25}{5-x}$

26. $\dfrac{x^2-16}{4-x}$

27. $\dfrac{x^3+5x^2+6x}{x^2-4}$

28. $\dfrac{x^2+8x-20}{x^2+11x+10}$

29. $\dfrac{y^2-7y+12}{y^2+3y-18}$

30. $\dfrac{x+1}{x^2-3x-4}$

31. $\dfrac{2-x+2x^2-x^3}{x-2}$

32. $\dfrac{x^2-9}{x^3+x^2-9x-9}$

33. $\dfrac{z^3-27}{z^2+3z+9}$

34. $\dfrac{y^3-2y^2-8y}{y^3+8}$

In Exercises 35–48, perform the indicated operations and simplify.

35. $\dfrac{5}{x-1} \cdot \dfrac{x-1}{25(x-2)}$

36. $\dfrac{x+13}{x^3(3-x)} \cdot \dfrac{x(x-3)}{5}$

37. $\dfrac{(x-9)(x+7)}{x+1} \cdot \dfrac{x}{9-x}$

38. $\dfrac{(x+5)(x-3)}{x+2} \cdot \dfrac{1}{(x+5)(x+2)}$

39. $\dfrac{r}{r-1} \cdot \dfrac{r^2-1}{r^2}$

40. $\dfrac{4y-16}{5y+15} \cdot \dfrac{2y+6}{4-y}$

41. $\dfrac{t^2 - t - 6}{t^2 + 6t + 9} \cdot \dfrac{t + 3}{t^2 - 4}$

42. $\dfrac{y^3 - 8}{2y^3} \cdot \dfrac{4y}{y^2 - 5y + 6}$

43. $\dfrac{x^2 + x - 2}{x^3 + x^2} \cdot \dfrac{x}{x^2 + 3x + 2}$

44. $\dfrac{x^3 - 8}{x + 1} \cdot \dfrac{x^2 - 1}{x^3 - 3x^2 + 2x}$

45. $\dfrac{3(x + y)}{4} \div \dfrac{x + y}{2}$

46. $\dfrac{x + 2}{5(x - 3)} \div \dfrac{x - 2}{5(x - 3)}$

47. $\dfrac{\left[\dfrac{x^2}{(x + 1)^2}\right]}{\left[\dfrac{x}{(x + 1)^3}\right]}$

48. $\dfrac{\left(\dfrac{x^2 - 1}{x}\right)}{\left[\dfrac{(x - 1)^2}{x}\right]}$

In Exercises 49–52, find the least common denominator of the expressions.

49. $\dfrac{1}{x^2}, \dfrac{1}{x - 1}, \dfrac{1}{x^2 - x}$

50. $\dfrac{1}{x}, \dfrac{1}{x^2 + 3x}, \dfrac{1}{x + 3}$

51. $\dfrac{10}{x + 5}, \dfrac{x + 4}{x - 7}, \dfrac{x + 5}{x^2 - 2x - 35}$

52. $\dfrac{x - 1}{x + 2}, \dfrac{8}{x^2 - x - 6}, \dfrac{x}{x - 3}$

In Exercises 53–62, perform the indicated operations and simplify.

53. $\dfrac{4x}{x - 2} + \dfrac{x}{x - 2}$

54. $\dfrac{3x - 2}{x + 1} + \dfrac{2 - x}{x + 1}$

55. $\dfrac{3x}{x - 4} + \dfrac{x}{4 - x}$

56. $\dfrac{4}{3 - x} + \dfrac{5}{x - 3}$

57. $4 - \dfrac{3}{x - 5}$

58. $\dfrac{4}{x + 2} - 6$

59. $\dfrac{2}{x^2 - 4} - \dfrac{1}{x^2 - 3x + 2}$

60. $\dfrac{x}{x^2 + x - 2} - \dfrac{1}{x + 2}$

61. $-\dfrac{1}{x} + \dfrac{2}{x^2 + 1} + \dfrac{1}{x^3 + x}$

62. $\dfrac{2}{x + 1} + \dfrac{2}{x - 1} + \dfrac{1}{x^2 - 1}$

In Exercises 63–68, simplify the complex fraction.

63. $\dfrac{\left(\dfrac{x}{2} - 1\right)}{(x - 2)}$

64. $\dfrac{(x - 3)}{\left(\dfrac{x}{4} - \dfrac{4}{x}\right)}$

65. $\dfrac{\left(\dfrac{1}{x} - \dfrac{1}{x + 1}\right)}{\left(\dfrac{1}{x + 1}\right)}$

66. $\dfrac{\left(\dfrac{5}{y} - \dfrac{6}{2y + 1}\right)}{\left(\dfrac{5}{y} + 4\right)}$

67. $\dfrac{\left(\sqrt{x} - \dfrac{1}{2\sqrt{x}}\right)}{\sqrt{x}}$

68. $\dfrac{\left(\dfrac{1}{\sqrt{2y}} + \sqrt{2y}\right)}{\sqrt{2y}}$

Monthly Payment In Exercises 69 and 70, use the formula for the approximate annual interest rate r of a monthly installment loan

$$r = \frac{\left[\dfrac{24(NM - P)}{N}\right]}{\left(P + \dfrac{NM}{12}\right)}$$

where N is the total number of payments, M is the monthly payment, and P is the amount financed.

69. (a) Approximate the annual interest rate r for a four-year car loan of \$18,000 that has monthly payments of \$475.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

70. (a) Approximate the annual interest rate r for a five-year car loan of \$20,000 that has monthly payments of \$475.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

71. Refrigeration When food is placed in a refrigerator, the time required for the food to cool depends on the amount of food, the air circulation in the refrigerator, the original temperature of the food, and the temperature of the refrigerator. One model for the temperature of food that starts at 75°F and is placed in a 40°F refrigerator is

$$T = 10\left(\frac{4t^2 + 16t + 75}{t^2 + 4t + 10}\right), \quad t \geq 0$$

where T is the temperature (in degrees Fahrenheit) and t is the time (in hours). Sketch a bar graph showing the temperature of the food when $t = 0$, 1, 2, 3, 4, and 5 hours. According to the model, will the food reach a temperature of 40°F after 6 hours?

72. Oxygen Level The mathematical model

$$O = \frac{t^2 - t + 1}{t^2 + 1}, \quad t \geq 0$$

gives the percent of the normal level of oxygen in a pond, where t is the time in weeks after organic waste is dumped into the pond. Sketch a bar graph showing the oxygen level of the pond when $t = 0, 1, 2, 3, 4$, and 5 weeks. What conclusions can you make from your bar graph?

Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 64. Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 0.1	Review Exercises
■ Classify real numbers as natural numbers, integers, rational numbers, or irrational numbers.	*1, 2*
■ Order real numbers.	*3, 4*
■ Use and interpret inequality notation.	*5–10*
■ Interpret absolute value notation.	*11–14, 19, 20*
■ Find the distance between two numbers on the real number line.	*15–18*

Section 0.2

- Identify the terms of an algebraic expression. — *21–24*
- Evaluate an algebraic expression. — *25, 26*
- Identify basic rules of algebra. — *27–30*
- Perform operations on real numbers. — *31–36*
- Use the least common denominator method to add and subtract fractions. — *33, 34*
- Use a calculator to evaluate an algebraic expression. — *37, 38*
- Round decimal numbers. — *37, 38*

Section 0.3

- Use properties of exponents to evaluate and simplify expressions with exponents.

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad (a^m)^n = a^{mn} \qquad a^{-n} = \frac{1}{a^n}$$

$$a^0 = 1 \qquad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \qquad |a^2| = |a|^2 = a^2$$

39–46

- Use scientific notation. — *47–50*
- Use a calculator to evaluate expressions involving powers. — *51, 52*
- Use interest formulas to solve an application problem.

Simple interest: $A = P(1 + rt)$

Compound interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$ — *53, 54*

*Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

Section 0.4

Review Exercises

- Simplify and evaluate expressions involving radicals.

$$\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \qquad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} \qquad \left(\sqrt[n]{a}\right)^n = a \qquad \text{For } n \text{ even, } \sqrt[n]{a^n} = |a|$$

For n odd, $\sqrt[n]{a^n} = a$ — *55–60*

- Rationalize a denominator by using its conjugate. — *61, 62*
- Combine radicals. — *63–68*
- Use properties of rational exponents. — *69, 70*
- Use a calculator to evaluate a radical. — *71, 72*

Section 0.5

- Write a polynomial in standard form. — *73–82*
- Add and subtract polynomials by combining like terms. — *73–76*
- Multiply polynomials using FOIL or a vertical format. — *77–82*
- Use special products to multiply polynomials.

$$(u + v)(u - v) = u^2 - v^2 \qquad (u \pm v)^2 = u^2 \pm 2uv + v^2$$

$$(u \pm v)^3 = u^3 \pm 3u^2v + 3uv^2 \pm v^3$$ — *78, 81, 82*

- Use polynomials to solve an application problem. — *83–86*

Section 0.6

- Factor a polynomial by removing common factors. — *87, 89, 94*
- Factor a polynomial in a special form.

$$u^2 - v^2 = (u + v)(u - v) \qquad u^2 \pm 2uv + v^2 = (u \pm v)^2$$

$$u^3 \pm v^3 = (u \pm v)(u^2 \mp uv + v^2)$$ — *87, 90–94*

- Factor a trinomial as the product of two binomials. — *88, 89*
- Factor a polynomial by grouping. — *90, 93*

Section 0.7

- Find the domain of an algebraic expression by finding values of the variable that make a denominator zero or a radicand negative. — *95–100*
- Simplify a rational expression by dividing out common factors from the numerator and denominator. — *101–106*
- Perform operations with rational expressions by using properties of fractions. — *107–112*
- Simplify a complex fraction. — *113–116*

Study Strategies

- **Use the Skills Review Exercises** Each section exercise set in this text (except the set for Section 0.1) begins with a set of skills review exercises. You should begin each homework session by quickly working through all of these exercises (all are answered in the back of the text). The "old" skills covered in these exercises are needed to master the "new" skills in the section exercise set. The skills review exercises remind you that mathematics is cumulative—to be successful in this course, you must retain "old" skills.
- **Use the Additional Study Aids** The additional study aids were prepared specifically to help you master the concepts discussed in the text. They are the *Student Solutions Manual*, the *Graphing Calculator Keystroke Guide*, and the *Instructional DVD*.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, determine which numbers in the set are (a) natural numbers, (b) integers, (c) rational numbers, and (d) irrational numbers.

1. $\{11, -14, -\frac{8}{9}, \frac{5}{2}, \sqrt{6}, 0.4\}$

2. $\{\sqrt{15}, -22, -\frac{10}{3}, 0, 5.2, \frac{3}{7}\}$

In Exercises 3 and 4, plot the two real numbers on the real number line and place the appropriate inequality sign (< or >) between them.

3. $-4, -3$

4. $\frac{1}{5}, \frac{1}{6}$

In Exercises 5 and 6, give a verbal description of the subset of real numbers that is represented by the inequality, and sketch the subset on the real number line.

5. $x \leq -6$

6. $x > 5$

In Exercises 7–10, use inequality notation to describe the subset of real numbers.

7. x is nonnegative.

8. x is at most 7.

9. x is greater than 2 and less than or equal to 5.

10. x is less than or equal to -2 or x is greater than 2.

In Exercises 11 and 12, evaluate the expression.

11. $-|-14|$

12. $|-4 - 2|$

In Exercises 13 and 14, place the correct symbol (<, >, or =) between the two real numbers.

13. $|-12| \quad -|12|$

14. $|9| \quad |-9|$

In Exercises 15–18, find the distance between a and b.

15. $a = -14, \quad b = -18$

16. $a = -1, \quad b = -5$

17. $a = 2, \quad b = -8$

18. $a = 10, \quad b = -3$

In Exercises 19 and 20, use absolute value notation to describe the sentence.

19. The distance between x and 7 is at least 4.

20. The distance between x and -22 is no more than 10.

In Exercises 21–24, identify the terms of the algebraic expression.

21. $4 + x - 2x^2$

22. $16x^2 - 4$

23. $3x^3 + 7x - 4$

24. $3x^3 - 9x$

In Exercises 25 and 26, evaluate the expression for each value of x.

25. $-4x^2 - 6x$ (a) $x = -1$ (b) $x = 0$

26. $12 - 5x^2$ (a) $x = -2$ (b) $x = 3$

In Exercises 27–30, identify the rule of algebra illustrated by the statement.

27. $5(x^2 + x) = 5x^2 + 5x$

28. $x + (2x + 3) = (x + 2x) + 3$

29. $3x + 7 = 7 + 3x$

30. $(x^2 - 1)\left(\frac{1}{x^2 - 1}\right) = 1$

In Exercises 31–36, perform the indicated operation(s). Write fractional answers in simplest form.

31. $-3 - 2(4 - 5)$

32. $12(-3 + 5) - 20$

33. $\frac{1}{2} + \frac{1}{3} - \frac{1}{6}$

34. $\frac{5}{12} + \frac{3}{5}$

35. $5^2 \cdot 5^{-1}$

36. $(4^2)^2$

In Exercises 37 and 38, use a calculator to evaluate the expression. (Round to two decimal places.)

37. $4\left(\frac{1}{6} - \frac{1}{7}\right)$

38. $-2 + 3\left(\frac{1}{2} - \frac{1}{3}\right)$

In Exercises 39–42, evaluate the expression for the value of x.

39. $-2x^2, \quad x = -1$

40. $-\frac{(-x)^2}{6}, \quad x = -3$

41. $\frac{x^2 - 4}{x - 4}, \quad x = 3$

42. $\frac{2x^3 - x + 2}{x - 7}, \quad x = -2$

In Exercises 43–46, simplify the expression.

43. $\frac{(4x)^2}{2x}$

44. $(-x)^2(-3x)^3$

45. $\frac{10x^2}{2x^6}$

46. $2x(5x^2)^3$

In Exercises 47 and 48, write the number in scientific notation.

47. Population of the United States: 300,400,000 *(Source: U.S. Census Bureau)*

48. Number of Meters in One Foot: 0.3048

In Exercises 49 and 50, write the number in decimal notation.

49. Diameter of the Sun: 8.644×10^5 miles

50. Length of an E. Coli Bacterium: 2×10^{-6} meter

In Exercises 51 and 52, use a calculator to evaluate the expression. (Round to three decimal places.)

51. (a) $1800(1 + 0.08)^{24}$

(b) $0.0024(7{,}658{,}400)$

52. (a) $50{,}000\left(1 + \dfrac{0.075}{12}\right)^{48}$

(b) $\dfrac{28{,}000{,}000 + 34{,}000{,}000}{87{,}000{,}000}$

In Exercises 53 and 54, complete the table by finding the balance.

53. Balance in an Account You deposit \$1500 in an account with an annual interest rate of 6.5% compounded monthly.

Year	5	10	15	20	25
Balance					

54. Balance in an Account You deposit \$12,000 in an account with an annual interest rate of 6% compounded quarterly.

Year	5	10	15	20	25
Balance					

In Exercises 55 and 56, fill in the missing form.

	Radical Form	*Rational Exponent Form*
55.	$\sqrt{16} = 4$	
56.		$16^{1/4} = 2$

In Exercises 57 and 58, evaluate the expression.

57. $\sqrt{169}$ **58.** $\sqrt[3]{125}$

In Exercises 59 and 60, simplify by removing all possible factors from the radical.

59. $\sqrt{4x^4}$ **60.** $\sqrt[3]{\dfrac{2x^3}{27}}$

In Exercises 61 and 62, rewrite the expression by rationalizing the denominator. Simplify your answer.

61. $\dfrac{1}{2 - \sqrt{3}}$

62. $\dfrac{2}{3 + \sqrt{5}}$

In Exercises 63–68, simplify the expression.

63. $2\sqrt{x} - 5\sqrt{x}$

64. $\sqrt{72} + \sqrt{128}$

65. $\sqrt{5}\sqrt{2}$

66. $\sqrt{3}\sqrt{4}$

67. $(64)^{-2/3}$

68. $4^{1/3} \cdot 4^{5/3}$

In Exercises 69 and 70, use rational exponents to reduce the index of the radical.

69. $\sqrt[4]{5^2}$ **70.** $\sqrt[8]{x^4}$

In Exercises 71 and 72, use a calculator to approximate the number. (Round your answer to three decimal places.)

71. $\sqrt{127}$ **72.** $\sqrt[3]{52}$

In Exercises 73–82, perform the indicated operation(s) and write the resulting polynomial in standard form.

73. $2(x - 3) - 4(2x - 8)$

74. $3(x^2 - 5x + 2) + 3x(2 - 4x)$

75. $x(x - 2) - 2(3x + 7)$

76. $2x(x + 1) + 3(x^2 - x)$

77. $(x + 1)(x - 2)$

78. $(2x - 5)(2x + 5)$

79. $(x + 4)(x^2 - 4x + 16)$

80. $(x - 2)(x^2 + 6x + 9)$

81. $(x + 4)^2$

82. $(2x + 1)^3$

83. Home Prices The average sale price (in thousands of dollars) of a newly manufactured residential mobile home in the United States from 2000 to 2005 can be represented by the polynomial

$3.17x + 45.7$

where x represents the year, with $x = 0$ corresponding to 2000. Evaluate the polynomial when $x = 5$. Then describe your result in everyday terms. *(Source: U.S. Census Bureau)*

84. Home Prices The median sale price (in thousands of dollars) of a new one-family home in the southern United States from 2000 to 2005 can be represented by the polynomial

$9.38x + 145.4$

where x represents the year, with $x = 0$ corresponding to 2000. Evaluate the polynomial when $x = 5$. Then describe your result in everyday terms. *(Source: U.S. Census Bureau and U.S. Department of Housing and Urban Development)*

85. Cell Phone Subscribers The numbers of cell phone subscribers (in millions) in the United States from 2000 to 2005 can be represented by the polynomial

$19.18x + 106.6$

where x represents the year, with $x = 0$ corresponding to 2000. Evaluate the polynomial when $x = 0$ and $x = 5$. Then describe your results in everyday terms. *(Source: Cellular Telecommunications & Internet Association)*

86. Cell Sites The numbers of cellular telecommunications sites in the United States from 2000 to 2005 can be represented by the polynomial

$-1297.79x^2 + 22{,}637.7x + 104{,}230$

where x represents the year, with $x = 0$ corresponding to 2000. Evaluate the polynomial when $x = 0$ and $x = 5$. Then describe your results in everyday terms. *(Source: Cellular Telecommunications & Internet Association)*

In Exercises 87–94, completely factor the expression.

87. $4x^2 - 36$

88. $x^2 - 4x - 5$

89. $-3x^2 - 6x + 3x^3$

90. $x^3 - 4x^2 - 2x + 8$

91. $x^3 - 16x$

92. $8x^3 - 125$

93. $x^3 - 2x^2 - 9x + 18$

94. $2x^5 - 16x^3$

In Exercises 95–98, find the domain of the expression.

95. $\dfrac{2x + 1}{x - 3}$

96. $\dfrac{x - 3}{x + 1}$

97. $2x^2 - 11x + 5$

98. $4\sqrt{2x}$

In Exercises 99 and 100, find the missing factor and state any domain restrictions necessary to make the two fractions equivalent.

99. $\dfrac{4}{3x} = \dfrac{4(\quad)}{9x^2}$

100. $\dfrac{5}{7} = \dfrac{5(\quad)}{7(x + 2)}$

In Exercises 101–106, write the rational expression in simplest form.

101. $\dfrac{x^2 - 4}{2x + 4}$

102. $\dfrac{2x^2 + 4x}{2x}$

103. $\dfrac{x^2 - 2x - 15}{x + 3}$

104. $\dfrac{x^3 + 2x^2 - 3x}{x - 1}$

105. $\dfrac{x^3 - 9x}{x^3 - 4x^2 + 3x}$

106. $\dfrac{x^3 + 64}{x^2 - x - 20}$

In Exercises 107–112, perform the operation and simplify.

107. $\dfrac{2x - 1}{x + 1} \cdot \dfrac{x^2 - 1}{2x^2 - 7x + 3}$

108. $\dfrac{x + 2}{x - 4} \div \dfrac{2x + 4}{8x}$

109. $\dfrac{x}{x - 1} + \dfrac{2x}{x - 2}$

110. $\dfrac{2}{x + 2} - \dfrac{3}{x - 2}$

111. $\dfrac{2}{x - 1} + \dfrac{4}{x + 1} + \dfrac{8}{x^2 - 1}$

112. $\dfrac{1}{x - 1} + \dfrac{2}{x} - \dfrac{1}{x^2 - x}$

In Exercises 113–116, simplify the complex fraction.

113. $\dfrac{\left(\dfrac{x^2 - 1}{x}\right)}{\left[\dfrac{(x - 1)^2}{x}\right]}$

114. $\dfrac{(x - 4)}{\left(\dfrac{x}{4} - \dfrac{4}{x}\right)}$

115. $\dfrac{\left(\dfrac{1}{x} - \dfrac{1}{y}\right)}{\left(\dfrac{1}{x} + \dfrac{1}{y}\right)}$

116. $\dfrac{\left(\dfrac{1}{2x - 3} - \dfrac{1}{2x + 3}\right)}{\left(\dfrac{1}{2x} - \dfrac{1}{2x + 3}\right)}$

Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

1. Evaluate the expression $-3x^2 - 5x$ when $x = -3$.

2. Complete the table at the left given that \$4000 is deposited in an account with an annual interest rate of 7.5% compounded monthly. What can you conclude from the table?

Year	Balance
5	
10	
15	
20	
25	

Table for 2

In Exercises 3–8, simplify the expression.

3. $8(-2x^2)^3$

4. $3\sqrt{x} - 7\sqrt{x}$

5. $5^{1/4} \cdot 5^{7/4}$

6. $\sqrt{48} - \sqrt{80}$

7. $\sqrt{12x^3}$

8. $\dfrac{2}{5 - \sqrt{7}}$

In Exercises 9 and 10, write the polynomial in standard form.

9. $(3x + 7)^2$

10. $3x(x + 5) - 2x(4x - 7)$

In Exercises 11–14, completely factor the expression.

11. $5x^2 - 80$

12. $4x^2 + 12x + 9$

13. $x^3 - 6x^2 - 3x + 18$

14. $x^3 + 2x^2 - 4x - 8$

15. Simplify: $\dfrac{x^2 - 16}{3x + 12}$.

16. Multiply and simplify: $\dfrac{3x - 5}{x + 3} \cdot \dfrac{x^2 + 7x + 12}{9x^2 - 25}$.

17. Add and simplify: $\dfrac{x}{x - 3} + \dfrac{3x}{x - 4}$.

18. Subtract and simplify: $\dfrac{3}{x + 5} - \dfrac{4}{x - 2}$.

In Exercises 19 and 20, find the domain of the expression.

19. $\sqrt{x - 2}$

20. $\dfrac{3}{x + 1}$

21. Simplify the complex fraction $\dfrac{\left(\dfrac{2x - 9}{x - 1}\right)}{\left(\dfrac{3}{x - 1} + \dfrac{1 - x}{x + 2}\right)}$.

22. **Movie Price** The average price of a movie ticket in the United States from 1995 to 2005 can be approximated by the polynomial $0.224x + 3.09$, where x is the year, with $x = 5$ corresponding to 1995. Evaluate the polynomial when $x = 5$ and $x = 15$. Then describe your results in everyday terms. *(Source: Exhibitor Relations Co., Inc.)*

1 Equations and Inequalities



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The force of gravity on the moon is about one-sixth the force of gravity on Earth. So, objects fall at a different rate on the moon than on Earth. You can use a quadratic equation to model the height with respect to time of a falling object on the moon. (See Section 1.4, Example 6.)

1.1 Linear Equations
1.2 Mathematical Modeling
1.3 Quadratic Equations
1.4 The Quadratic Formula
1.5 Other Types of Equations
1.6 Linear Inequalities
1.7 Other Types of Inequalities

Applications

Equations and inequalities are used to model and solve many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Blood Oxygen Level, Exercise 69, page 113
- Life Expectancy, Exercise 70, page 124
- Make a Decision: Company Profits, Exercise 65, page 146

Section 1.1
Linear Equations

- Classify an equation as an identity or a conditional equation.
- Solve a linear equation in one variable.
- Use a linear model to solve an application problem.

Equations and Solutions

An **equation** is a statement that two algebraic expressions are equal. Some examples of equations in x are

$$3x - 5 = 7, \quad x^2 - x - 6 = 0, \quad \text{and} \quad \sqrt{2x} = 4.$$

To **solve** an equation in x means to find all values of x for which the equation is **true.** Such values are called **solutions.** For instance, $x = 4$ is a solution of the equation $3x - 5 = 7$, because $3(4) - 5 = 7$ is a true statement.

An equation that is true for *every* real number in the domain of the variable is called an **identity.** Two examples of identities are

$$x^2 - 9 = (x + 3)(x - 3) \quad \text{and} \quad \frac{x}{3x^2} = \frac{1}{3x}, \quad x \neq 0.$$

The first equation is an identity because it is a true statement for all real values of x. The second is an identity because it is true for all nonzero real values of x.

An equation that is true for just *some* (or even none) of the real numbers in the domain of the variable is called a **conditional equation.** For example, the equation $x^2 - 9 = 0$ is conditional because $x = 3$ and $x = -3$ are the only values in the domain that satisfy the equation.

Example 1 Classifying Equations

Determine whether each equation is an identity or a conditional equation.

a. $2(x + 3) = 2x + 6$ **b.** $2(x + 3) = x + 6$ **c.** $2(x + 3) = 2x + 3$

SOLUTION

a. This equation is an identity because it is true for every real value of x.

b. This equation is a conditional equation because $x = 0$ is the only value in the domain for which the equation is true.

c. This equation is a conditional equation because there are no real number values of x for which the equation is true.

✓CHECKPOINT 1

Determine whether the equation $4(x + 1) = 4x + 4$ is an identity or a conditional equation. ■

Equations are used in algebra for two distinct purposes: (1) *identities* are usually used to state mathematical properties and (2) *conditional equations* are usually used to model and solve problems that occur in real life.

Linear Equations in One Variable

The most common type of conditional equation is a **linear equation.**

Definition of a Linear Equation

A **linear equation** in one variable x is an equation that can be written in the standard form

$$ax + b = 0$$

where a and b are real numbers with $a \neq 0$.

A linear equation in x has exactly one solution. To see this, consider the following steps. (Remember that $a \neq 0$.)

$ax + b = 0$ Original equation

$ax = -b$ Subtract b from each side.

$x = -\frac{b}{a}$ Divide each side by a.

So, the equation $ax + b = 0$ has exactly one solution, $x = -b/a$.

To solve a linear equation in x, you should isolate x by forming a sequence of **equivalent** (and usually simpler) equations, each having the same solution as the original equation. The operations that yield equivalent equations come from the basic rules of algebra reviewed in Section 0.2.

Forming Equivalent Equations

A given equation can be transformed into an equivalent equation by one or more of the following steps.

	Given Equation	*Equivalent Equation*
1. Remove symbols of grouping, combine like terms, or simplify one or both sides of the equation.	$2x - x = 4$	$x = 4$
	$3(x - 2) = 5$	$3x - 6 = 5$
2. Add (or subtract) the same quantity to (from) *each* side of the equation.	$x + 1 = 6$	$x = 5$
3. Multiply (or divide) *each* side of the equation by the same *nonzero* quantity.	$2x = 6$	$x = 3$
4. Interchange sides of the equation.	$2 = x$	$x = 2$

The steps for solving a linear equation in x written in standard form are shown in Example 2.

Example 2 Solving a Linear Equation

Solve $3x - 6 = 0$.

SOLUTION

$$3x - 6 = 0 \qquad \text{Write original equation.}$$
$$3x = 6 \qquad \text{Add 6 to each side.}$$
$$x = 2 \qquad \text{Divide each side by 3.}$$

✓CHECKPOINT 2

Solve $5 + 5x = 15$. ■

After solving an equation, you should **check each solution** in the *original* equation. For instance, in Example 2, you can check that 2 is a solution by substituting 2 for x in the original equation $3x - 6 = 0$, as follows.

CHECK

$$3x - 6 = 0 \qquad \text{Write original equation.}$$
$$3(2) - 6 \stackrel{?}{=} 0 \qquad \text{Substitute 2 for } x.$$
$$6 - 6 = 0 \qquad \text{Solution checks. ✓}$$

Example 3 Solving a Linear Equation

Solve $6(x - 1) + 4 = 3(7x + 1)$.

SOLUTION

$$6(x - 1) + 4 = 3(7x + 1) \qquad \text{Write original equation.}$$
$$6x - 6 + 4 = 21x + 3 \qquad \text{Distributive Property}$$
$$6x - 2 = 21x + 3 \qquad \text{Simplify.}$$
$$-15x = 5 \qquad \text{Add 2 to and subtract } 21x \text{ from each side.}$$
$$x = -\tfrac{1}{3} \qquad \text{Divide each side by } -15.$$

The solution is $x = -\frac{1}{3}$. You can check this as follows.

CHECK

$$6(x - 1) + 4 = 3(7x + 1) \qquad \text{Write original equation.}$$
$$6\left(-\tfrac{1}{3} - 1\right) + 4 \stackrel{?}{=} 3\left[7\left(-\tfrac{1}{3}\right) + 1\right] \qquad \text{Substitute } -\tfrac{1}{3} \text{ for } x.$$
$$6\left(-\tfrac{4}{3}\right) + 4 \stackrel{?}{=} 3\left(-\tfrac{7}{3} + 1\right) \qquad \text{Add fractions.}$$
$$-8 + 4 \stackrel{?}{=} -7 + 3 \qquad \text{Simplify.}$$
$$-4 = -4 \qquad \text{Solution checks. ✓}$$

✓CHECKPOINT 3

Solve $2(x + 2) + 6 = 4(2x - 3)$. ■

STUDY TIP

You may think a solution to a problem looks easy when it is worked out in class, but you may not know where to begin when solving the problem on your own. Keep in mind that many problems involve some trial and error before a solution is found.

Some equations in one variable have *infinitely many solutions*. To recognize an equation of this type, perform the regular steps for solving the equation. If, when writing equivalent equations, you reach a statement that is true for all values in the domain of the variable, then the equation is an identity and has infinitely many solutions.

Example 4 An Equation with Infinitely Many Solutions

Solve $x + 4(x - 2) = 3x + 2(x - 4)$.

SOLUTION

$$x + 4(x - 2) = 3x + 2(x - 4)$$ Write original equation.

$$x + 4x - 8 = 3x + 2x - 8$$ Distributive Property

$$5x - 8 = 5x - 8$$ Simplify.

$$-8 = -8$$ Subtract $5x$ from each side.

Because the last equation is true for every real value of x, the original equation is an identity and you can conclude that it has infinitely many solutions.

✓CHECKPOINT 4

Solve $x + 5 + 3(2x + 1) = 7x + 8$. ■

It is also possible for an equation in one variable to have *no solution*. When solving an equation of this type, you will reach a statement that is not true for any value of the variable.

Example 5 An Equation with No Solution

Solve $4x - 9 + 2(x + 8) = 1 + 6(x + 4)$.

SOLUTION

$$4x - 9 + 2(x + 8) = 1 + 6(x + 4)$$ Write original equation.

$$4x - 9 + 2x + 16 = 1 + 6x + 24$$ Distributive Property

$$6x + 7 = 6x + 25$$ Simplify.

$$7 \neq 25$$ Subtract $6x$ from each side.

Because the statement $7 = 25$ is not true, you can conclude that the original equation has no solution.

✓CHECKPOINT 5

Solve $1 + 4(x + 1) = 4(2 + x)$. ■

Equations in one variable with infinitely many solutions or no solution are not linear because they cannot be written in the standard form $ax + b = 0$. Note that a linear equation in x has exactly one solution.

Equations Involving Fractional Expressions

To solve an equation involving fractional expressions, you can multiply every term in the equation by the least common denominator (LCD) of the terms.

Example 6 An Equation Involving Fractional Expressions

$$\frac{x}{3} + \frac{3x}{4} = 2 \qquad \text{Original equation}$$

$$(12)\frac{x}{3} + (12)\frac{3x}{4} = (12)2 \qquad \text{Multiply each term by least common denominator.}$$

$$4x + 9x = 24 \qquad \text{Simplify.}$$

$$13x = 24 \qquad \text{Combine like terms.}$$

$$x = \frac{24}{13} \qquad \text{Divide each side by 13.}$$

The solution is $x = \frac{24}{13}$. Check this in the original equation.

✓CHECKPOINT 6

Solve $\frac{4x}{3} - \frac{x}{12} = 5$. ■

When multiplying or dividing an equation by a *variable expression*, it is possible to introduce an **extraneous** solution—one that does not satisfy the original equation. In such cases a check is especially important.

Example 7 An Equation with an Extraneous Solution

Solve $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$.

SOLUTION The least common denominator is $x^2 - 4 = (x + 2)(x - 2)$. Multiply each term by this LCD and simplify.

$$\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4} \qquad \text{Write original equation.}$$

$$\frac{1}{x-2}(x+2)(x-2) = \frac{3}{x+2}(x+2)(x-2) - \frac{6x}{x^2-4}(x+2)(x-2)$$

$$x + 2 = 3(x - 2) - 6x, \quad x \neq \pm 2 \qquad \text{Simplify.}$$

$$x + 2 = 3x - 6 - 6x \qquad \text{Distributive Property}$$

$$4x = -8 \qquad \text{Combine like terms and simplify.}$$

$$x = -2 \qquad \text{Extraneous solution}$$

By checking $x = -2$, you can see that it yields a denominator of zero for the fraction $3/(x + 2)$. So, $x = -2$ is extraneous, and the equation has *no solution.*

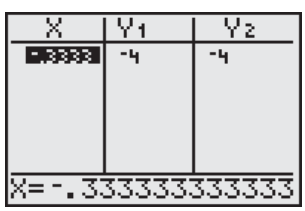
TECHNOLOGY

Use the *table* feature of your graphing utility to check the solution in Example 3. In the equation editor, enter the expression to the left of the equal sign in y_1 and enter the expression to the right of the equal sign in y_2 as follows.

$$y_1 = 6(x - 1) + 4$$

$$y_2 = 3(7x + 1)$$

Set the *table* feature to ASK mode. When you enter the solution $-\frac{1}{3}$ for x, both y_1 and y_2 are -4, as shown.



Similarly, a graphing utility can help you determine if a solution is extraneous. For instance, enter the equation from Example 7 into the graphing utility's equation editor. Then, use the *table* feature in ASK mode to enter -2 for x. You will see that the graphing utility displays ERROR in the y_2 column. So, the solution $x = -2$ is extraneous.

✓CHECKPOINT 7

Solve $\frac{1}{x-4} + \frac{1}{x} = \frac{4}{x(x-4)}$. ■

TECHNOLOGY

When using the *equation editor* of a graphing utility, you must enter equations in terms of x. So, if you wanted to enter an equation like the one shown in Example 8, you would replace y with x as shown.

$$y_1 = (3x - 2)/(2x + 1)$$
$$y_2 = (6x - 9)/(4x + 3)$$

An equation with a *single fraction* on each side can be cleared of denominators by **cross-multiplying,** which is equivalent to multiplying each side of the equation by the least common denominator and then simplifying.

Example 8 Cross–Multiplying to Solve an Equation

Solve $\dfrac{3y - 2}{2y + 1} = \dfrac{6y - 9}{4y + 3}$.

SOLUTION

$$\frac{3y - 2}{2y + 1} = \frac{6y - 9}{4y + 3}$$ Write original equation.

$$(3y - 2)(4y + 3) = (6y - 9)(2y + 1)$$ Cross-multiply.

$$12y^2 + y - 6 = 12y^2 - 12y - 9$$ Multiply.

$$13y = -3$$ Isolate y-term on left.

$$y = -\frac{3}{13}$$ Divide each side by 13.

The solution is $y = -\frac{3}{13}$. Check this in the original equation.

✓CHECKPOINT 8

Solve $\dfrac{3x - 6}{x + 10} = \dfrac{3}{4}$. ■

Example 9 Using a Calculator to Solve an Equation

Solve $\dfrac{1}{9.38} - \dfrac{3}{x} = \dfrac{5}{0.3714}$.

STUDY TIP

Because of roundoff error, a check of a decimal solution may not yield exactly the same values for each side of the original equation. The difference, however, should be quite small.

SOLUTION Roundoff error will be minimized if you solve for x before performing any calculations. The least common denominator is $(9.38)(0.3714)(x)$.

$$\frac{1}{9.38} - \frac{3}{x} = \frac{5}{0.3714}$$

$$(9.38)(0.3714)(x)\left(\frac{1}{9.38} - \frac{3}{x}\right) = (9.38)(0.3714)(x)\left(\frac{5}{0.3714}\right)$$

$$0.3714x - 3(9.38)(0.3714) = 5(9.38)(x), \quad x \neq 0$$

$$[0.3714 - 5(9.38)]x = 3(9.38)(0.3714)$$

$$x = \frac{3(9.38)(0.3714)}{0.3714 - 5(9.38)}$$

$$x \approx -0.225$$ Round to three decimal places.

The solution is $x \approx -0.225$. Check this in the original equation.

✓CHECKPOINT 9

Solve $\dfrac{5}{x} + \dfrac{1}{2.7} = \dfrac{4}{0.6}$. ■

Application

Example 10
MAKE A DECISION **Hourly Earnings** Ⓡ

The mean hourly earnings y (in dollars) of employees at outpatient care centers in the United States from 2000 to 2005 can be modeled by the linear equation

$$y = 0.782t + 15.20, \quad 0 \le t \le 5$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to estimate the year in which the mean hourly earnings were \$16.75. *(Source: U.S. Bureau of Labor Statistics)*

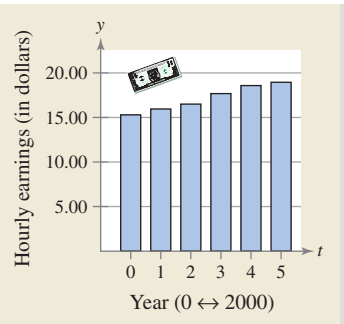


FIGURE 1.1

SOLUTION To determine when the mean hourly earnings were \$16.75, solve the model for t when $y = 16.75$

$$y = 0.782t + 15.20 \qquad \text{Write original model.}$$
$$16.75 = 0.782t + 15.20 \qquad \text{Substitute 16.75 for } y \text{ so you can solve for } t.$$
$$1.55 = 0.782t \qquad \text{Subtract 15.20 from each side.}$$
$$t = \frac{1.55}{0.782} \approx 2 \qquad \text{Divide each side by 0.782.}$$

Because $t = 0$ corresponds to 2000, it follows that $t = 2$ corresponds to 2002. See Figure 1.1. So, mean hourly earnings were \$16.75 in 2002.

✓CHECKPOINT 10

The mean hourly earnings y (in dollars) of the employees at a factory from 2000 to 2008 can be modeled by the linear equation

$$y = 0.825t + 18.60, \quad 0 \le t \le 8$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to estimate the year in which the mean hourly earnings of the employees at the factory were \$21.90. ■

CONCEPT CHECK

1. Is the equation $x(8 - x) = 15$ a linear equation? Explain.
2. Explain the difference between an identity and a conditional equation.
3. Can the equation $ax + b = 0$ have two solutions? Explain.
4. Does the equation $\frac{4x}{x - 3} = 8 + \frac{12}{x - 3}$ have an extraneous solution? Explain.

The symbol Ⓡ indicates an example that uses or is derived from real-life data.

Skills Review 1.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2 and 0.7.

In Exercises 1–10, perform the indicated operations and simplify your answer.

1. $(2x - 4) - (5x + 6)$

2. $(3x - 5) + (2x - 7)$

3. $2(x + 1) - (x + 2)$

4. $-3(2x - 4) + 7(x + 2)$

5. $\frac{x}{3} + \frac{x}{5}$

6. $x - \frac{x}{4}$

7. $\frac{1}{x + 1} - \frac{1}{x}$

8. $\frac{2}{x} + \frac{3}{x}$

9. $\frac{4}{x} + \frac{3}{x - 2}$

10. $\frac{1}{x + 1} - \frac{1}{x - 1}$

Exercises 1.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, determine whether the equation is an identity or a conditional equation.

1. $2(x - 1) = 2x - 2$

2. $3(x + 2) = 3x + 6$

3. $2(x - 1) = 3x + 4$

4. $3(x + 2) = 2x + 4$

5. $2(x + 1) = 2x + 1$

6. $3(x + 4) = 3x + 4$

In Exercises 7–16, determine whether each value of x is a solution of the equation.

	Equation	*Values*	
7.	$5x - 3 = 3x + 5$	(a) $x = 0$	(b) $x = -5$
		(c) $x = 4$	(d) $x = 10$
8.	$7 - 3x = 5x - 17$	(a) $x = -3$	(b) $x = 0$
		(c) $x = 8$	(d) $x = 3$
9.	$3x^2 + 2x - 5 = 2x^2 - 2$	(a) $x = -3$	(b) $x = 1$
		(c) $x = 4$	(d) $x = -5$
10.	$5x^3 + 2x - 3 = 4x^3 + 2x - 11$	(a) $x = 2$	(b) $x = -2$
		(c) $x = 0$	(d) $x = 10$
11.	$\frac{5}{2x} - \frac{4}{x} = 3$	(a) $x = -\frac{1}{2}$	(b) $x = 4$
		(c) $x = 0$	(d) $x = \frac{1}{4}$
12.	$3 + \frac{1}{x + 2} = 4$	(a) $x = -1$	(b) $x = -2$
		(c) $x = 0$	(d) $x = 5$
13.	$(x + 5)(x - 3) = 20$	(a) $x = 3$	(b) $x = -2$
		(c) $x = 0$	(d) $x = -7$
14.	$(3x + 5)(2x - 7) = 0$	(a) $x = -\frac{5}{3}$	(b) $x = -\frac{2}{7}$
		(c) $x = \frac{2}{3}$	(d) $x = \frac{3}{2}$
15.	$\sqrt{2x - 3} = 3$	(a) $x = 6$	(b) $x = -3$
		(c) $x = -\frac{1}{3}$	(d) $x = -2$
16.	$\sqrt[3]{x - 8} = 3$	(a) $x = 2$	(b) $x = -5$
		(c) $x = 35$	(d) $x = 8$

In Exercises 17–54, solve the equation and check your solution. (Some equations have no solution.)

17. $x + 10 = 15$

18. $9 - x = 13$

19. $7 - 2x = 15$

20. $7x + 2 = 16$

21. $8x - 5 = 3x + 10$

22. $7x + 3 = 3x - 13$

23. $2(x + 5) - 7 = 3(x - 2)$

24. $2(13t - 15) + 3(t - 19) = 0$

25. $6[x - (2x + 3)] = 8 - 5x$

26. $3[2x - (x + 7)] = 5(x - 3)$

27. $\frac{5x}{4} + \frac{1}{2} = x - \frac{1}{2}$

28. $\frac{x}{5} - \frac{x}{2} = 3$

29. $\frac{3}{2}(z + 5) - \frac{1}{4}(z + 24) = 0$

30. $\frac{3x}{2} + \frac{1}{4}(x - 2) = 10$

31. $0.25x + 0.75(10 - x) = 3$

32. $0.60x + 0.40(100 - x) = 50$

33. $x + 8 = 2(x - 2) - x$

34. $3(x + 3) = 5(1 - x) - 1$

35. $\dfrac{100 - 4u}{3} = \dfrac{5u + 6}{4} + 6$

36. $\dfrac{17 + y}{y} + \dfrac{32 + y}{y} = 100$

37. $\dfrac{5x - 4}{5x + 4} = \dfrac{2}{3}$

38. $\dfrac{10x + 3}{5x + 6} = \dfrac{1}{2}$

39. $10 - \dfrac{13}{x} = 4 + \dfrac{5}{x}$

40. $\dfrac{15}{x} - 4 = \dfrac{6}{x} + 3$

41. $\dfrac{1}{x - 3} + \dfrac{1}{x + 3} = \dfrac{10}{x^2 - 9}$

42. $\dfrac{1}{x - 2} + \dfrac{3}{x + 3} = \dfrac{4}{x^2 + x - 6}$

43. $\dfrac{6}{(x - 3)(x - 1)} = \dfrac{3}{x - 3} + \dfrac{4}{x - 1}$

44. $\dfrac{2}{(x - 4)(x - 2)} = \dfrac{1}{x - 4} + \dfrac{2}{x - 2}$

45. $\dfrac{7}{2x + 1} - \dfrac{8x}{2x - 1} = -4$

46. $\dfrac{4}{u - 1} + \dfrac{6}{3u + 1} = \dfrac{15}{3u + 1}$

47. $\dfrac{3}{x(x - 3)} + \dfrac{4}{x} = \dfrac{1}{x - 3}$

48. $3 = 2 + \dfrac{2}{z + 2}$

49. $(x + 2)^2 + 5 = (x + 3)^2$

50. $(x + 1)^2 + 2(x - 2) = (x + 1)(x - 2)$

51. $(x + 2)^2 - x^2 = 4(x + 1)$

52. $4(x + 1) - 3x = x + 5$

53. $(2x + 1)^2 = 4(x^2 + x + 1)$

54. $(2x - 1)^2 = 4(x^2 - x + 6)$

55. A student states that the solution to the equation

$$\frac{2}{x(x - 2)} + \frac{5}{x} = \frac{1}{x - 2}$$

is $x = 2$. Describe and correct the student's error.

56. A student states that the equation

$$-3(x + 2) = -3x + 6$$

is an identity. Describe and correct the student's error.

57. Explain why a solution of an equation involving fractional expressions may be extraneous.

58. Describe two methods you can use to check a solution of an equation involving fractional expressions.

59. What is meant by "equivalent equations"? Give an example of two equivalent equations.

60. For what value(s) of b does the equation

$$7x + 3 = 7x + b$$

have infinitely many solutions? no solution?

In Exercises 61–66, use a calculator to solve the equation. (Round your solution to three decimal places.)

61. $0.275x + 0.725(500 - x) = 300$

62. $2.763 - 4.5(2.1x - 5.1432) = 6.32x + 5$

63. $\dfrac{x}{0.6321} + \dfrac{x}{0.0692} = 1000$

64. $(x + 5.62)^2 + 10.83 = (x + 7)^2$

65. $\dfrac{2}{7.398} - \dfrac{4.405}{x} = \dfrac{1}{x}$

66. $\dfrac{x}{2.625} + \dfrac{x}{4.875} = 1$

67. What method or methods would you recommend for checking the solutions to Exercises 61–66 using your graphing utility?

68. In Exercises 61–66, your answers are rounded to three decimal places. What effect does rounding have as you check a solution?

In Exercises 69–72, evaluate the expression in two ways. (a) Calculate entirely on your calculator using appropriate parentheses, and then round the answer to two decimal places. (b) Round both the numerator and the denominator to two decimal places before dividing, and then round the final answer to two decimal places. Does the second method introduce an additional roundoff error?

69. $\dfrac{1 + 0.73205}{1 - 0.73205}$

70. $\dfrac{1 + 0.86603}{1 - 0.86603}$

71. $\dfrac{333 + \dfrac{1.98}{0.74}}{4 + \dfrac{6.25}{3.15}}$

72. $\dfrac{1.73205 - 1.19195}{3 - (1.73205)(1.19195)}$

The symbol T indicates when to use graphing technology or a symbolic computer algebra system to solve a problem or an exercise. The solutions of other exercises may also be facilitated by use of appropriate technology.

73. Personal Income The per capita personal income in the United States from 1998 to 2005 can be approximated by the linear equation

$y = 944.7t + 19{,}898, \quad 8 \le t \le 15$

where t represents the year, with $t = 8$ corresponding to 1998. Use the model to estimate the year in which the per capita personal income was \$32,000. *(Source: U.S. Department of Commerce, Bureau of Economic Analysis)*

74. Annual Sales The annual sales S (in billions of dollars) of Microsoft Corporation from 1996 to 2006 can be approximated by the linear equation

$S = 3.54t - 13.1, \quad 6 \le t \le 16$

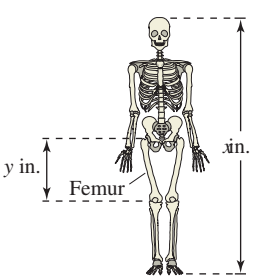
where t represents the year, with $t = 6$ corresponding to 1996. Use the model to estimate the year in which Microsoft's annual sales were about \$20,000,000,000. *(Source: Microsoft Corporation)*

Human Height In Exercises 75 and 76, use the following information. The relationship between the length of an adult's femur (thigh bone) and the height of the adult can be approximated by the linear equations

$y = 0.432x - 10.44$ Female

$y = 0.449x - 12.15$ Male

where y is the length of the femur in inches and x is the height of the adult in inches (see figure).



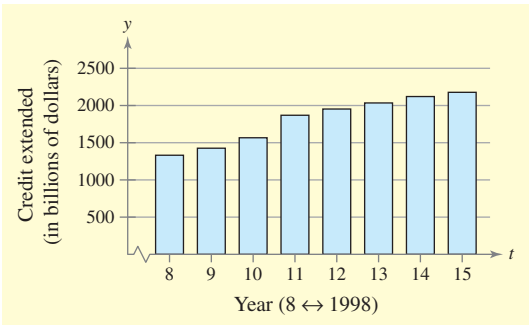
75. An anthropologist discovers a femur belonging to an adult human female. The bone is 15 inches long. Estimate the height of the female.

76. ***MAKE A DECISION*** From the foot bones of an adult human male, an anthropologist estimates that the male was 65 inches tall. A few feet away from the site where the foot bones were discovered, the anthropologist discovers an adult male femur that is 17 inches long. Is it possible that the leg and foot bones came from the same person? Explain.

Consumer Credit In Exercises 77 and 78, use the following information. From 1998 to 2005, the annual credit y (in billions of dollars) extended to consumers in the United States (other than real estate loans) can be approximated by the equation

$y = 129.51t + 320.5, \quad 8 \le t \le 15$

where t is the year, with $t = 8$ corresponding to 1998. *(Source: Federal Reserve Board)*



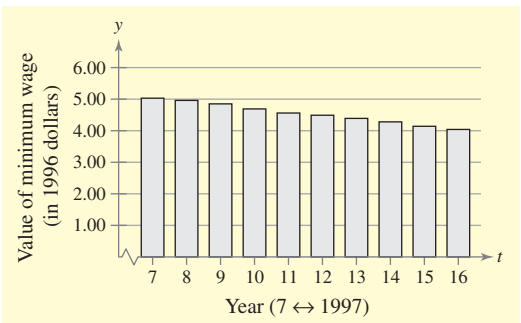
77. In which year was the credit extended to consumers about \$2 trillion?

78. Use the model to predict the year in which the credit extended to consumers will be about \$2.9 trillion.

Minimum Wage In Exercises 79 and 80, use the following information. From 1997 to 2006, the federal minimum wage was \$5.15 per hour. Adjusting for inflation, the federal minimum wage's *value in 1996 dollars* during these years can be approximated by the linear equation

$y = -0.112t + 5.83, \quad 7 \le t \le 16$

where t is the year, with $t = 7$ corresponding to 1997. *(Source: U.S. Department of Labor)*



79. In which year was the value of the federal minimum wage about \$4.60 in 1996 dollars?

80. According to the model, did the value of the federal minimum wage in 1996 dollars fall below \$4.00 by 2007? Explain.

Section 1.2

Mathematical Modeling

- Construct a mathematical model from a verbal model.
- Model and solve percent and mixture problems.
- Use common formulas to solve geometry and simple interest problems.
- Develop a general problem-solving strategy.

Introduction to Problem Solving

In this section, you will use algebra to solve real-life problems. To do this, you will construct one or more equations that represent each real-life problem. This procedure is called **mathematical modeling.**

A good approach to mathematical modeling is to use two stages. First, use the verbal description of the problem to form a *verbal model*. Then, assign labels to each of the quantities in the verbal model and use the labels to form a *mathematical model* or an *algebraic equation*.

Verbal description → Verbal model → Algebraic equation

When you are trying to construct a verbal model, it is sometimes helpful to look for a *hidden equality*. For instance, in the following example the hidden equality equates your annual income to 24 pay periods and one bonus check.

Example 1 Using a Verbal Model

You accept a job with an annual income of $36,500. This includes your salary and a $500 year-end bonus. You are paid twice a month. What is your salary per pay period?

SOLUTION Because there are 12 months in a year and you are paid twice a month, it follows that there are 24 pay periods during the year.

Verbal Model: Income for year = 24 pay periods + Bonus

Labels:
Income for year = 36,500 (dollars)
Salary for each pay period = x (dollars)
Bonus = 500 (dollars)

Equation: $36{,}500 = 24x + 500$

Using the techniques discussed in Section 1.1, you can find that the solution is $x = \$1500$. Check whether a salary of $1500 per pay period is reasonable for the situation.

✓CHECKPOINT 1

In Example 1, suppose you are paid weekly. What is your salary per pay period? ■

Translating Key Words and Phrases

Key Words and Phrases	*Verbal Description*	*Algebraic Statement*
Consecutive		
Next, subsequent	Consecutive integers	$n, n + 1$
Addition		
Sum, plus, greater, increased by, more than, exceeds, total of	The sum of 5 and x Seven more than y	$5 + x$ $y + 7$
Subtraction		
Difference, minus, less than, decreased by, subtracted from, reduced by, the remainder	Four decreased by b Three less than z Five subtracted from w	$4 - b$ $z - 3$ $w - 5$
Multiplication		
Product, multiplied by, twice, times, percent of	Two times x	$2x$
Division		
Quotient, divided by, per	The quotient of x and 8	$\frac{x}{8}$

STUDY TIP

In Example 2, notice that part of the labeling process is to list the unit of measure for each labeled quantity. Developing this habit helps in checking the validity of a verbal model.

Example 2 Constructing Mathematical Models

a. A salary of \$28,000 is increased by 9%. Write an equation that represents the new salary.

Verbal Model: New salary = 9%(original salary) + Original salary

Labels:
Original salary = 28,000 (dollars)
New salary = S (dollars)
Percent = 0.09 (percent in decimal form)

Equation: $S = 0.09(28{,}000) + 28{,}000$

b. A laptop computer is marked down 20% to \$1760. Write an equation you can use to find the original price.

Verbal Model: Original price − 20%(original price) = Sale price

Labels:
Original price = p (dollars)
Sale price = 1760 (dollars)
Percent = 0.2 (percent in decimal form)

Equation: $p - 0.2p = 1760$

✓CHECKPOINT 2

A salary of \$40,000 is increased by 5%. Write an equation that you can use to find the new salary. ■

Using Mathematical Models

Study the next several examples carefully. Your goal should be to develop a *general problem-solving strategy.*

Example 3 Finding the Percent of a Raise

You accept a job that pays \$8 an hour. You are told that after a two-month probationary period, your hourly wage will be increased to \$9 an hour. What percent raise will you receive after the two-month period?

SOLUTION

Verbal Model: Raise = Percent · Old wage

Labels:
Old wage = 8 (dollars)
Raise = 1 (dollar)
Percent = r (percent in decimal form)

Equation: $1 = r \cdot 8$

By solving this equation, you can find that you will receive a raise of $\frac{1}{8} = 0.125$, or 12.5%.

✓CHECKPOINT 3

You buy stock at \$25 per share. You sell the stock at \$30 per share. What is the percent increase of the stock's value? ■

Example 4 Finding the Percent of a Salary

Your annual salary is \$35,000. In addition to your salary, your employer also provides the following benefits. The total of this benefits package is equal to what percent of your annual salary?

Social Security (employer's portion):	6.2% of salary	\$2170
Worker's compensation:	0.5% of salary	\$175
Unemployment compensation:	0.75% of salary	\$262.50
Medical insurance:	\$2600 per year	\$2600
Retirement contribution:	5% of salary	\$1750

SOLUTION

Verbal Model: Benefits package = Percent · Salary

Labels:
Salary = 35,000 (dollars)
Benefits package = 6957.50 (dollars)
Percent = r (percent in decimal form)

Equation: $6957.50 = r \cdot 35{,}000$

By solving this equation, you can find that your benefits package is equal to $r = 6957.50/35{,}000$, or about 19.9% of your salary.



Charles Gupton/Getty Images

In 2005, 15.3% of the population of the United States had no health insurance. *(Source: Centers for Disease Control and Prevention, National Health Interview Survey)*

✓CHECKPOINT 4

Your income last year was \$42,000. Throughout that year you paid a total of \$648 for parking fees. The total of the parking fees was equal to what percent of your income? ■

Example 5 Finding the Dimensions of a Room

A rectangular family room is twice as long as it is wide, and its perimeter is 84 feet. Find the dimensions of the family room.

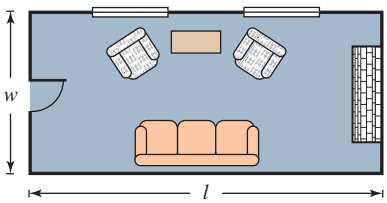


FIGURE 1.2

SOLUTION For this problem, it helps to sketch a diagram, as shown in Figure 1.2.

Verbal Model: $2 \cdot \text{Length} + 2 \cdot \text{Width} = \text{Perimeter}$

Labels:
Perimeter $= 84$ (feet)
Width $= w$ (feet)
Length $= l = 2w$ (feet)

Equation:
$$2(2w) + 2w = 84$$
$$4w + 2w = 84$$
$$6w = 84$$
$$w = 14 \text{ feet}$$
$$l = 2w = 28 \text{ feet}$$

The dimensions of the room are 14 feet by 28 feet.

✓CHECKPOINT 5

A rectangular driveway is three times as long as it is wide, and its perimeter is 120 feet. Find the dimensions of the driveway. ■

Example 6 MAKE A DECISION A Distance Problem

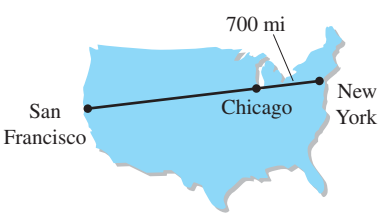


FIGURE 1.3

A plane travels nonstop from New York to San Francisco, a distance of 2600 miles. It takes 1.5 hours to fly from New York to Chicago, a distance of about 700 miles (see Figure 1.3). Assuming the plane flies at a constant speed, how long does the entire trip take? What time (EST) should the plane leave New York to arrive in San Francisco by 5 P.M. PST (8 P.M. EST)?

SOLUTION To solve this problem, use the formula that relates distance, rate, and time. That is, (distance) = (rate)(time). Because it took the plane 1.5 hours to travel a distance of about 700 miles, you can conclude that its rate (or speed) is

$$\text{Rate} = \frac{\text{distance}}{\text{time}} = \frac{700 \text{ miles}}{1.5 \text{ hours}} \approx 466.67 \text{ miles per hour.}$$

Because the entire trip is about 2600 miles, the time for the entire trip is

$$\text{Time} = \frac{\text{distance}}{\text{rate}} = \frac{2600 \text{ miles}}{466.67 \text{ miles per hour}} \approx 5.57 \text{ hours.}$$

Because 0.57 hour represents about 34 minutes, you can conclude that the trip takes about 5 hours and 34 minutes. The plane must leave New York by 2:26 P.M. in order to arrive in San Francisco by 8 P.M. EST.

✓CHECKPOINT 6

A small boat travels at full speed to an island 11 miles away. It takes 0.3 hour to travel the first 3 miles. How long does the entire trip take? ■

Another way to solve the distance problem in Example 6 is to use the concept of **ratio and proportion.** To do this, let x represent the time required to fly from New York to San Francisco, set up the following proportion, and solve for x.

$$\frac{\text{Time to San Francisco}}{\text{Time to Chicago}} = \frac{\text{Distance to San Francisco}}{\text{Distance to Chicago}}$$

$$\frac{x}{1.5} = \frac{2600}{700}$$

$$x = 1.5 \cdot \frac{2600}{700}$$

$$x \approx 5.57$$

Notice how ratio and proportion are used with a property from geometry to solve the problem in the following example.

Example 7 An Application Involving Similar Triangles

To determine the height of Petronas Tower 1 (in Kuala Lumpur, Malaysia), you measure the shadow cast by the building to be 113 meters long, as shown in Figure 1.4. Then you measure the shadow cast by a 100-centimeter post and find that its shadow is 25 centimeters long. Use this information to determine the height of Petronas Tower 1.

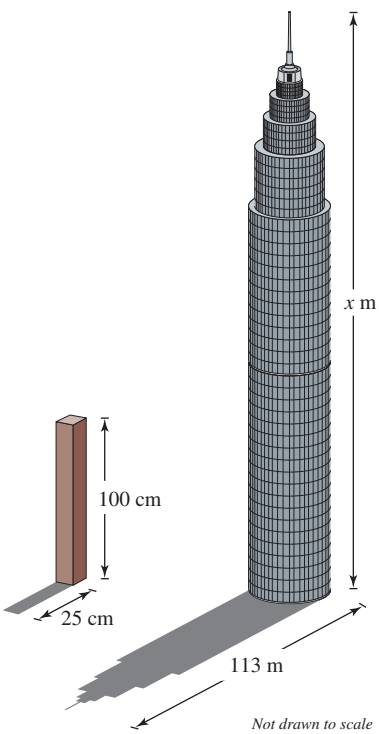


FIGURE 1.4

SOLUTION To find the height of the tower, you can use a property from geometry that states that the ratios of corresponding sides of similar triangles are equal.

Verbal Model:
$$\frac{\text{Height of tower}}{\text{Length of tower's shadow}} = \frac{\text{Height of post}}{\text{Length of post's shadow}}$$

Labels:
Height of tower $= x$ (meters)
Length of tower's shadow $= 113$ (meters)
Height of post $= 100$ (centimeters)
Length of post's shadow $= 25$ (centimeters)

Equation:
$$\frac{x}{113} = \frac{100}{25}$$

$$x = 113 \cdot \frac{100}{25}$$

$$x = 113 \cdot 4$$

$$x = 452 \text{ meters}$$

The Petronas Tower 1 is 452 meters high.

✓CHECKPOINT 7

A tree casts a shadow that is 24 feet long. At the same time, a four-foot tall mailbox casts a shadow that is 3 feet long. How tall is the tree? ■

Mixture Problems

The next example is called a **mixture problem** because it involves two different unknown quantities that are *mixed* in a specific way. Watch for a *hidden product* in the verbal model.

TECHNOLOGY

You can write a program for a programmable calculator to solve simple interest problems. Sample programs for various calculators may be found at the website for this text at *college.hmco.com/info/larsonapplied*. Use a program with Example 8 to find how much interest was earned on just the portion of the money invested at $5\frac{1}{2}\%$.

Example 8 A Simple Interest Problem

You invested a total of \$10,000 in accounts that earned $4\frac{1}{2}\%$ and $5\frac{1}{2}\%$ simple interest. In 1 year, the two accounts earned \$508.75 in interest. How much did you invest in each account?

SOLUTION The formula for simple interest is $I = Prt$, where I is the interest, P is the principal, r is the annual interest rate (in decimal form), and t is the time in years.

Verbal Model: Interest from $4\frac{1}{2}\%$ + Interest from $5\frac{1}{2}\%$ = Total interest

You can let x represent the amount invested at $4\frac{1}{2}\%$. Because the total amount invested at $4\frac{1}{2}\%$ and $5\frac{1}{2}\%$ is \$10,000, you can let $10{,}000 - x$ represent the amount invested at $5\frac{1}{2}\%$.

Labels:

Amount invested at $4\frac{1}{2}\% = x$ (dollars)

Amount invested at $5\frac{1}{2}\% = 10{,}000 - x$ (dollars)

Interest from $4\frac{1}{2}\% = Prt = (x)(0.045)(1)$ (dollars)

Interest from $5\frac{1}{2}\% = Prt = (10{,}000 - x)(0.055)(1)$ (dollars)

Total interest $= 508.75$ (dollars)

Equation:

$$0.045x + 0.055(10{,}000 - x) = 508.75$$
$$0.045x + 550 - 0.055x = 508.75$$
$$-0.01x = -41.25$$
$$x = \$4125$$

So, the amount invested at $4\frac{1}{2}\%$ is \$4125 and the amount invested at $5\frac{1}{2}\%$ is

$$10{,}000 - x = 10{,}000 - 4125 = \$5875.$$

Check these results in the original statement of the problem, as follows.

CHECK

Interest from $4\frac{1}{2}\%$ + Interest from $5\frac{1}{2}\%$ = Total interest

$$0.045(4125) + 0.055(10{,}000 - 4125) \stackrel{?}{=} 508.75$$
$$185.625 + 323.125 \stackrel{?}{=} 508.75$$
$$508.75 = 508.75$$

Solution checks. ✓

✓CHECKPOINT 8

You invested a total of \$1000 in accounts that earned 4% and 5% simple interest. In 1 year you earned a total of \$48 in interest. How much did you invest in each account? ■

In Example 8, did you recognize the hidden products in the two terms on the left side of the equation? Both hidden products come from the common formula

$$\text{Interest} = \text{Principal} \cdot \text{Rate} \cdot \text{Time}$$
$$I = Prt.$$

Common Formulas

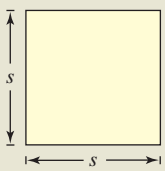
Many common types of geometric, scientific, and investment problems use ready-made equations, called **formulas.** Knowing formulas such as those in the following lists will help you translate and solve a wide variety of real-life problems involving perimeter, area, volume, temperature, interest, and distance.

Common Formulas for Area, Perimeter, and Volume

Square

$A = s^2$

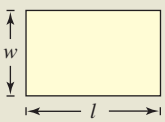
$P = 4s$



Rectangle

$A = lw$

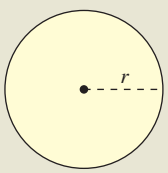
$P = 2l + 2w$



Circle

$A = \pi r^2$

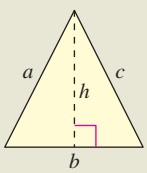
$C = 2\pi r$



Triangle

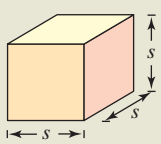
$A = \frac{1}{2}bh$

$P = a + b + c$



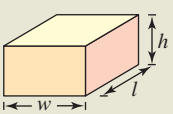
Cube

$V = s^3$



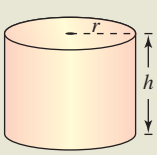
Rectangular Solid

$V = lwh$



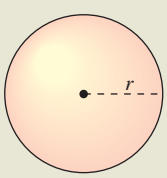
Circular Cylinder

$V = \pi r^2 h$



Sphere

$V = \frac{4}{3}\pi r^3$



Miscellaneous Common Formulas

Temperature: F = degrees Fahrenheit, C = degrees Celsius

$$F = \frac{9}{5}C + 32$$

Simple interest: I = interest, P = principal, r = interest rate, t = time

$$I = Prt$$

Distance: d = distance traveled, r = rate, t = time

$$d = rt$$

When working with applied problems, you often need to rewrite common formulas. For instance, the formula

$$P = 2l + 2w$$

for the perimeter of a rectangle can be rewritten or solved for w to produce

$$w = \tfrac{1}{2}(P - 2l).$$

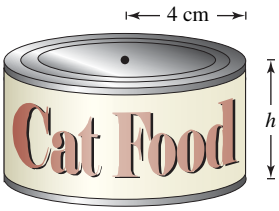


FIGURE 1.5

✓CHECKPOINT 9

One cubic foot of water fills a cylindrical pipe with a radius of 0.5 foot. What is the height of the pipe? ■

Example 9 **Using a Formula**

A cylindrical can has a volume of 200 cubic centimeters and a radius of 4 centimeters, as shown in Figure 1.5. Find the height of the can.

SOLUTION The formula for the *volume of a cylinder* is $V = \pi r^2 h$. To find the height of the can, solve for h.

$$h = \frac{V}{\pi r^2}$$

Then, using $V = 200$ and $r = 4$, find the height.

$$h = \frac{200}{\pi(4)^2}$$ Substitute 200 for V and 4 for r.

$$= \frac{200}{16\pi}$$ Simplify denominator.

$$\approx 3.98$$ Use a calculator.

So, the height of the can is about 3.98 centimeters. You can use unit analysis to check that your answer is reasonable.

$$\frac{200 \text{ cm}^3}{16\pi \text{ cm}^2} \approx 3.98 \text{ cm}$$

Strategy for Solving Word Problems

1. *Search* for the hidden equality—two expressions said to be equal or known to be equal. A sketch may be helpful.
2. *Write* a verbal model that equates these two expressions. Identify any *hidden* products.
3. *Assign* numbers to the known quantities and letters (or algebraic expressions) to the unknown quantities.
4. *Rewrite* the verbal model as an algebraic equation using the assigned labels.
5. *Solve* the resulting algebraic equation.
6. *Check* to see that the answer satisfies the word problem as stated. (Remember that "solving for x" or some other variable may not completely answer the question.)

CONCEPT CHECK

1. Write a verbal model for the volume of a rectangular solid.
2. Describe and correct the error in the statement.
 The product of 10 and 5 less than x is $10(5 - x)$.
3. Two spherical balloons, each with radius r, are filled with air. Write an algebraic equation that represents the total volume of air in the balloons.
4. Using the formula for the volume of a rectangular solid, what information do you need to find the length of a block of ice?

Skills Review 1.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 1.1.

In Exercises 1–10, solve the equation (if possible) and check your answer.

1. $3x - 42 = 0$

2. $64 - 16x = 0$

3. $2 - 3x = 14 + x$

4. $7 + 5x = 7x - 1$

5. $5[1 + 2(x + 3)] = 6 - 3(x - 1)$

6. $2 - 5(x - 1) = 2[x + 10(x - 1)]$

7. $\frac{x}{3} + \frac{x}{2} = \frac{1}{3}$

8. $\frac{2}{x} + \frac{2}{5} = 1$

9. $1 - \frac{2}{z} = \frac{z}{z + 3}$

10. $\frac{x}{x + 1} - \frac{1}{2} = \frac{4}{3}$

Exercises 1.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Creating a Mathematical Model In Exercises 1–10, write an algebraic expression for the verbal expression.

1. The sum of two consecutive natural numbers

2. The product of two natural numbers whose sum is 25

3. Distance Traveled The distance traveled in t hours by a car traveling at 50 miles per hour

4. Travel Time The travel time for a plane that is traveling at a rate of r miles per hour for 200 miles

5. Acid Solution The amount of acid in x gallons of a 20% acid solution

6. Discount The sale price of an item that is discounted by 20% of its list price L

7. Geometry The perimeter of a rectangle whose width is x and whose length is twice the width

8. Geometry The area of a triangle whose base is 20 inches and whose height is h inches

9. Total Cost The total cost to buy x units at \$25 per unit with a total shipping fee of \$1200

10. Total Revenue The total revenue obtained by selling x units at \$3.59 per unit

In Exercises 11–16, write an equation that represents the statement.

11. The sum of 5 and x equals 8.

12. The difference of n and 7 is 4.

13. The quotient of r and 2 is 9.

14. The product of x and 6 equals -9.

15. The sum of a number n and twice the number is 15.

16. The product of 3 less than x and 8 is 40.

Using a Mathematical Model In Exercises 17–22, write a mathematical model for the number problem, and solve the problem.

17. Find two consecutive numbers whose sum is 525.

18. Find three consecutive natural numbers whose sum is 804.

19. One positive number is five times another positive number. The difference between the two numbers is 148. Find the numbers.

20. One positive number is one-fifth of another number. The difference between the two numbers is 76. Find the numbers.

21. Find two consecutive integers whose product is five less than the square of the smaller number.

22. Find two consecutive natural numbers such that the difference of their reciprocals is one-fourth the reciprocal of the smaller number.

23. Weekly Paycheck Your weekly paycheck is 12% *more* than your coworker's. Your two paychecks total \$848. Find the amount of each paycheck.

24. Weekly Paycheck Your weekly paycheck is 12% *less* than your coworker's. Your two paychecks total \$848. Find the amount of each paycheck.

25. Monthly Profit The profit for a company in February was 5% *higher* than it was in January. The total profit for the two months was \$129,000. Find the profit for each month.

26. Monthly Profit The profit for a company in February was 5% *lower* than it was in January. The total profit for the two months was \$129,000. Find the profit for each month.

Movie Sequels In Exercises 27–32, use the following information. The movie industry frequently releases sequels and/or prequels to successful movies. The revenue of each *Star Wars* movie is shown. Compare the revenue of the two given *Star Wars* movies by finding the percent increase or decrease in the domestic gross. *(Source: Infoplease.com)*

Movie	Domestic gross (in dollars)
Star Wars (1977)	$460,998,007
The Empire Strikes Back (1980)	$290,271,960
Return of the Jedi (1983)	$309,209,079
Episode I: The Phantom Menace (1999)	$431,088,295
Episode II: Attack of the Clones (2002)	$310,675,583
Episode III: Revenge of the Sith (2005)	$380,262,555

27. *Star Wars* (1977) to *The Empire Strikes Back* (1980)

28. *The Empire Strikes Back* (1980) to *Return of the Jedi* (1983)

29. *Return of the Jedi* (1983) to *Episode I: The Phantom Menace* (1999)

30. *Episode I: The Phantom Menace* (1999) to *Episode II: Attack of the Clones* (2002)

31. *Episode II: Attack of the Clones* (2002) to *Episode III: Revenge of the Sith* (2005)

32. *Star Wars* (1977) to *Episode III: Revenge of the Sith* (2005)

Size Inflation In Exercises 33–36, use the following information. Restaurants tend to serve food in larger portions now than they have in the past. Several examples are shown in the table. Find the percent increase in size from the past to 2006 for the indicated food item. *(Source: The Portion Teller, McDonald's, Little Caesars, and Pizza Hut)*

Food or drink item	Past size	2006 size
Small soft drink (McDonald's)	7 fl oz	16 fl oz
Small French fries (McDonald's)	2.4 oz	2.6 oz
Large French fries (McDonald's)	3.5 oz	6 oz
Pizza (Little Caesars, Pizza Hut)	10 in.	12 in.

33. McDonald's small soft drink

34. McDonald's small French fries

35. McDonald's large French fries

36. Little Caesars or Pizza Hut standard pizza

37. Comparing Calories A lunch consisting of a Big Mac, large fries, and large soft drink at McDonald's contains 1440 calories. A lunch consisting of a small hamburger, small fries, and a small soft drink at McDonald's contains 660 calories. Find the percent change in calories from the larger to the smaller lunch. *(Source: McDonald's Corporation)*

38. Comparing Calories One slice (or one-tenth) of a 14-inch Little Caesars pizza with bacon, pepperoni, Italian sausage, and extra cheese has 315 calories. The same slice without the extra toppings has 200 calories. Find the percent change in calories from a slice with the extra toppings to a slice without them. *(Source: Little Caesars)*

39. Salary You accept a new job with a starting salary of $35,000. You receive an 8% raise at the start of your second year, a 7.8% raise at the start of your third year, and a 9.4% raise at the start of your fourth year.

(a) Find your salary for the second year.

(b) Find your salary for the third year.

(c) Find your salary for the fourth year.

40. Salary You accept a new job with a starting salary of $48,000. You receive a 4% raise at the start of your second year, a 5.5% raise at the start of your third year, and an 11.4% raise at the start of your fourth year.

(a) Find your salary for the second year.

(b) Find your salary for the third year.

(c) Find your salary for the fourth year.

41. World Internet Users The number of Internet users in the world reached 500 million in 2001. By the end of 2003, the number increased 43.8%. By the end of 2004, the number increased 13.6% from 2003. By the end of 2006 the number increased 33.8% from 2004. *(Source: Internet World Stats)*

(a) Find the number of users at the end of 2003.

(b) Find the number of users at the end of 2004.

(c) Find the number of users at the end of 2006.

(d) Find the percent increase in the number of users from 2001 to 2006.

42. Sporting Goods Sales In 2002, the total sales of sporting goods in the United States was $77,726,000,000. In 2003, the total sales increased 2.6% from 2002. In 2004, the total sales increased 6.1% from 2003. In 2005, the total sales increased 2.5% from 2004. *(Source: National Sporting Goods Association)*

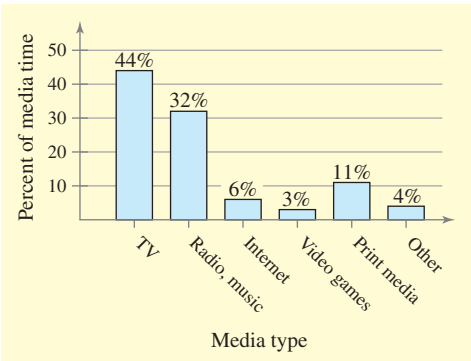
(a) Find the total sporting goods sales in 2003.

(b) Find the total sporting goods sales in 2004.

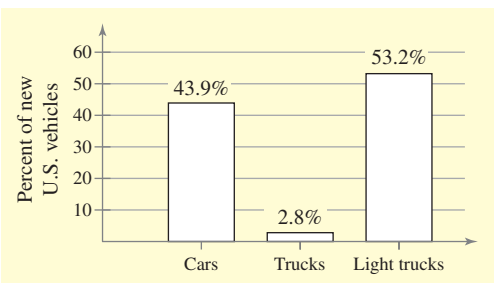
(c) Find the total sporting goods sales in 2005.

(d) Find the percent increase in total sales from 2002 to 2005.

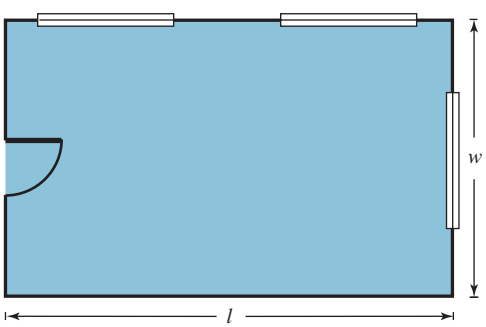
43. Media Usage It was projected that by 2009, the average person would spend 3555 hours per year using some type of media. Use the bar graph to determine the number of hours the average person will spend watching television, listening to the radio or recorded music, using the Internet, playing non-Internet video games, reading print media, and using other types of media in 2009. *(Source: Veronis Schuler Stevenson)*



44. New Vehicle Sales In 2005, the number of motor vehicles sold in the U.S. was about 17,445,000. Use the bar graph to determine how many cars, trucks, and light trucks were sold in 2005. *(Source: U.S. Bureau of Economic Analysis)*



45. Geometry A room is 1.5 times as long as it is wide, and its perimeter is 75 feet (see figure). Find the dimensions of the room.



46. Geometry A picture frame has a total perimeter of 3 feet (see figure). The width of the frame is 0.62 times its length. Find the dimensions of the frame.



47. Simple Interest You invest $2500 at 7% simple interest. How many years will it take for the investment to earn $1000 in interest?

48. Simple Interest An investment earns $3200 interest over a seven-year period. What is the rate of simple interest on a $4800 principal investment?

49. Course Grade To get an A in a course, you need an average of 90% or better on four tests that are worth 100 points each. Your scores on the first three tests were 87, 92, and 84. What must you score on the fourth test to get an A for the course?

50. Course Grade To get an A in a course, you need an average of 90% or better on four tests. The first three tests are worth 100 points each and the fourth is worth 200 points. Your scores on the first three tests are 87, 92, and 84. What must you score on the fourth test to get an A for the course?

51. List Price The price of a swimming pool has been discounted 15%. The sale price is $1200. Find the original list price of the swimming pool.

52. List Price The price of a home theater system has been discounted 10%. The sale price is $499. Find the original price of the system.

53. Discount Rate A satellite radio system for your car has been discounted by $30. The sale price is $119. What percent of the original list price is the discount?

54. Discount Rate The price of a shirt has been discounted by $20. The sale price is $29.95. What percent of the original list price is the discount?

55. Wholesale Price A store marks up a power drill 60% from its wholesale price. In a clearance sale, the price is discounted by 25%. The sale price is $21.60. What was the wholesale price of the power drill?

56. Wholesale Price A store marks up a picture frame 80% from its wholesale price. In a clearance sale, the price is discounted by 40%. The sale price is $28.08. What was the wholesale price of the picture frame?

Weekly Salary In Exercises 57 and 58, use the following information to write a mathematical model and solve. Due to economic factors, your employer has reduced your weekly wage by 15%. Before the reduction, your weekly salary was $425.

57. What is your reduced salary?

58. What percent raise must you receive to bring your weekly salary back up to $425? Explain why the percent raise is different from the percent reduction.

59. Travel Time You are driving to a college 150 miles from home. It takes 28 minutes to travel the first 30 miles. At this rate, how long is your entire trip?

60. Travel Time Two friends fly from Denver to Orlando (a distance of 1526 miles). It takes 1 hour and 15 minutes to fly the first 500 miles. At this rate, how long is the entire flight?

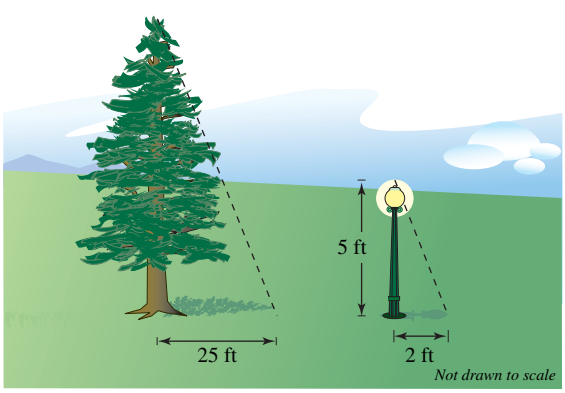
61. Travel Time Two cars start at the same time at a given point and travel in the same direction at constant speeds of 40 miles per hour and 55 miles per hour. After how long are the cars 5 miles apart?

62. Catch-Up Time Students are traveling in two cars to a football game 135 miles away. One car travels at an average speed of 45 miles per hour. The second car starts $\frac{1}{2}$ hour later and travels at an average speed of 55 miles per hour. How long will it take the second car to catch up to the first car?

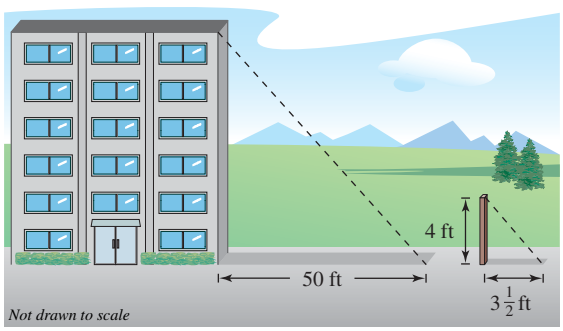
63. Radio Waves Radio waves travel at the same speed as light, 3.0×10^8 meters per second. Find the time required for a radio wave to travel from mission control in Houston to NASA astronauts on the surface of the moon 3.84×10^8 meters away.

64. Distance to a Star Find the distance (in miles) to a star that is 50 light years (distance traveled by light in 1 year) away. (Light travels at 186,000 miles per second.)

65. Height of a Tree To determine the height of a tree, you measure its shadow and the shadow of a five-foot lamppost, as shown in the figure. How tall is the tree?



66. Height of a Building To determine the height of a building, you measure the building's shadow and the shadow of a four-foot stake, as shown in the figure. How tall is the building?



67. Projected Expenses From January through May, a company's expenses totaled $325,450. If the monthly expenses continue at this rate, what will be the total expenses for the year?

68. Projected Revenue From January through August, a company's revenues totaled $549,680. If the monthly revenue continues at this rate, what will be the total revenue for the year?

69. Investment Mix You invest $15,000 in two funds paying 6.5% and 7.5% simple interest. The total annual interest is $1020. How much do you invest in each fund?

70. Investment Mix You invest $30,000 in two funds paying 3% and $4\frac{1}{2}$% simple interest. The total annual interest is $1230. How much do you invest in each fund?

71. Stock Mix You invest $5000 in two stocks. In one year, the value of stock A increases by 9.8% and the value of stock B increases by 6.2%. The total value of the stocks is now $5389.20. How much did you originally invest in each stock?

72. Stock Mix You invest $4000 in two stocks. In one year, the value of stock A increases by 5.4% and the value of stock B increases by 12.8%. The total value of the stocks is now $4401. How much did you originally invest in each stock?

73. Comparing Investment Returns You invest $12,000 in a fund paying $9\frac{1}{2}$% simple interest and $8000 in a fund for which the interest rate varies. At the end of the year the total interest for both funds is $2054.40. What simple interest rate yields the same interest amount as the variable rate fund?

74. Comparing Investment Returns You have $10,000 in an account earning simple interest that is linked to the prime rate. The prime rate drops for the last quarter of the year, so your rate drops by $1\frac{1}{2}$% for the same period. Your total annual interest is $1112.50. What is your interest rate for the first three quarters and for the last quarter?

Production Limit In Exercises 75 and 76, use the following information. *Variable costs* depend on the number of units produced. *Fixed costs* are the same regardless of how many units are produced. Find the greatest number of units the company can produce each month.

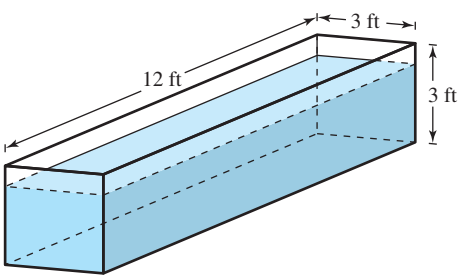
75. The company has fixed monthly costs of \$15,000 and variable monthly costs of \$8.75 per unit. The company has \$90,000 available each month to cover costs.

76. The company has fixed monthly costs of \$10,000 and variable monthly costs of \$9.30 per unit. The company has \$85,000 available each month to cover costs.

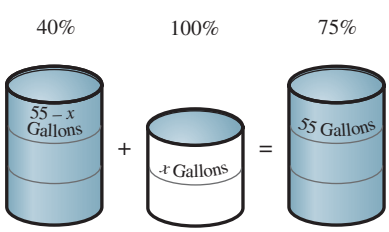
77. Length of a Tank The diameter of a cylindrical propane gas tank is 4 feet (see figure). The total volume of the tank is 603.2 cubic feet. Find the length of the tank.



78. Water Depth A trough is 12 feet long, 3 feet deep, and 3 feet wide (see figure). Find the depth of the water when the trough contains 70 gallons of water. (1 gallon ≈ 0.13368 cubic foot.)



79. Mixture A 55-gallon barrel contains a mixture with a concentration of 40%. How much of this mixture must be withdrawn and replaced by 100% concentrate to bring the mixture up to 75% concentration? (See figure.)



80. Mixture A farmer mixes gasoline and oil to make 2 gallons of mixture for his two-cycle chain saw engine. This mixture is 32 parts gasoline and 1 part two-cycle oil. How much gasoline must be added to bring the mixture to 40 parts gasoline and 1 part oil?

New York City Marathon In Exercises 81 and 82, the length of the New York City Marathon course is 26 miles, 385 yards. Find the average speed of the record holding runner. (Note that 1 mile = 5280 feet = 1760 yards.)

81. Men's record time: 2 hours, $7\frac{3}{4}$ minutes

82. Women's record time: 2 hours, $22\frac{1}{2}$ minutes

In Exercises 83–100, solve for the indicated variable.

83. Area of a Triangle

Solve for h in $A = \frac{1}{2}bh$.

84. Perimeter of a Rectangle

Solve for l in $P = 2l + 2w$.

85. Volume of a Rectangular Prism

Solve for l in $V = lwh$

86. Ideal Gas Law

Solve for T in $PV = nRT$.

87. Volume of a Right Circular Cylinder

Solve for h in $V = \pi r^2 h$

88. Kinetic Energy

Solve for m in $E = \frac{1}{2}mv^2$.

89. Markup

Solve for C in $S = C + RC$.

90. Discount

Solve for L in $S = L - RL$.

91. Investment at Simple Interest

Solve for r in $A = P + Prt$.

92. Investment at Compound Interest

Solve for P in $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

93. Area of a Trapezoid

Solve for b in $A = \frac{1}{2}(a + b)h$.

94. Area of a Sector of a Circle

Solve for θ in $A = \frac{\pi r^2 \theta}{360}$.

95. Arithmetic Progression

Solve for n in $L = a + (n - 1)d$.

96. Geometric Progression

Solve for r in $S = \dfrac{rL - a}{r - 1}$.

97. Lateral Surface Area of a Cylinder

Solve for h in $A = 2\pi rh$.

98. Surface Area of a Cone

Solve for l in $S = \pi r^2 + \pi rl$.

99. Lensmaker's Equation

Solve for R_1 in $\dfrac{1}{f} = (n - 1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

100. Capacitance in Series Circuits

Solve for C_1 in $C = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$

101. Monthly Sales The table below shows the monthly sales of a sales team for the first quarter of the year. Find the average monthly sales for each salesperson. Then find the team's average sales for each month.

Name	January	February	March
Williams	$15,000	$18,800	$22,300
Gonzalez	$20,900	$17,500	$25,600
Walters	$18,600	$25,000	$16,400
Gilbert	$18,100	$18,700	$23,000
Hart	$13,000	$20,500	$20,000

102. Monthly Sales The table below shows the monthly sales of a sales team for the second quarter of the year. Find the average monthly sales for each salesperson. Then find the team's average sales for each month.

Name	April	May	June
Williams	$25,000	$28,800	$21,000
Gonzalez	$26,200	$27,800	$29,500
Walters	$26,600	$23,400	$26,900
Gilbert	$27,100	$22,200	$29,000
Hart	$23,100	$27,400	$22,800

103. Monthly Sales The table below shows the monthly sales of a sales team for the third quarter of the year. Use a spreadsheet software program to find the average monthly sales for each salesperson. Then find the team's average sales for each month.

Name	July	August	September
Williams	$24,400	$29,500	$21,200
Gonzalez	$26,100	$22,900	$19,600
Walters	$29,200	$28,600	$18,400
Gilbert	$25,000	$27,600	$29,800
Hart	$31,400	$28,700	$24,200
Reyes	$27,300	$26,400	$21,200
Sanders	$8,200	$12,400	$20,300

104. Monthly Sales The table below shows the monthly sales of a sales team for the fourth quarter of the year. Use a spreadsheet software program to find the average monthly sales for each salesperson. Then find the team's average sales for each month.

Name	October	November	December
Williams	$20,000	$25,100	$23,900
Gonzalez	$24,200	$23,600	$18,500
Walters	$31,900	$23,800	$18,400
Gilbert	$24,600	$23,100	$30,700
Hart	$32,400	$19,100	$28,600
Reyes	$24,700	$24,500	$23,400
Sanders	$18,700	$22,100	$23,200

105. Applied problems in textbooks usually give just the amount of information that is necessary to solve the problem. In real life, however, you must often sort through the given information and discard facts that are irrelevant to the problem. Such an irrelevant fact is called a **red herring.** Find any red herrings in the following problem.

Beneath the surface of the ocean, pressure changes at a rate of approximately 4.4 pounds per square inch for every 10-foot change in depth. A diver takes 30 minutes to ascend 25 feet from a depth of 150 feet. What change in pressure does the diver experience?

The symbol (S) indicates an exercise in which you are instructed to use a spreadsheet.

Section 1.3

Quadratic Equations

- Solve a quadratic equation by factoring.
- Solve a quadratic equation by extracting square roots.
- Construct and use a quadratic model to solve an application problem.

Solving Quadratic Equations by Factoring

In the first two sections of this chapter, you studied linear equations in one variable. In this and the next section, you will study quadratic equations.

Definition of a Quadratic Equation

A **quadratic equation** in x is an equation that can be written in the general form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$. Another name for a quadratic equation in x is a **second-degree polynomial equation in x.**

There are three basic techniques for solving quadratic equations: factoring, extracting square roots, and the *Quadratic Formula*. (The Quadratic Formula is discussed in the next section.) The first technique is based on the following property.

Zero-Factor Property

If $ab = 0$, then $a = 0$ or $b = 0$.

To use this property, rewrite the left side of the general form of a quadratic equation as the product of two linear factors. Then find the solutions of the quadratic equation by setting each linear factor equal to zero.

STUDY TIP

The Zero-Factor Property applies *only* to equations written in general form (in which one side of the equation is zero). So, be sure that all terms are collected on one side *before* factoring. For instance, in the equation

$$(x - 5)(x + 2) = 8$$

it is *incorrect* to set each factor equal to 8. Can you solve this equation correctly?

Example 1 Solving a Quadratic Equation by Factoring

Solve $x^2 - 3x - 10 = 0$.

SOLUTION

$x^2 - 3x - 10 = 0$			Write original equation.
$(x - 5)(x + 2) = 0$			Factor.
$x - 5 = 0$	⇨	$x = 5$	Set 1st factor equal to 0.
$x + 2 = 0$	⇨	$x = -2$	Set 2nd factor equal to 0.

The solutions are $x = 5$ and $x = -2$. Check these in the original equation.

✓CHECKPOINT 1

Solve $x^2 - x - 12 = 0$. ■

Example 2 Solving a Quadratic Equation by Factoring

$$6x^2 - 3x = 0 \qquad \text{Original equation}$$
$$3x(2x - 1) = 0 \qquad \text{Factor out common factor.}$$
$$3x = 0 \Rightarrow x = 0 \qquad \text{Set 1st factor equal to 0.}$$
$$2x - 1 = 0 \Rightarrow x = \tfrac{1}{2} \qquad \text{Set 2nd factor equal to 0.}$$

The solutions are $x = 0$ and $x = \frac{1}{2}$. Check these by substituting in the original equation, as follows.

CHECK

$$6x^2 - 3x = 0 \qquad \text{Write original equation.}$$
$$6(0)^2 - 3(0) \stackrel{?}{=} 0 \qquad \text{Substitute 0 for } x.$$
$$0 - 0 = 0 \qquad \text{First solution checks. } \checkmark$$
$$6\left(\tfrac{1}{2}\right)^2 - 3\left(\tfrac{1}{2}\right) \stackrel{?}{=} 0 \qquad \text{Substitute } \tfrac{1}{2} \text{ for } x.$$
$$\tfrac{6}{4} - \tfrac{3}{2} = 0 \qquad \text{Second solution checks. } \checkmark$$

✓CHECKPOINT 2

Solve $4x^2 - 8x = 0$. ■

If the two factors of a quadratic expression are the same, the corresponding solution is a **double** or **repeated** solution.

Example 3 A Quadratic Equation with a Repeated Solution

Solve $9x^2 - 6x = -1$.

SOLUTION

$$9x^2 - 6x = -1 \qquad \text{Write original equation.}$$
$$9x^2 - 6x + 1 = 0 \qquad \text{Write in general form.}$$
$$(3x - 1)^2 = 0 \qquad \text{Factor.}$$
$$3x - 1 = 0 \qquad \text{Set repeated factor equal to 0.}$$
$$x = \tfrac{1}{3} \qquad \text{Solution}$$

The only solution is $x = \frac{1}{3}$. Check this by substituting in the original equation, as follows.

$$9x^2 - 6x = -1 \qquad \text{Write original equation.}$$
$$9\left(\tfrac{1}{3}\right)^2 - 6\left(\tfrac{1}{3}\right) \stackrel{?}{=} -1 \qquad \text{Substitute } \tfrac{1}{3} \text{ for } x.$$
$$1 - 2 \stackrel{?}{=} -1 \qquad \text{Simplify.}$$
$$-1 = -1 \qquad \text{Solution checks. } \checkmark$$

✓CHECKPOINT 3

Solve $x^2 + 4x = -4$. ■

TECHNOLOGY

To check the solution in Example 3 with your graphing utility, you should first write the equation in general form.

$$9x^2 - 6x + 1 = 0$$

Then enter the expression $9x^2 - 6x + 1$ into y_1 of the equation editor. Now you can use the ASK mode of the *table* feature of your graphing utility to check the solution. For instructions on how to use the *table* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

Extracting Square Roots

There is a shortcut for solving equations of the form $u^2 = d$, where $d > 0$. By factoring, you can see that this equation has two solutions.

$$u^2 = d \qquad \text{Write original equation.}$$
$$u^2 - d = 0 \qquad \text{Write in general form.}$$
$$(u + \sqrt{d})(u - \sqrt{d}) = 0 \qquad \text{Factor.}$$
$$u + \sqrt{d} = 0 \Rightarrow u = -\sqrt{d} \qquad \text{Set 1st factor equal to 0.}$$
$$u - \sqrt{d} = 0 \Rightarrow u = \sqrt{d} \qquad \text{Set 2nd factor equal to 0.}$$

Solving an equation of the form $u^2 = d$ without going through the steps of factoring is called **extracting square roots.**

Extracting Square Roots

The equation $u^2 = d$, where $d > 0$, has exactly two solutions:

$u = \sqrt{d}$ and $u = -\sqrt{d}$.

These solutions can also be written as $u = \pm\sqrt{d}$.

Example 4 Extracting Square Roots

Solve $4x^2 = 12$.

SOLUTION

$$4x^2 = 12 \qquad \text{Write original equation.}$$
$$x^2 = 3 \qquad \text{Divide each side by 4.}$$
$$x = \pm\sqrt{3} \qquad \text{Extract square roots.}$$

The solutions are $x = \sqrt{3}$ and $x = -\sqrt{3}$. Check these in the original equation.

✓CHECKPOINT 4

Solve $2x^2 = 8$. ■

Example 5 Extracting Square Roots

Solve $(x - 3)^2 = 7$.

SOLUTION

$$(x - 3)^2 = 7 \qquad \text{Write original equation.}$$
$$x - 3 = \pm\sqrt{7} \qquad \text{Extract square roots.}$$
$$x = 3 \pm \sqrt{7} \qquad \text{Add 3 to each side.}$$

The solutions are $x = 3 \pm \sqrt{7}$. Check these in the original equation.

✓CHECKPOINT 5

Solve $(x - 1)^2 = 16$. ■

Applications

Quadratic equations often occur in problems dealing with area. Here is a simple example.

> A square room has an area of 144 square feet. Find the dimensions of the room.

To solve this problem, you can let x represent the length of each side of the room. Then, by solving the equation

$$x^2 = 144$$

you can conclude that each side of the room is 12 feet long. Note that although the equation $x^2 = 144$ has two solutions, $x = -12$ and $x = 12$, the negative solution makes no sense (for this problem), so you should choose the positive solution.

Example 6 Finding the Dimensions of a Room

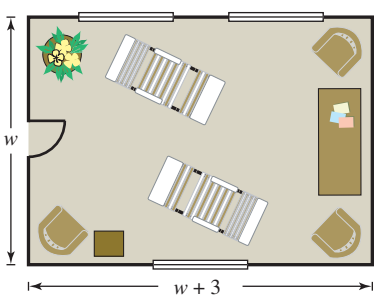


FIGURE 1.6

A sunroom is 3 feet longer than it is wide (see Figure 1.6) and has an area of 154 square feet. Find the dimensions of the room.

SOLUTION You can begin by using the same type of problem-solving strategy that was presented in Section 1.2.

Verbal Model: Width of room · Length of room = Area of room

Labels:
Area of room = 154 (square feet)
Width of room = w (feet)
Length of room = $w + 3$ (feet)

Equation:

$$w(w + 3) = 154$$
$$w^2 + 3w - 154 = 0$$
$$(w - 11)(w + 14) = 0$$
$$w - 11 = 0 \Rightarrow w = 11$$
$$w + 14 = 0 \Rightarrow w = -14$$

Choosing the positive value, you can conclude that the width is 11 feet and the length is $w + 3 = 14$ feet. You can check this in the original statement of the problem as follows.

CHECK

The length of 14 feet is 3 feet more than the width of 11 feet. ✓

The area of the sunroom is $(11)(14) = 154$ square feet. ✓

✓CHECKPOINT 6

A rectangular kitchen is 8 feet longer than it is wide and has an area of 84 square feet. Find the dimensions of the kitchen. ■

Another application of quadratic equations involves an object that is falling (or is vertically projected into the air). The equation that gives the height of such an object is called a **position equation,** and on Earth's surface it has the form

$$s = -16t^2 + v_0t + s_0.$$

In this equation, s represents the height of the object (in feet), v_0 represents the initial velocity of the object (in feet per second), s_0 represents the initial height of the object (in feet), and t represents the time (in seconds).

The position equation shown above ignores air resistance. This implies that it is appropriate to use the position equation only to model falling objects that have little air resistance and that fall over short distances.

Example 7 *MAKE A DECISION* Falling Object

A construction worker accidentally drops a wrench from a height of 235 feet and yells "Look out below!" (see Figure 1.7). Could a person at ground level hear this warning in time to get out of the way of the falling wrench?

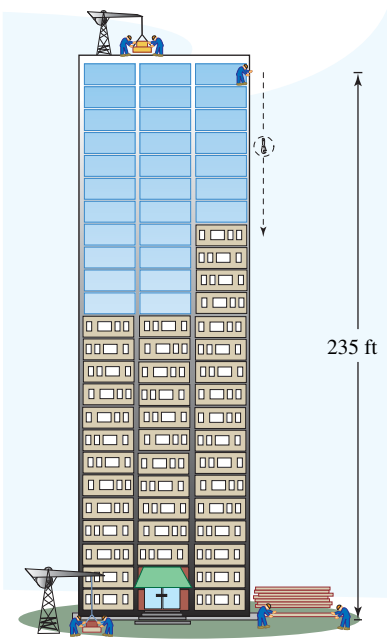


FIGURE 1.7

SOLUTION Because sound travels at about 1100 feet per second, it follows that a person at ground level hears the warning within 1 second of the time the wrench is dropped. To set up a mathematical model for the height of the wrench, use the position equation

$$s = -16t^2 + v_0t + s_0.$$

Because the object is dropped rather than thrown, the initial velocity is $v_0 = 0$ feet per second. Moreover, because the initial height is $s_0 = 235$ feet, you have the following model.

$$s = -16t^2 + (0)t + 235 = -16t^2 + 235$$

After falling for 1 second, the height of the wrench is $-16(1)^2 + 235 = 219$ feet. After falling for 2 seconds, the height of the wrench is $-16(2)^2 + 235 = 171$ feet. To find the number of seconds it takes the wrench to hit the ground, let the height s be zero and solve the equation for t.

$$s = -16t^2 + 235 \qquad \text{Write position equation.}$$

$$0 = -16t^2 + 235 \qquad \text{Substitute 0 for height.}$$

$$16t^2 = 235 \qquad \text{Add } 16t^2 \text{ to each side.}$$

$$t^2 = \frac{235}{16} \qquad \text{Divide each side by 16.}$$

$$t = \frac{\sqrt{235}}{4} \qquad \text{Extract positive square root.}$$

$$t \approx 3.83 \qquad \text{Use a calculator.}$$

The wrench will take about 3.83 seconds to hit the ground. If the person hears the warning within 1 second after the wrench is dropped, the person still has almost 3 seconds to get out of the way.

✓CHECKPOINT 7

You drop a rock from a height of 144 feet. How long does it take the rock to hit the ground? ■

A third type of application using a quadratic equation involves the hypotenuse of a right triangle. Recall from geometry that the sides of a right triangle are related by a formula called the **Pythagorean Theorem.** This theorem states that if a and b are the lengths of the legs of the triangle and c is the length of the hypotenuse (see Figure 1.8),

$$a^2 + b^2 = c^2. \qquad \text{Pythagorean Theorem}$$

Notice how this formula is used in the next example.

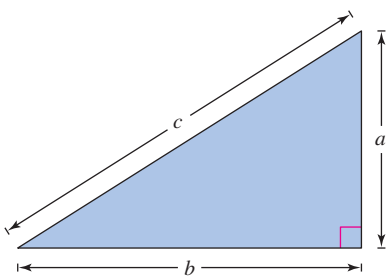


FIGURE 1.8

Example 8 Cutting Across the Lawn

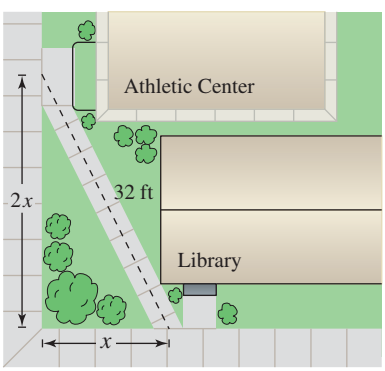


FIGURE 1.9

An L-shaped sidewalk from the athletic center to the library on a college campus is shown in Figure 1.9. The sidewalk was constructed so that the length of one sidewalk forming the L is twice as long as the other. The length of the diagonal sidewalk that cuts across the grounds between the two buildings is 32 feet. How many feet does a person save by walking on the diagonal sidewalk?

SOLUTION Using the Pythagorean Theorem, you have

$$\begin{aligned}
a^2 + b^2 &= c^2 && \text{Pythagorean Theorem}\\
x^2 + (2x)^2 &= 32^2 && \text{Substitute for } a, b, \text{ and } c.\\
5x^2 &= 1024 && \text{Combine like terms.}\\
x^2 &= 204.8 && \text{Divide each side by 5.}\\
x &= \pm\sqrt{204.8} && \text{Take the square root of each side.}\\
x &= \sqrt{204.8}. && \text{Extract positive square root.}
\end{aligned}$$

The total distance covered by walking on the L-shaped sidewalk is

$$\begin{aligned}
x + 2x &= 3x\\
&= 3\sqrt{204.8}\\
&\approx 42.9 \text{ feet.}
\end{aligned}$$

Walking on the diagonal sidewalk saves a person about $42.9 - 32 = 10.9$ feet.

✓CHECKPOINT 8

In Example 8, suppose the length of one sidewalk forming the L is three times as long as the other. How many feet does a person save by walking on the 32-foot diagonal sidewalk? ■

A fourth type of application of a quadratic equation is one in which a quantity y is changing over time t according to a quadratic model. In the next example, we exchange y for E, because E is a better descriptor in the model.

Example 9 Carbon Dioxide Emissions

From 2001 to 2005, yearly emissions E (in billions of metric tons) of carbon dioxide (CO_2) from energy consumption at power plants in the United States can be modeled by

$$E = 0.0053t^2 + 2.38, \quad 1 \le t \le 5$$

where t represents the year, with $t = 1$ corresponding to 2001 (see Figure 1.10). Use the model to approximate the year that CO_2 emissions were about 2,420,000,000 metric tons. *(Source: Energy Information Administration)*

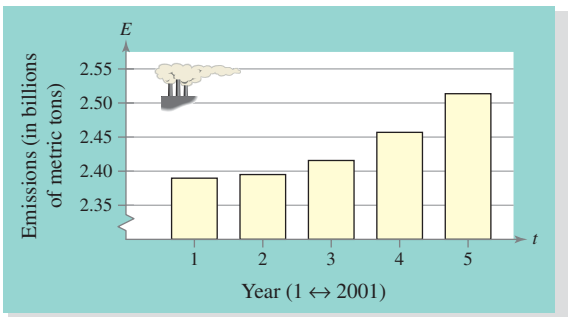


FIGURE 1.10

SOLUTION To solve this problem, let the CO_2 emissions E be 2.42 billion and solve the equation for t.

$$0.0053t^2 + 2.38 = 2.42 \qquad \text{Substitute 2.42 for } E.$$
$$0.0053t^2 = 0.04 \qquad \text{Subtract 2.38 from each side.}$$
$$t^2 \approx 7.547 \qquad \text{Divide each side by 0.0053.}$$
$$t \approx \sqrt{7.547} \qquad \text{Extract positive square root.}$$
$$t \approx 2.747 \qquad \text{Simplify.}$$

The solution is $t \approx 3$. Because $t = 1$ represents 2001, you can conclude that, according to the model, CO_2 emissions were about 2.42 billion metric tons in the year 2003.

✓CHECKPOINT 9

In Example 9, use the model to predict the year that CO_2 emissions will be about 3.0 billion metric tons. ■

CONCEPT CHECK

1. When using a quadratic model to solve an application problem, when can you reject one of the solutions?
2. Does the quadratic equation $x^2 = d$, where $d > 0$, have a repeated solution? Explain.
3. Which method would you use to solve the quadratic equation $(x - 5)^2 = 16$? Explain your reasoning.
4. Describe and correct the error in the solution:

$$x^2 - 2x = 3$$
$$x(x - 2) = 3$$
$$x = 3$$
$$(x - 2) = 3 \implies x = 5$$

Skills Review 1.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4 and 0.6.

In Exercises 1–4, simplify the expression.

1. $\sqrt{\frac{7}{50}}$ **2.** $\sqrt{32}$

3. $\sqrt{7^2 + 3 \cdot 7^2}$ **4.** $\sqrt{\frac{1}{4} + \frac{3}{8}}$

In Exercises 5–10, factor the expression.

5. $3x^2 + 7x$ **6.** $4x^2 - 25$

7. $16 - (x - 11)^2$ **8.** $x^2 + 7x - 18$

9. $10x^2 + 13x - 3$ **10.** $6x^2 - 73x + 12$

Exercises 1.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, write the quadratic equation in general form.

1. $2x^2 = 3 - 5x$ **2.** $4x^2 - 2x = 9$

3. $x^2 = 25x$ **4.** $10x^2 = 90$

5. $(x - 3)^2 = 2$ **6.** $12 - 3(x + 7)^2 = 0$

7. $x(x + 2) = 3x^2 + 1$ **8.** $x(x + 5) = 2(x + 5)$

9. $\frac{3x^2 - 10}{5} = 12x$ **10.** $\frac{x^2 - 7}{3} = 2x$

In Exercises 11–22, solve the quadratic equation by factoring.

11. $x^2 - 2x - 8 = 0$ **12.** $x^2 - 10x + 9 = 0$

13. $6x^2 + 3x = 0$ **14.** $9x^2 - 1 = 0$

15. $x^2 + 10x + 25 = 0$ **16.** $16x^2 + 56x + 49 = 0$

17. $3 + 5x - 2x^2 = 0$ **18.** $2x^2 = 19x + 33$

19. $x^2 + 4x = 12$ **20.** $x^2 + 4x = 21$

21. $-x^2 - 7x = 10$ **22.** $-x^2 + 8x = 12$

In Exercises 23–40, solve the quadratic equation by extracting square roots. List both the exact answer *and* a decimal answer that has been rounded to two decimal places.

23. $x^2 = 16$ **24.** $x^2 = 144$

25. $x^2 = 7$ **26.** $x^2 = 27$

27. $3x^2 = 36$ **28.** $9x^2 = 25$

29. $(x - 12)^2 = 18$ **30.** $(x + 13)^2 = 21$

31. $(x + 2)^2 = 12$ **32.** $(x + 5)^2 = 20$

33. $12x^2 = 300$ **34.** $6x^2 = 250$

35. $5x^2 = 190$

36. $15x^2 = 620$

37. $3x^2 + 2(x^2 - 4) = 15$

38. $x^2 + 3(x^2 - 5) = 10$

39. $6x^2 - 3(x^2 + 1) = 23$

40. $2x^2 + 5(x^2 - 2) = 29$

In Exercises 41–62, solve the quadratic equation using any convenient method.

41. $x^2 = 64$ **42.** $7x^2 = 32$

43. $x^2 - 2x + 1 = 0$ **44.** $x^2 - 6x + 5 = 0$

45. $16x^2 - 9 = 0$ **46.** $11x^2 + 33x = 0$

47. $4x^2 - 12x + 9 = 0$ **48.** $x^2 - 14x + 49 = 0$

49. $(x + 4)^2 = 49$ **50.** $(x - 3)^2 = 36$

51. $4x = 4x^2 - 3$ **52.** $80 + 6x = 9x^2$

53. $50 + 5x = 3x^2$ **54.** $144 - 73x + 4x^2 = 0$

55. $12x = x^2 + 27$ **56.** $26x = 8x^2 + 15$

57. $50x^2 - 60x + 10 = 0$ **58.** $9x^2 + 12x + 3 = 0$

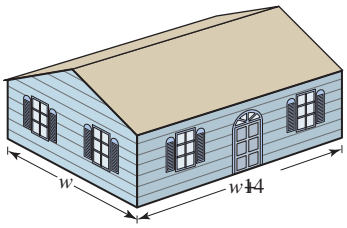
59. $(x + 3)^2 - 4 = 0$ **60.** $(x - 2)^2 - 9 = 0$

61. $(x + 1)^2 = x^2$ **62.** $(x + 1)^2 = 4x^2$

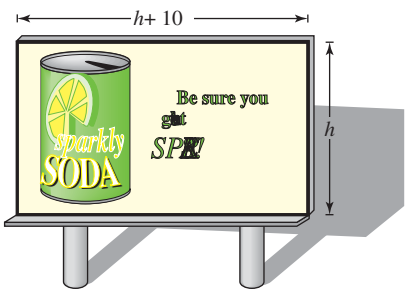
63. Consider the expression $(x + 2)^2$. How would you convince someone in your class that $(x + 2)^2 \neq x^2 + 4$? Give an argument based on the rules of algebra. Give an argument using your graphing utility.

64. Consider the expression $\sqrt{a^2 + b^2}$. How would you convince someone in your class that $\sqrt{a^2 + b^2} \neq a + b$? Give an argument based on the rules of algebra or geometry. Give an argument using your graphing utility.

65. Geometry A one-story building is 14 feet longer than it is wide (see figure). The building has 1632 square feet of floor space. What are the dimensions of the building?

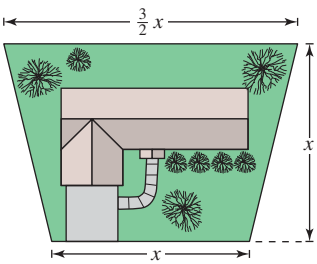


66. Geometry A billboard is 10 feet longer than it is high (see figure). The billboard has 336 square feet of advertising space. What are the dimensions of the billboard?

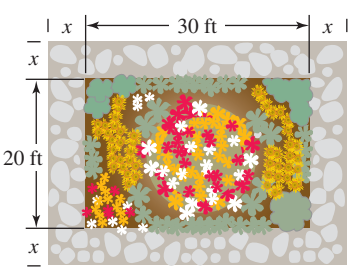


67. Geometry A triangular sign has a height that is equal to its base. The area of the sign is 4 square feet. Find the base and height of the sign.

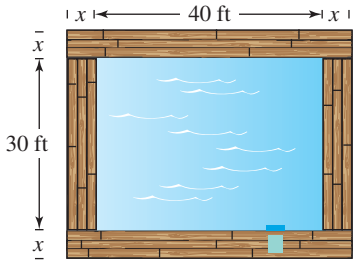
68. Geometry The building lot shown in the figure has an area of 8000 square feet. What are the dimensions of the lot?



69. Geometry A rectangular garden that is 30 feet long and 20 feet wide is surrounded on all four sides by a rock path that is x feet wide. The total area of the garden and the rock path is 1200 square feet. What is the width of the path?



70. Geometry A rectangular pool is 30 feet wide and 40 feet long. It is surrounded on all four sides by a wooden deck that is x feet wide. The total area enclosed within the perimeter of the deck is 3000 square feet. What is the width of the deck?



In Exercises 71–76, assume that air resistance is negligible, which implies that the position equation $s = -16t^2 + v_0t + s_0$ is a reasonable model.

71. Falling Object A rock is dropped from the top of a 200-foot cliff that overlooks the ocean. How long will it take for the rock to hit the water?

72. Royal Gorge Bridge The Royal Gorge Bridge near Canon City, Colorado is the highest suspension bridge in the world. The bridge is 1053 feet above the Arkansas river. A rock is dropped from the bridge. How long does it take the rock to hit the water?

73. Olympic Diver The high-dive platform in the Olympics is 10 meters above the water. A diver wants to perform an armstand dive, which means she will drop to the water from a handstand position. How long will the diver be in the air? (*Hint:* 1 meter ≈ 3.2808 feet)

74. The Owl and the Mouse An owl is circling a field and sees a mouse. The owl folds its wings and begins to dive. If the owl starts its dive from a height of 100 feet, how long does the mouse have to escape?

75. Wind Resistance At the same time a skydiver jumps from an airplane 13,000 feet above the ground, a steel ball is dropped from the plane. Because of air resistance, it takes the skydiver 67 seconds to freefall to a height of 3000 feet where the parachute opens. The steel ball has relatively no air resistance, so its height can be modeled by the position equation. How much faster does the ball reach a height of 3000 feet than the skydiver?

76. Wind Resistance At the same time a skydiver jumps from an airplane 8900 feet above the ground, a steel ball is dropped from the plane. Because of air resistance, it takes the skydiver 44 seconds to freefall to a height of 2500 feet where the parachute opens. The steel ball has relatively no air resistance and its height can be modeled by the position equation. How much faster does the ball reach a height of 2500 feet than the skydiver?

77. Geometry The hypotenuse of an isosceles right triangle is 6 centimeters long. How long are the legs? (An isosceles right triangle is one whose two legs are of equal length.)

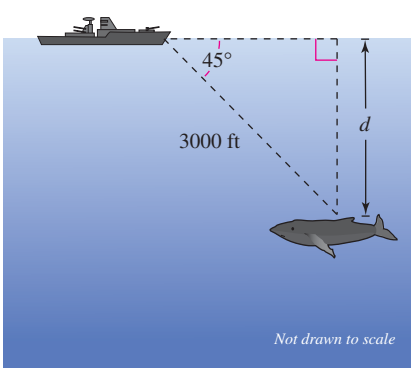
78. Geometry An equilateral triangle has a height of 3 feet. How long are each of its legs? (*Hint:* Use the height of the triangle to partition the triangle into two right triangles of the same size.)

79. Flying Distance A commercial jet flies to three cities whose locations form the vertices of a right triangle (see figure). The air distance from Atlanta to Buffalo is about 703 miles and the air distance from Atlanta to Chicago is about 583 miles. Approximate the air distance from Atlanta to Buffalo *by way of* Chicago.



In Exercises 80 and 81, use the following information. The sum of the angles of a triangle is 180°. Also, if two angles of a triangle are equal, the lengths of the sides opposite the angles are equal.

80. Depth of a Whale The sonar of a research ship detects a whale that is 3000 feet from the ship. The angle formed by the ocean surface and a line from the ship to the whale is 45° (see figure). How deep is the whale?

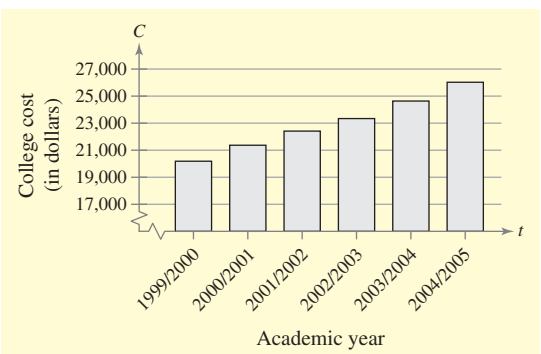


81. Depth of a Whale Shark A research ship is tracking the movements of a whale shark that is 700 meters from the ship. The angle formed by the ocean surface and a line from the ship to the whale shark is 45°. How deep is the whale shark?

82. College Costs The average yearly cost C of attending a private college full time for the academic years 1999/2000 to 2004/2005 in the United States can be approximated by the model

$$C = 45.6t^2 + 15{,}737, \quad 10 \le t \le 15$$

where $t = 10$ corresponds to the 1999/2000 academic year (see figure). Use the model to predict the year in which the average cost of attending a private college full time is about \$30,000. *(Source: U.S. National Center for Education Statistics)*



83. Total Revenue The demand equation for a product is $p = 36 - 0.0003x$, where p is the price per unit and x is the number of units sold. The total revenue R for selling x units is given by

$$R = xp = x(36 - 0.0003x).$$

How many units must be sold to produce a revenue of \$1,080,000?

84. Total Revenue The demand equation for a product is $p = 40 - 0.0005x$, where p is the price per unit and x is the number of units sold. The total revenue R for selling x units is given by

$$R = xp = x(40 - 0.0005x).$$

How many units must be sold to produce a revenue of \$800,000?

85. Production Cost A company determines that the average monthly cost C (in dollars) of raw materials for manufacturing a product line can be modeled by

$$C = 35.65t^2 + 7205, \quad t \ge 0$$

where t is the year, with $t = 0$ corresponding to 2000. Use the model to estimate the year in which the average monthly cost reaches \$12,000.

86. Monthly Cost A company determines that the average monthly cost C (in dollars) for staffing temporary positions can be modeled by

$$C = 135.47t^2 + 13{,}702, \quad t \geq 0$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to predict the year in which the average monthly cost is about $25,000.

87. *MAKE A DECISION: U.S. POPULATION* The resident population P (in thousands) of the United States from 1800 to 1890 can be approximated by the model

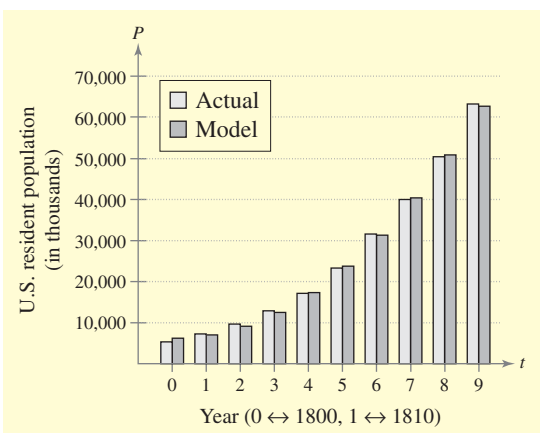
$$P = 694.59t^2 + 6179, \quad 0 \leq t \leq 9$$

where t represents the year, with $t = 0$ corresponding to 1800, $t = 1$ corresponding to 1810, and so on (see figure).

(a) Assume this model had continued to be valid up through the present time. In what year would the resident population of the United States have reached 250,000,000?

(b) Judging from the figure, would you say that this model is a good representation of the resident population through 1890?

(c) How about through 2006, when the United States resident population was approximately 300,000,000 people? *(Source: U.S. Census Bureau)*



88. U.S. Population The resident population P (in thousands) of the United States from 1900 to 2000 can be approximated by the model

$$P = 1951.00t^2 + 97{,}551, \quad 0 \leq t \leq 10$$

where t represents the year, with $t = 0$ corresponding to 1900, $t = 1$ corresponding to 1910, and so on (see figure). Assume this model continues to be valid. In what year will the resident population of the United States reach 330,000,000? *(Source: U.S. Census Bureau)*

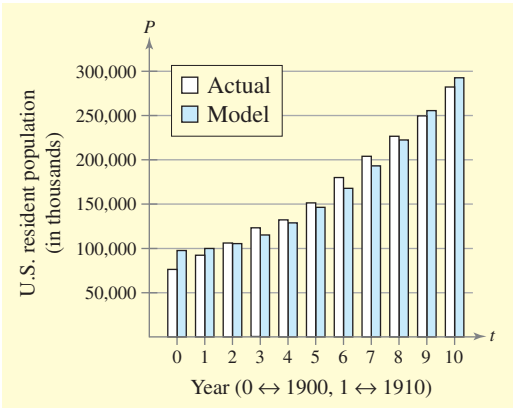


Figure for 88

89. *MAKE A DECISION* The U.S. Census Bureau predicts that the population in 2050 will be 419,854,000. Does the model in Exercise 88 appear to be a valid model for the year 2050?

90. *MAKE A DECISION* The enrollment E in an early childhood development program for a school district from 1995 to 2008 can be approximated by the model $E = 1.678t^2 + 1025$, $5 \leq t \leq 18$, where t represents the year, with $t = 5$ corresponding to 1995. Use the model to approximate the year in which the early childhood enrollment reached 1450 children. Can you use the model to estimate early childhood enrollment for the year 1980? Explain.

91. *MAKE A DECISION* The temperature T (in degrees Fahrenheit) during a certain day can be approximated by $T = 0.31t^2 + 32.9$, $7 \leq t \leq 15$, where t represents the hour of the day, with $t = 7$ corresponding to 7 A.M. Use the model to predict the time when the temperature was 85° F. Can you use this model to predict the temperature at 7 P.M.? Explain.

92. Hydrofluorocarbon Emissions From 2000 to 2005, yearly emissions E (in millions of metric tons) of hydrofluorocarbons (HFCs) in the United States can be modeled by $E = 1.26t^2 + 99.98$, $0 \leq t \leq 5$, where t represents the year, with $t = 0$ corresponding to 2000. Use the model to estimate the year in which HFC emissions were about 124,000,000 metric tons. *(Source: Energy Information Administration)*

93. Blue Oak The blue oak tree, native to California, is known for its slow rate of growth. Fencing enclosures protect seedlings from herbivore damage and promote faster growth. The height H (in inches) of an enclosed blue oak tree can be approximated by the model

$$H = 0.74t^2 + 25, \quad 0 \leq t \leq 5$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to approximate the year in which the height of the tree was about 32 inches.

Section 1.4

The Quadratic Formula

- Develop the Quadratic Formula by completing the square.
- Use the discriminant to determine the number of real solutions of a quadratic equation.
- Solve a quadratic equation using the Quadratic Formula.
- Use the Quadratic Formula to solve an application problem.

Development of the Quadratic Formula

In Section 1.3 you studied two methods for solving quadratic equations. These two methods are efficient for special quadratic equations that are factorable or that can be solved by extracting square roots. There are, however, many quadratic equations that cannot be solved efficiently by either of these two techniques. Fortunately, there is a general formula that can be used to solve *any* quadratic equation. It is called the **Quadratic Formula.** This formula is derived using a process called **completing the square.**

$$ax^2 + bx + c = 0 \qquad \text{General form, } a \neq 0$$

$$ax^2 + bx = -c \qquad \text{Subtract } c \text{ from each side.}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divide each side by } a.$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{Complete the square.}$$

$\left(\text{half of } \frac{b}{a}\right)^2$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{Simplify.}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Extract square roots.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Solutions}$$

STUDY TIP

The Quadratic Formula is one of the most important formulas in algebra, and you should memorize it. It might help to try to memorize a verbal statement of the rule. For instance, you might try to remember the following verbal statement of the Quadratic Formula: "The opposite of b, plus or minus the square root of b squared minus $4ac$, all divided by $2a$."

The Quadratic Formula

The solutions of

$$ax^2 + bx + c = 0, \quad a \neq 0$$

are given by the **Quadratic Formula,**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The Discriminant

In the Quadratic Formula, the quantity under the radical sign, $b^2 - 4ac$, is called the **discriminant** of the quadratic expression $ax^2 + bx + c$.

$b^2 - 4ac$ Discriminant

It can be used to determine the number of the solutions of a quadratic equation.

Solutions of a Quadratic Equation

The solutions of a quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

can be classified by the discriminant, $b^2 - 4ac$, as follows.

1. If $b^2 - 4ac > 0$, the equation has *two* distinct real solutions.
2. If $b^2 - 4ac = 0$, the equation has *one* repeated real solution.
3. If $b^2 - 4ac < 0$, the equation has *no* real solutions.

If the discriminant of a quadratic equation is negative, as in case 3 above, then its square root is imaginary (not a real number) and the Quadratic Formula yields two complex solutions. You will study complex solutions in Section 3.5.

Example 1 Using the Discriminant

Use the discriminant to determine the number of real solutions of each of the following quadratic equations.

a. $4x^2 - 20x + 25 = 0$

b. $13x^2 + 7x + 1 = 0$

c. $5x^2 = 8x$

SOLUTION

a. Using $a = 4$, $b = -20$, and $c = 25$, the discriminant is

$$b^2 - 4ac = (-20)^2 - 4(4)(25) = 400 - 400 = 0.$$

Because $b^2 - 4ac = 0$, there is *one* repeated real solution.

b. Using $a = 13$, $b = 7$, and $c = 1$, the discriminant is

$$b^2 - 4ac = (7)^2 - 4(13)(1) = 49 - 52 = -3.$$

Because $b^2 - 4ac < 0$, there are *no* real solutions.

c. In general form, this equation is $5x^2 - 8x = 0$, with $a = 5$, $b = -8$, and $c = 0$, which implies that the discriminant is

$$b^2 - 4ac = (-8)^2 - 4(5)(0) = 64.$$

Because $b^2 - 4ac > 0$, there are *two* distinct real solutions.

✓CHECKPOINT 1

Use the discriminant to determine the number of real solutions of $x^2 + 6x + 9 = 0$. ■

Using the Quadratic Formula

When using the Quadratic Formula, remember that *before* the formula can be applied, you must first write the quadratic equation in general form.

Example 2 Two Distinct Solutions

Solve $x^2 + 3x = 9$.

SOLUTION

$$x^2 + 3x = 9$$ Write original equation.

$$x^2 + 3x - 9 = 0$$ Write in general form.

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-9)}}{2(1)}$$ Quadratic Formula

$$x = \frac{-3 \pm \sqrt{45}}{2}$$ Simplify.

$$x = \frac{-3 \pm 3\sqrt{5}}{2}$$ Simplify.

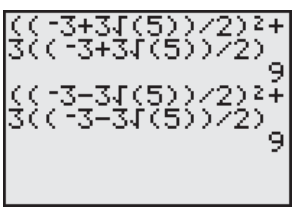
The solutions are

$$x = \frac{-3 + 3\sqrt{5}}{2} \quad \text{and} \quad x = \frac{-3 - 3\sqrt{5}}{2}.$$

Check these in the original equation.

TECHNOLOGY

You can check the solutions to Example 2 using a calculator.



✓CHECKPOINT 2

Solve $x^2 + 2x - 2 = 0$. ■

Example 3 One Repeated Solution

Solve $8x^2 - 24x + 18 = 0$.

SOLUTION Begin by dividing each side by the common factor 2.

$$8x^2 - 24x + 18 = 0$$ Write original equation.

$$4x^2 - 12x + 9 = 0$$ Divide each side by 2.

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$ Quadratic Formula

$$x = \frac{12 \pm \sqrt{0}}{8}$$ Simplify.

$$x = \frac{3}{2}$$ Repeated solution

The only solution is $x = \frac{3}{2}$. Check this in the original equation.

✓CHECKPOINT 3

Solve $9x^2 - 6x = -1$. ■

The discriminant in Example 3 is a perfect square (zero in this case), and you could have factored the quadratic as

$$4x^2 - 12x + 9 = 0$$
$$(2x - 3)^2 = 0$$

and concluded that the solution is $x = \frac{3}{2}$. Because factoring is easier than applying the Quadratic Formula, try factoring first when solving a quadratic equation. If, however, factors cannot easily be found, then use the Quadratic Formula. For instance, try solving the quadratic equation

$$x^2 - x - 12 = 0$$

in two ways—by factoring and by the Quadratic Formula—to see that you get the same solutions either way.

When using a calculator with the Quadratic Formula, you should get in the habit of using the memory key to store intermediate steps. This will save steps and minimize roundoff error.

Example 4 Using a Calculator with the Quadratic Formula

Solve $16.3x^2 - 197.6x + 7.042 = 0$.

SOLUTION In this case, $a = 16.3$, $b = -197.6$, $c = 7.042$, and you have

$$x = \frac{-(-197.6) \pm \sqrt{(-197.6)^2 - 4(16.3)(7.042)}}{2(16.3)}.$$

To evaluate these solutions, begin by calculating the square root of the discriminant, as follows.

Scientific Calculator Keystrokes

197.6 [+/−] [x²] [−] 4 [×] 16.3 [×] 7.042 [=] [√]

Graphing Calculator Keystrokes

[√] [(] [(] [(−)] 197.6 [)] [x²] [−] 4 [×] 16.3 [×] 7.042 [)] [ENTER]

In either case, the result is 196.434777. Storing this result and using the recall key, you can find the following two solutions.

$$x \approx \frac{197.6 + 196.434777}{2(16.3)} \approx 12.087$$ Add stored value.

$$x \approx \frac{197.6 - 196.434777}{2(16.3)} \approx 0.036$$ Subtract stored value.

✓CHECKPOINT 4

Solve $4.7x^2 - 3.2x - 5.9 = 0$. ■

TECHNOLOGY

Try to calculate the value of x in Example 4 by using additional parentheses instead of storing the intermediate result, 196.434777, in your calculator.

Applications

In Section 1.3, you studied four basic types of applications involving quadratic equations: area, falling bodies, the Pythagorean Theorem, and quadratic models. The solution to each of these types of problems can involve the Quadratic Formula. For instance, Example 5 shows how the Quadratic Formula can be used to analyze a quadratic model for a patient's blood oxygen level.

Example 5 Blood Oxygen Level

Doctors treated a patient at an emergency room from 1:00 P.M. to 5:00 P.M. The patient's blood oxygen level L (in percent) during this time period can be modeled by

$$L = -0.25t^2 + 3.0t + 87, \quad 1 \le t \le 5$$

where t represents the time of day, with $t = 1$ corresponding to 1:00 P.M. Use the model to estimate the time when the patient's blood oxygen level was 95%.

TECHNOLOGY

A program can be written to solve equations using the Quadratic Formula. A program for several models of graphing utilities can be found on the website for this text at *college.hmco.com/info/larsonapplied.* Use a program to solve Example 5.

SOLUTION To find the hour when the patient's blood oxygen level was 95%, solve the equation

$$95 = -0.25t^2 + 3.0t + 87.$$

To begin, write the equation in general form.

$$-0.25t^2 + 3.0t - 8 = 0$$

Then apply the Quadratic Formula with $a = -0.25$, $b = 3.0$, and $c = -8$.

$$t = \frac{-3 \pm \sqrt{3^2 - 4(-0.25)(-8)}}{2(-0.25)}$$

$$= \frac{-3 \pm \sqrt{1}}{-0.5} = 4 \text{ or } 8$$

Of the two possible solutions, only $t = 4$ makes sense in the context of the problem, because $t = 8$ is not in the domain of L. Because $t = 1$ corresponds to 1:00 P.M., it follows that $t = 4$ corresponds to 4:00 P.M. So, from the model you can conclude that the patient's blood oxygen level was 95% at about 4:00 P.M. Figure 1.11 shows the patients oxygen level recorded every 30 minutes.

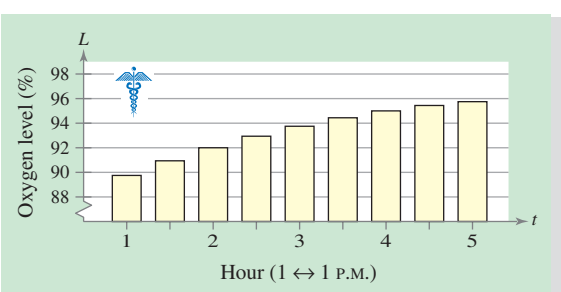


FIGURE 1.11

✓CHECKPOINT 5

In Example 5, use the model to estimate the time when the patient's blood oxygen level was 92%. ■

STUDY TIP

Note in the position equation

$$s = -16t^2 + v_0t + s_0$$

that the initial velocity v_0 is positive when an object is rising and negative when an object is falling.

In Section 1.3, you learned that the position equation for a falling object is of the form

$$s = -16t^2 + v_0t + s_0$$

where s is the height (in feet) of the object, v_0 is the initial velocity (in feet per second), t is the time (in seconds), and s_0 is the initial height (in feet). This equation is valid only for free-falling objects near Earth's surface. Because of differences in gravitational force, position equations are different on other planets or moons. The next example looks at a position equation for a falling object on our moon.

Example 6 **Throwing an Object on the Moon** R

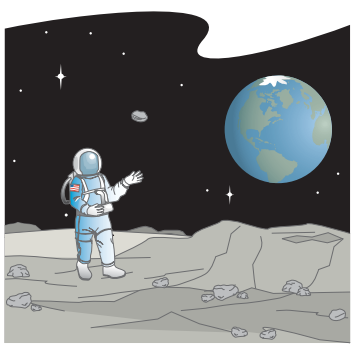


FIGURE 1.12

An astronaut standing on the surface of the moon throws a rock straight up at 27 feet per second from a height of 6 feet (see Figure 1.12). The height s (in feet) of the rock is given by

$$s = -2.7t^2 + 27t + 6$$

where t is the time (in seconds). How much time elapses before the rock strikes the lunar surface?

SOLUTION Because s gives the height of the rock at any time t, you can find the time that the rock hits the surface of the moon by setting s equal to zero and solving for t.

$$-2.7t^2 + 27t + 6 = 0 \qquad \text{Substitute 0 for } s.$$

$$t = \frac{-27 \pm \sqrt{(27)^2 - 4(-2.7)(6)}}{2(-2.7)} \qquad \text{Quadratic Formula}$$

$$\approx 10.2 \text{ seconds} \qquad \text{Choose positive solution.}$$

So, about 10.2 seconds elapse before the rock hits the lunar surface.

TECHNOLOGY

Use the *last entry* feature of your graphing calculator to find the time in the air on Earth for the rock in Example 6. Simply replace -2.7 with -16 in the expression for t. For specific keystrokes on using the *last entry* feature, go to the text website at *college.hmco.com/info/larsonapplied*.

✓CHECKPOINT 6

In Example 6, suppose the rock is thrown straight up at 13 feet per second from a height of 4 feet. The height s (in feet) of the rock is given by $s = -2.7t^2 + 13t + 4$. How much time (in seconds) elapses before the rock strikes the lunar surface? ■

CONCEPT CHECK

1. When using the quadratic formula to solve $4x^2 = 2 - 3x$, what are the values of a, b, and c?
2. The quadratic equation $ax^2 + bx + c = 0$ has two distinct solutions. Does $b^2 - 4ac = 0$? Explain.
3. The area A (in square feet) of a parking lot is represented by $A = x^2 + 9x + 300$. Is it possible for the parking lot to have an area of 275 square feet? Explain.
4. The discriminants of two quadratic equations are 5 and -10. Can the equations have the same solution? Explain.

Skills Review 1.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4 and 1.3.

In Exercises 1–4, simplify the expression.

1. $\sqrt{9 - 4(3)(-12)}$

2. $\sqrt{36 - 4(2)(3)}$

3. $\sqrt{12^2 - 4(3)(4)}$

4. $\sqrt{15^2 + 4(9)(12)}$

In Exercises 5–10, solve the quadratic equation by factoring.

5. $x^2 - x - 2 = 0$

6. $2x^2 + 3x - 9 = 0$

7. $x^2 - 4x = 5$

8. $2x^2 + 13x = 7$

9. $x^2 = 5x - 6$

10. $x(x - 3) = 4$

Exercises 1.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, use the discriminant to determine the number of real solutions of the quadratic equation.

1. $4x^2 - 4x + 1 = 0$

2. $2x^2 - x - 1 = 0$

3. $3x^2 + 4x + 1 = 0$

4. $x^2 + 2x + 4 = 0$

5. $2x^2 - 5x = -5$

6. $3 - 6x = -3x^2$

7. $\frac{1}{5}x^2 + \frac{6}{5}x - 8 = 0$

8. $\frac{1}{3}x^2 - 5x + 25 = 0$

In Exercises 9–30, use the Quadratic Formula to solve the quadratic equation.

9. $2x^2 + x - 1 = 0$

10. $2x^2 - x - 1 = 0$

11. $16x^2 + 8x - 3 = 0$

12. $25x^2 - 20x + 3 = 0$

13. $2 + 2x - x^2 = 0$

14. $x^2 - 10x + 22 = 0$

15. $x^2 + 14x + 44 = 0$

16. $6x = 4 - x^2$

17. $x^2 + 8x - 4 = 0$

18. $4x^2 - 4x - 4 = 0$

19. $12x - 9x^2 = -3$

20. $16x^2 + 22 = 40x$

21. $36x^2 + 24x = 7$

22. $3x + x^2 - 1 = 0$

23. $4x^2 + 4x = 7$

24. $16x^2 - 40x + 5 = 0$

25. $28x - 49x^2 = 4$

26. $9x^2 + 24x + 16 = 0$

27. $8t = 5 + 2t^2$

28. $25h^2 + 80h + 61 = 0$

29. $(y - 5)^2 = 2y$

30. $(x + 6)^2 = -2x$

In Exercises 31–36, use a calculator to solve the quadratic equation. (Round your answer to three decimal places.)

31. $5.1x^2 - 1.7x - 3.2 = 0$

32. $10.4x^2 + 8.6x + 1.2 = 0$

33. $7.06x^2 - 4.85x + 0.50 = 0$

34. $2x^2 - 2.50x - 0.42 = 0$

35. $-0.003x^2 + 0.025x - 0.98 = 0$

36. $-0.005x^2 + 0.101x - 0.193 = 0$

In Exercises 37–46, solve the quadratic equation using any convenient method.

37. $2x^2 + 7 = 2x^2 - x - 4$

38. $x^2 - 2x + 5 = x^2 - 5$

39. $4x^2 - 15 = 25$

40. $3x^2 - 16 = 38$

41. $x^2 + 3x + 1 = 0$

42. $x^2 + 3x - 4 = 0$

43. $(x - 1)^2 = 9$

44. $2x^2 - 4x - 6 = 0$

45. $3x^2 + 5x - 11 = 4(x - 2)$

46. $2x^2 + 4x - 9 = 2(x - 1)^2$

Writing Real-Life Problems In Exercises 47–50, solve the number problem *and* write a real-life problem that could be represented by this verbal model. For instance, an applied problem that could be represented by Exercise 47 is as follows.

The sum of the length and width of a one-story house is 100 feet. The house has 2500 square feet of floor space. What are the length and width of the house?

47. Find two numbers whose sum is 100 and whose product is 2500.

48. One number is 1 more than another number. The product of the two numbers is 72. Find the numbers.

49. One number is 1 more than another number. The sum of their squares is 113. Find the numbers.

50. One number is 2 more than another number. The product of the two numbers is 440. Find the numbers.

Cost Equation In Exercises 51–54, use the cost equation to find the number of units x that a manufacturer can produce for the cost C. (Round your answer to the nearest positive integer.)

51. $C = 0.125x^2 + 20x + 5000$ $C = \$14{,}000$

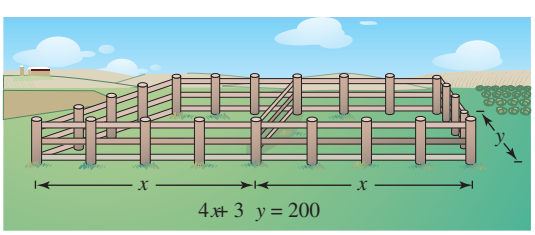
52. $C = 0.5x^2 + 15x + 5000$ $C = \$11{,}500$

53. $C = 800 + 0.04x + 0.002x^2$ $C = \$1680$

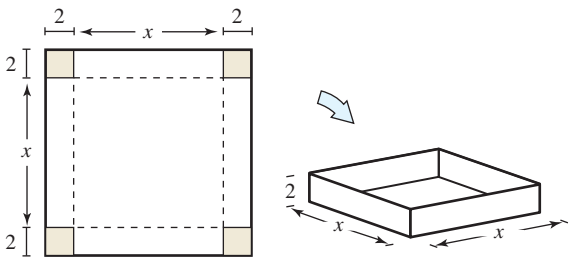
54. $C = 312.5 - 10x + 0.4x^2$ $C = \$900$

55. Seating Capacity A rectangular classroom seats 72 students. If the seats were rearranged with three more seats in each row, the classroom would have two fewer rows. Find the original number of seats in each row.

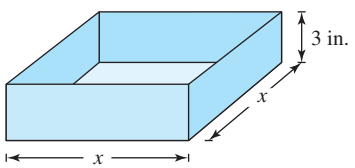
56. Geometry A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals (see figure). Find the dimensions such that the total enclosed area will be 1400 square feet.



57. Geometry An open box is to be made from a square piece of material by cutting two-inch squares from the corners and turning up the sides (see figure). The volume of the finished box is to be 200 cubic inches. Find the size of the original piece of material.



58. Geometry An open box (see figure) is to be constructed from 108 square inches of material. Find the dimensions of the square base. (*Hint:* The surface area is $S = x^2 + 4xh$.)



59. Eiffel Tower You throw a coin straight up from the top of the Eiffel Tower in Paris with a velocity of 20 miles per hour. The building has a height of 984 feet.

(a) Use the position equation to write a mathematical model for the height of the coin.

(b) Find the height of the coin after 4 seconds.

(c) How long will it take before the coin strikes the ground?

60. Sports Some Major League Baseball pitchers can throw a fastball at speeds of up to and over 100 miles per hour. Assume a Major League Baseball pitcher throws a baseball straight up into the air at 100 miles per hour from a height of 6 feet 3 inches.

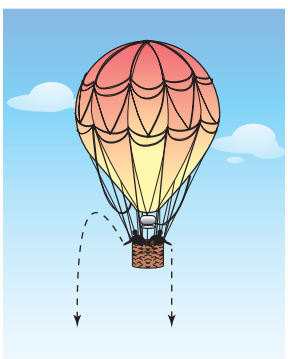
(a) Use the position equation to write a mathematical model for the height of the baseball.

(b) Find the height of the baseball after 4 seconds, 5 seconds, and 6 seconds. What must have occurred sometime in the interval $4 \le t \le 6$? Explain.

(c) How many seconds is the baseball in the air?

61. On the Moon An astronaut on the moon throws a rock straight upward into space. The height s (in feet) of the rock is given by $s = -2.7t^2 + 40t + 5$, where the initial velocity is 40 feet per second, the initial height is 5 feet, and t is the time (in seconds). How long will it take the rock to hit the surface of the moon? If the rock had been thrown with the same initial velocity and height on Earth, how long would it have remained in the air?

62. Hot Air Balloon Two people are floating in a hot air balloon 200 feet above a lake. One person tosses out a coin with an initial velocity of 20 feet per second. One second later, the balloon is 20 feet higher and the other person drops another coin (see figure). The position equation for the first coin is $s = -16t^2 + 20t + 200$, and the position equation for the second coin is $s = -16t^2 + 220$. Which coin will hit the water first? (*Hint:* Remember that the first coin was tossed one second before the second coin was dropped.)



Falling Objects In Exercises 63 and 64, use the following information. The position equation for falling objects on Earth is of the form

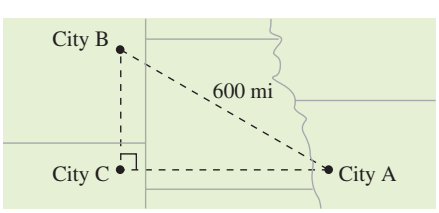
$$s = -16t^2 + v_0t + s_0$$

where s is the height of the object (in feet), v_0 is the initial velocity (in feet per second), t is the time (in seconds), and s_0 is the initial height (in feet).

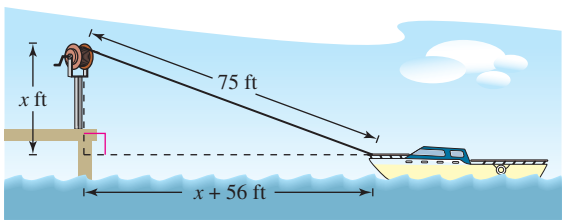
63. ***MAKE A DECISION*** Would a rock thrown upward from an initial height of 6 feet with an initial velocity of 27 feet per second take longer to reach the ground on Earth or on the moon? (See Example 6.)

64. ***MAKE A DECISION*** Would a rock thrown downward from an initial height of 6 feet with an initial velocity of -14 feet per second take longer to reach the ground on Earth or on the moon? (See Example 6.)

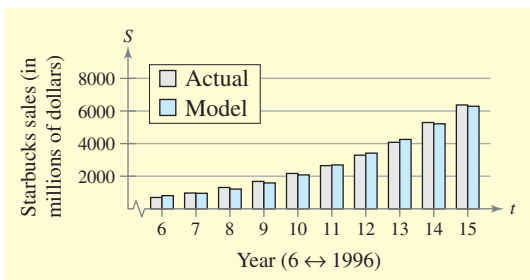
65. Flying Distance A small commuter airline flies to three cities whose locations form the vertices of a right triangle (see figure). The total flight distance (from City A to City B to City C and back to City A) is 1400 miles. It is 600 miles between the two cities that are farthest apart. Find the other two distances between cities.



66. Distance from a Dock A windlass is used to tow a boat to a dock. The figure shows a situation in which there is 75 feet of rope extended to the boat. How far is the boat from the dock?



67. Starbucks The total sales S (in millions of dollars) for Starbucks from 1996 to 2005 can be approximated by the model $S = 58.155t^2 - 612.3t + 2387.1$, $6 \le t \le 15$, where t represents the year, with $t = 6$ corresponding to 1996. The figure shows the actual sales and the sales represented by the model. *(Source: Starbucks Corporation)*

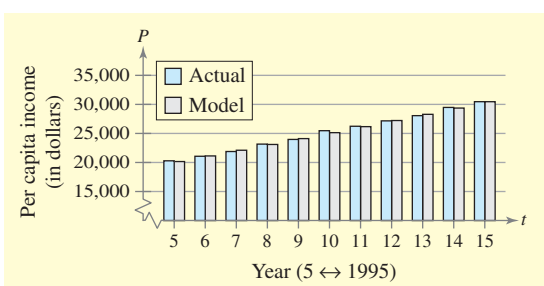


(a) Use the model to estimate the year when total sales were about \$4 billion.

(b) Use the model to predict the year when the total sales were about \$6.2 billion.

(c) Starbucks sales were expected to reach \$9.45 billion in 2007. Does the model agree? Explain your reasoning.

68. Per Capita Income The per capita income P (in dollars) in the United States from 1995 to 2005 can be approximated by the model $P = 7.14t^2 + 887.5t + 15{,}544$, $5 \le t \le 15$, where t represents the year, with $t = 5$ corresponding to 1995. The figure shows the actual per capita income and the per capita income represented by the model. *(Source: U.S. Bureau of Economic Analysis)*



(a) Use the model to estimate the year in which the per capita income was about $26,500.

(b) Use the model to predict the year in which the per capita income is about $34,000.

69. Blood Oxygen Level Doctors treated a patient at an emergency room from 2:00 P.M. to 7:00 P.M. The patient's blood oxygen level L (in percent) during this time period can be modeled by

$$L = -0.270t^2 + 3.59t + 83.1, \quad 2 \le t \le 7$$

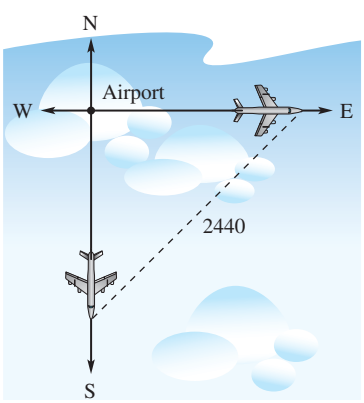
where t represents the time of day, with $t = 2$ corresponding to 2:00 P.M. Use the model to estimate the time (rounded to the nearest hour) when the patient's blood oxygen level was 93%.

70. Prescription Drugs The total amounts A (in billions of dollars) projected by the industry to be spent on prescription drugs in the United States from 2002 to 2012 can be approximated by the model.

$$A = 0.89t^2 + 15.9t + 126, \quad 2 \le t \le 12$$

where t represents the year, with $t = 2$ corresponding to 2002. Use the model to predict the year in which the total amount spent on prescription drugs will be about $374,000,000,000. *(Source: U.S. Center for Medicine and Medicaid Services)*

71. Flying Speed Two planes leave simultaneously from the same airport, one flying due east and the other due south (see figure). The eastbound plane is flying 50 miles per hour faster than the southbound plane. After 3 hours the planes are 2440 miles apart. Find the speed of each plane.

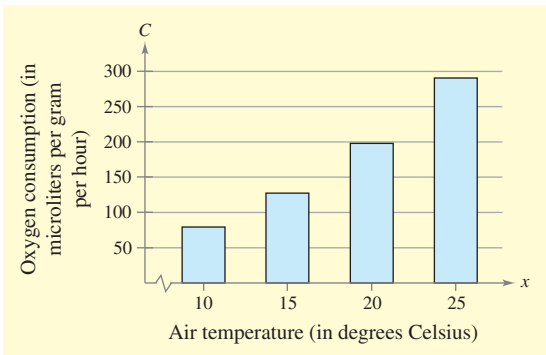


72. Flying Speed Two planes leave simultaneously from the same airport, one flying due east and the other due south. The eastbound plane is flying 100 miles per hour faster than the southbound plane. After 2 hours the planes are 1500 miles apart. Find the speed of each plane.

73. Biology The metabolic rate of an ectothermic organism increases with increasing temperature within a certain range. The graph shows experimental data for the oxygen consumption C (in microliters per gram per hour) of a beetle at certain temperatures. The data can be approximated by the model

$$C = 0.45x^2 - 1.65x + 50.75, \quad 10 \le x \le 25$$

where x is the temperature in degrees Celsius.



(a) The oxygen consumption is 150 microliters per gram per hour. What is the air temperature?

(b) The temperature is increased from 10°C to 20°C. The oxygen consumption is increased by approximately what factor?

74. Total Revenue The demand equation for a product is

$$p = 60 - 0.0004x$$

where p is the price per unit and x is the number of units sold. The total revenue R for selling x units is given by

$$R = xp.$$

How many units must be sold to produce a revenue of $220,000?

75. Total Revenue The demand equation for a product is

$$p = 50 - 0.0005x$$

where p is the price per unit and x is the number of units sold. The total revenue R for selling x units is given by

$$R = xp.$$

How many units must be sold to produce a revenue of $250,000?

76. When the Quadratic Formula is used to solve certain problems, such as the problem in Example 5 on page 108, why is only one solution used?

77. Extended Application To work an extended application analyzing the sales per share of Starbucks Corporation from 1992 to 2005, visit this text's website at *college.hmco.com*. *(Data Source: Starbucks Corporation)*

Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, solve the equation and check your solution.

1. $3(x - 2) - 4(2x + 5) = 4$

2. $\frac{3x + 3}{5x - 2} = \frac{3}{4}$

3. $\frac{2}{x(x - 1)} + \frac{1}{x} = \frac{1}{x - 4}$

4. $(x + 3)^2 - x^2 = 6(x + 2)$

5. Describe how you can check your answers to Exercises 1-4 using your graphing utility.

In Exercises 6 and 7, solve the equation. (Round your solution to three decimal places.)

6. $\frac{x}{2.004} - \frac{x}{5.128} = 100$

7. $0.378x + 0.757(500 - x) = 215$

In Exercises 8 and 9, write an algebraic equation for the verbal description. Find the solution if possible and check.

8. A company has fixed costs of \$30,000 per month and variable costs of \$8.50 per unit manufactured. The company has \$200,000 available each month to cover monthly costs. How many units can the company manufacture?

9. The demand equation for a product is $p = 75 - 0.0002x$, where p is the price per unit and x is the number of units sold. The total revenue R for selling x units is given by $R = xp$. How many units must be sold to produce a revenue of \$300,000?

In Exercises 10–15, solve the quadratic equation by the indicated method.

10. *Factoring:* $3x^2 + 13x = 10$

11. *Extracting roots:* $3x^2 = 15$

12. *Extracting roots:* $(x + 3)^2 = 17$

13. *Quadratic Formula:* $2x + x^2 = 5$

14. *Quadratic Formula:* $3x^2 + 7x - 2 = 0$

15. *Quadratic Formula:* $3x^2 - 4.50x - 0.32 = 0$

In Exercises 16 and 17, use the discriminant to determine the number of real solutions of the quadratic equation.

16. $2x^2 - 4x + 9 = 0$

17. $4x^2 - 12x + 9 = 0$

18. Describe how you would convince a fellow student that $(x + 3)^2 = x^2 + 6x + 9$.

19. A rock is dropped from a height of 300 feet. How long will it take the rock to hit the ground?

20. An open box has a square base and a height of 6 inches. The volume of the box is 384 cubic inches. Find the dimensions of the box.

Section 1.5 Other Types of Equations

- Solve a polynomial equation by factoring.
- Rewrite and solve an equation involving radicals or rational exponents.
- Rewrite and solve an equation involving fractions or absolute values.
- Construct and use a nonquadratic model to solve an application problem.

Polynomial Equations

In this section you will extend the techniques for solving equations to nonlinear and nonquadratic equations. At this point in the text, you have only three basic methods for solving nonlinear equations—*factoring, extracting roots,* and the *Quadratic Formula.* So the main goal of this section is to learn to *rewrite* nonlinear equations in a form to which you can apply one of these methods.

STUDY TIP

When solving an equation, avoid dividing each side by a common variable factor to simplify. You may lose solutions. For instance, if you divide each side by x^2 in Example 1, you lose the solution $x = 0$. Also, when solving an equation by factoring, be sure to set each variable factor equal to zero to find all of the possible solutions.

Example 1 Solving a Polynomial Equation by Factoring

Solve $3x^4 = 48x^2$.

SOLUTION The basic approach is first to write the polynomial equation in general form with zero on one side, then to factor the other side, and finally to set each factor equal to zero and solve.

$3x^4 = 48x^2$			Write original equation.
$3x^4 - 48x^2 = 0$			Write in general form.
$3x^2(x^2 - 16) = 0$			Factor out common factor.
$3x^2(x + 4)(x - 4) = 0$			Difference of two squares
$3x^2 = 0$	⇨	$x = 0$	Set 1st factor equal to 0.
$x + 4 = 0$	⇨	$x = -4$	Set 2nd factor equal to 0.
$x - 4 = 0$	⇨	$x = 4$	Set 3rd factor equal to 0.

You can check these solutions by substituting in the original equation, as follows.

CHECK

$3x^4 = 48x^2$	Write original equation.
$3(0)^4 = 48(0)^2$	0 checks. ✓
$3(-4)^4 = 48(-4)^2$	-4 checks. ✓
$3(4)^4 = 48(4)^2$	4 checks. ✓

After checking, you can conclude that the solutions are $x = 0, x = -4$, and $x = 4$.

✓CHECKPOINT 1

Solve $3x^3 = 3x$. ■

Example 2 Solving a Polynomial Equation by Factoring

Solve $x^3 - 3x^2 - 3x + 9 = 0$.

SOLUTION

$$x^3 - 3x^2 - 3x + 9 = 0$$ Write original equation.

$$x^2(x - 3) - 3(x - 3) = 0$$ Group terms.

$$(x - 3)(x^2 - 3) = 0$$ Factor by grouping.

$$x - 3 = 0 \Rightarrow x = 3$$ Set 1st factor equal to 0.

$$x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$$ Set 2nd factor equal to 0.

The solutions are $x = 3$, $x = \sqrt{3}$, and $x = -\sqrt{3}$. Check these in the original equation. Notice that this polynomial has a degree of 3 and has three solutions.

✓CHECKPOINT 2

Solve $x^3 - x^2 - 2x + 2 = 0$. ■

DISCOVERY

What do you observe about the degrees of the polynomials in Examples 1, 2, and 3 and the possible numbers of solutions of the equations? Does your observation apply to the quadratic equations in Sections 1.3 and 1.4?

Occasionally, mathematical models involve equations that are of **quadratic type.** In general, an equation is of quadratic type if it can be written in the form

$$au^2 + bu + c = 0$$

where $a \neq 0$ and u is an algebraic expression.

Example 3 Solving an Equation of Quadratic Type

Solve $x^4 - 3x^2 + 2 = 0$.

SOLUTION This equation is of quadratic type with $u = x^2$.

$$(x^2)^2 - 3(x^2) + 2 = 0$$

To solve this equation, you can factor the left side of the equation as the product of two second-degree polynomials.

$$x^4 - 3x^2 + 2 = 0$$ Write original equation.

$$(x^2 - 1)(x^2 - 2) = 0$$ Partially factor.

$$(x + 1)(x - 1)(x^2 - 2) = 0$$ Completely factor.

$$x + 1 = 0 \Rightarrow x = -1$$ Set 1st factor equal to 0.

$$x - 1 = 0 \Rightarrow x = 1$$ Set 2nd factor equal to 0.

$$x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$$ Set 3rd factor equal to 0.

The solutions are $x = -1$, $x = 1$, $x = \sqrt{2}$, and $x = -\sqrt{2}$. Check these in the original equation. Notice that this polynomial has a degree of 4 and has four solutions.

✓CHECKPOINT 3

Solve $x^4 - 5x^2 + 4 = 0$. ■

Solving Equations Involving Radicals

The steps involved in solving the remaining equations in this section will often introduce *extraneous solutions,* as discussed in Section 1.1. Operations such as squaring each side of an equation, raising each side of an equation to a rational power, or multiplying each side of an equation by a variable quantity all create this potential danger. So, when you use any of these operations, checking of solutions is crucial.

Example 4 An Equation Involving a Radical

Solve $\sqrt{2x + 7} - x = 2$.

SOLUTION

$$\sqrt{2x + 7} - x = 2$$ Write original equation.

$$\sqrt{2x + 7} = x + 2$$ Isolate the square root.

$$2x + 7 = x^2 + 4x + 4$$ Square each side.

$$0 = x^2 + 2x - 3$$ Write in general form.

$$0 = (x + 3)(x - 1)$$ Factor.

$$x + 3 = 0 \Rightarrow x = -3$$ Set 1st factor equal to 0.

$$x - 1 = 0 \Rightarrow x = 1$$ Set 2nd factor equal to 0.

By checking these values, you can determine that the only solution is $x = 1$.

✓CHECKPOINT 4

Solve $\sqrt{3x} - 6 = 0$. ■

STUDY TIP

The basic technique used in Example 5 is to isolate the factor with the rational exponent and raise each side to the reciprocal power. In Example 4, this is equivalent to isolating the square root and squaring each side.

Example 5 An Equation Involving a Rational Exponent

Solve $4x^{3/2} - 8 = 0$.

SOLUTION

$$4x^{3/2} - 8 = 0$$ Write original equation.

$$4x^{3/2} = 8$$ Add 8 to each side.

$$x^{3/2} = 2$$ Isolate $x^{3/2}$.

$$x = 2^{2/3}$$ Raise each side to the $\frac{2}{3}$ power.

$$x \approx 1.587$$ Round to three decimal places.

CHECK

$$4x^{3/2} - 8 = 0$$ Write original equation.

$$4(2^{2/3})^{3/2} \stackrel{?}{=} 8$$ Substitute $2^{2/3}$ for x.

$$4(2) \stackrel{?}{=} 8$$ Power of a Power Property

$$8 = 8$$ Solution checks. ✓

✓CHECKPOINT 5

Solve $2x^{3/4} - 54 = 0$. ■

Equations Involving Fractions or Absolute Values

In Section 1.1, you learned how to solve equations involving fractions. Recall that the first step is to multiply each term of the equation by the least common denominator (LCD).

Example 6 An Equation Involving Fractions

Solve $\frac{2}{x} = \frac{3}{x-2} - 1$.

SOLUTION For this equation, the LCD of the three terms is $x(x-2)$, so begin by multiplying each term of the equation by this expression.

$$\frac{2}{x} = \frac{3}{x-2} - 1$$ Write original equation.

$$x(x-2)\frac{2}{x} = x(x-2)\frac{3}{x-2} - x(x-2)(1)$$ Multiply each term by LCD.

$$2(x-2) = 3x - x(x-2), \quad x \neq 0, 2$$ Simplify.

$$2x - 4 = -x^2 + 5x$$ Distributive Property

$$x^2 - 3x - 4 = 0$$ Write in general form.

$$(x-4)(x+1) = 0$$ Factor.

$$x - 4 = 0 \Rightarrow x = 4$$ Set 1st factor equal to 0.

$$x + 1 = 0 \Rightarrow x = -1$$ Set 2nd factor equal to 0.

Notice that the values $x = 0$ and $x = 2$ are excluded from the domains of the fractions because they result in division by zero. So, both $x = 4$ and $x = -1$ are possible solutions.

CHECK

$$\frac{2}{x} = \frac{3}{x-2} - 1$$ Write original equation.

$$\frac{2}{4} \stackrel{?}{=} \frac{3}{4-2} - 1$$ Substitute 4 for x.

$$\frac{1}{2} = \frac{3}{2} - 1$$ 4 checks. ✓

$$\frac{2}{-1} \stackrel{?}{=} \frac{3}{-1-2} - 1$$ Substitute -1 for x.

$$-2 = -1 - 1$$ -1 checks. ✓

The solutions are $x = 4$ and $x = -1$.

✓CHECKPOINT 6

Solve $\frac{3}{x} + \frac{1}{x-2} = 2$. ■

To solve an equation involving an absolute value, remember that the expression inside the absolute value signs can be positive or negative. This results in *two* separate equations, each of which must be solved. For instance, the equation

$$|x - 2| = 3$$

results in the two equations

$$x - 2 = 3 \quad \text{and} \quad -(x - 2) = 3$$

which implies that the original equation has two solutions: $x = 5$ and $x = -1$. When setting up the negative expression, it is important to remember to place parentheses around the entire expression that is inside the absolute value bars. After you set up the two equations, solve each one independently.

Example 7 An Equation Involving Absolute Value

Solve $|x^2 - 3x| = -4x + 6$.

SOLUTION Because the variable expression inside the absolute value signs can be positive or negative, you must solve the following two equations.

First Equation

$$x^2 - 3x = -4x + 6$$ Use positive expression.

$$x^2 + x - 6 = 0$$ Write in general form.

$$(x + 3)(x - 2) = 0$$ Factor.

$$x + 3 = 0 \Rightarrow x = -3$$ Set 1st factor equal to 0.

$$x - 2 = 0 \Rightarrow x = 2$$ Set 2nd factor equal to 0.

Second Equation

$$-(x^2 - 3x) = -4x + 6$$ Use negative expression.

$$x^2 - 7x + 6 = 0$$ Write in general form.

$$(x - 1)(x - 6) = 0$$ Factor.

$$x - 1 = 0 \Rightarrow x = 1$$ Set 1st factor equal to 0.

$$x - 6 = 0 \Rightarrow x = 6$$ Set 2nd factor equal to 0.

CHECK

$$|(-3)^2 - 3(-3)| = -4(-3) + 6$$ -3 checks. ✓

$$|(2)^2 - 3(2)| \neq -4(2) + 6$$ 2 does not check.

$$|(1)^2 - 3(1)| = -4(1) + 6$$ 1 checks. ✓

$$|(6)^2 - 3(6)| \neq -4(6) + 6$$ 6 does not check.

The solutions are $x = -3$ and $x = 1$.

✓CHECKPOINT 7

Solve $|x^2 - 3| = 5x - 3$. ■

Applications

It would be virtually impossible to categorize all of the many different types of applications that involve nonlinear and nonquadratic models. However, from the few examples and exercises that follow, we hope you will gain some appreciation for the variety of applications that involve such models.

Example 8 Reduced Rates



Stockbyte/Getty Images

A ski club charters a bus for a ski trip at a cost of $700. In an attempt to lower the bus fare per skier, the club invites five nonmembers to go along. As a result, the fare per skier decreases by $7. How many club members are going on the trip?

SOLUTION Begin the solution by creating a verbal model and assigning labels, as follows.

Verbal Model: Cost per skier $\cdot$ Number of skiers $=$ Cost of trip

Labels:

Cost of trip $= 700$ (dollars)

Number of ski club members $= x$ (people)

Number of skiers $= x + 5$ (people)

Original cost per member $= \dfrac{700}{x}$ (dollars per person)

Cost per skier $= \dfrac{700}{x} - 7$ (dollars per person)

Equation:

$$\left(\frac{700}{x} - 7\right)(x + 5) = 700$$ Original equation

$$\left(\frac{700 - 7x}{x}\right)(x + 5) = 700$$ Rewrite first factor.

$$(700 - 7x)(x + 5) = 700x, \; x \neq 0$$ Multiply each side by x.

$$700x + 3500 - 7x^2 - 35x = 700x$$ Multiply factors.

$$-7x^2 - 35x + 3500 = 0$$ Subtract $700x$ from each side.

$$x^2 + 5x - 500 = 0$$ Divide each side by -7.

$$(x + 25)(x - 20) = 0$$ Factor left side of equation.

$x + 25 = 0 \Rightarrow x = -25$ Set 1st factor equal to 0.

$x - 20 = 0 \Rightarrow x = 20$ Set 2nd factor equal to 0.

Only the positive x-value makes sense in the context of the problem. So, you can conclude that 20 ski club members are going on the trip. Check this in the original statement of the problem.

✓CHECKPOINT 8

In Example 8, suppose the ski club invites eight nonmembers to join the trip. As a result, the fare per skier decreases by $10. How many club members are going on the trip? ■

Interest earned on a savings account is calculated by one of three basic methods: simple interest, interest compounded n times per year, and interest compounded continuously. The next example uses the formula for interest that is compounded n times per year,

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

In this formula, A is the balance in the account, P is the principal (or original deposit), r is the annual interest rate (in decimal form), n is the number of compoundings per year, and t is the time in years. In Chapter 4, you will study the derivation of this formula for compound interest.

Example 9 Compound Interest

When you were born, your grandparents deposited $5000 in a savings account earning interest compounded quarterly. On your 25th birthday the balance of the account is $25,062.59. What is the average annual interest rate of the account?

SOLUTION

Formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Labels:

Balance $= A = 25{,}062.59$	(dollars)
Principal $= P = 5000$	(dollars)
Time $= t = 25$	(years)
Compoundings per year $= n = 4$	(compoundings)
Annual interest rate $= r$	(percent in decimal form)

Equation:

$$25{,}062.59 = 5000\left(1 + \frac{r}{4}\right)^{4(25)}$$ Substitute.

$$\frac{25{,}062.59}{5000} = \left(1 + \frac{r}{4}\right)^{100}$$ Divide each side by 5000.

$$5.0125 \approx \left(1 + \frac{r}{4}\right)^{100}$$ Use a calculator.

$$(5.0125)^{1/100} \approx 1 + \frac{r}{4}$$ Raise each side to reciprocal power.

$$1.01625 \approx 1 + \frac{r}{4}$$ Use a calculator.

$$0.01625 \approx \frac{r}{4}$$ Subtract 1 from each side.

$$0.065 \approx r$$ Multiply each side by 4.

The average annual interest rate is about $0.065 = 6.5\%$. Check this in the original statement of the problem.

✓CHECKPOINT 9

You placed $1000 in an account earning interest compounded monthly. After 3 years, the account balance is $1144.25. What is the annual interest rate? ■

Example 10 **Market Research**

The marketing department of a publishing company is asked to determine the price of a book. The department determines that the demand for the book depends on the price of the book according to the model

$$p = 40 - \sqrt{0.0001x + 1}, \quad 0 \le x \le 15{,}990{,}000$$

where p is the price per book in dollars and x is the number of books sold at the given price. For instance, in Figure 1.13, note that if the price were \$39, then (according to the model) no one would be willing to buy the book. On the other hand, if the price were \$17.60, 5 million copies could be sold. The publisher set the price at \$12.95. How many copies can the publisher expect to sell?

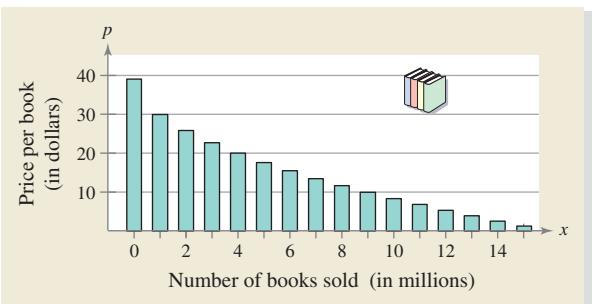


FIGURE 1.13

SOLUTION

$$\begin{aligned}
p &= 40 - \sqrt{0.0001x + 1} && \text{Write given model.}\\
12.95 &= 40 - \sqrt{0.0001x + 1} && \text{Set price at \$12.95.}\\
\sqrt{0.0001x + 1} &= 27.05 && \text{Isolate the radical.}\\
0.0001x + 1 &= 731.7025 && \text{Square each side.}\\
0.0001x &= 730.7025 && \text{Subtract 1 from each side.}\\
x &= 7{,}307{,}025 && \text{Divide each side by 0.0001.}
\end{aligned}$$

So, by setting the book's price at \$12.95, the publisher can expect to sell about 7.3 million copies.

✓CHECKPOINT 10

In Example 10, suppose the publisher set the price at \$19.95. How many copies can the publisher expect to sell? ■

CONCEPT CHECK

1. What method would you use to solve $x^3 + 3x^2 - 9x - 27 = 0$?
2. Explain why $x^6 + 2x^3 + 1 = 0$ is of the quadratic type, but $x^4 + 3x + 2 = 0$ is not.
3. How do you introduce an extraneous solution when solving $\sqrt{2x} + 4 = x$?
4. What two equations do you need to write in order to solve $|3x^2 - 5x| = 5$?

Skills Review 1.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.3 and 1.4.

In Exercises 1–10, find the real solution(s) of the equation.

1. $x^2 - 22x + 121 = 0$
2. $x(x - 20) + 3(x - 20) = 0$
3. $(x + 20)^2 = 625$
4. $5x^2 + x = 0$
5. $3x^2 + 4x - 4 = 0$
6. $12x^2 + 8x - 55 = 0$
7. $x^2 + 4x - 5 = 0$
8. $4x^2 + 4x - 15 = 0$
9. $x^2 - 3x + 1 = 0$
10. $x^2 - 4x + 2 = 0$

Exercises 1.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–20, find the real solution(s) of the polynomial equation. Check your solutions.

1. $x^3 - 2x^2 - 3x = 0$
2. $20x^3 - 125x = 0$
3. $4x^4 - 18x^2 = 0$
4. $2x^4 - 15x^3 + 18x^2 = 0$
5. $x^4 - 81 = 0$
6. $x^6 - 64 = 0$
7. $5x^3 + 30x^2 + 45x = 0$
8. $9x^4 - 24x^3 + 16x^2 = 0$
9. $x^3 - 7x^2 - 4x + 28 = 0$
10. $x^3 + 2x^2 + 3x + 6 = 0$
11. $x^4 - x^3 + x - 1 = 0$
12. $x^4 + 2x^3 - 8x - 16 = 0$
13. $x^4 - 12x^2 + 11 = 0$
14. $x^4 - 29x^2 + 100 = 0$
15. $x^4 + 5x^2 - 36 = 0$
16. $x^4 - 4x^2 + 3 = 0$
17. $4x^4 - 65x^2 + 16 = 0$
18. $36t^4 + 29t^2 - 7 = 0$
19. $x^6 + 7x^3 - 8 = 0$
20. $x^6 + 3x^3 + 2 = 0$

In Exercises 21–34, find the real solution(s) of the radical equation. Check your solutions.

21. $\sqrt{2x} - 10 = 0$
22. $4\sqrt{x} - 3 = 0$
23. $\sqrt{x - 10} - 4 = 0$
24. $\sqrt{5 - x} - 3 = 0$
25. $\sqrt[3]{2x + 5} + 3 = 0$
26. $\sqrt[3]{3x + 1} - 5 = 0$
27. $2x + 9\sqrt{x} - 5 = 0$
28. $6x - 7\sqrt{x} - 3 = 0$
29. $x = \sqrt{11x - 30}$
30. $2x - \sqrt{15 - 4x} = 0$
31. $-\sqrt{26 - 11x} + 4 = x$
32. $x + \sqrt{31 - 9x} = 5$
33. $\sqrt{x + 1} - 3x = 1$
34. $\sqrt{2x + 1} + x = 7$

In Exercises 35–40, find the real solution(s) of the equation involving rational exponents. Check your solutions.

35. $(x - 5)^{2/3} = 16$
36. $(x + 3)^{4/3} = 16$
37. $(x + 3)^{3/2} = 8$
38. $(x^2 + 2)^{2/3} = 9$
39. $(x^2 - 5)^{2/3} = 16$
40. $(x^2 - x - 22)^{4/3} = 16$

In Exercises 41–48, find the real solution(s) of the equation involving fractions. Check your solutions.

41. $\frac{1}{x} - \frac{1}{x + 1} = 3$
42. $\frac{x}{x^2 - 4} + \frac{1}{x + 2} = 3$
43. $\frac{20 - x}{x} = x$
44. $\frac{4}{x} - \frac{5}{3} = \frac{x}{6}$
45. $\frac{1}{x} = \frac{4}{x - 1} + 1$
46. $x + \frac{9}{x + 1} = 5$
47. $\frac{4}{x + 1} - \frac{3}{x + 2} = 1$
48. $\frac{x + 1}{3} - \frac{x + 1}{x + 2} = 0$

In Exercises 49–56, find the real solution(s) of the equation involving absolute value. Check your solutions.

49. $|x + 1| = 2$
50. $|x - 2| = 3$
51. $|2x - 1| = 5$
52. $|3x + 2| = 7$
53. $|x| = x^2 + x - 3$
54. $|x^2 + 6x| = 3x + 18$
55. $|x - 10| = x^2 - 10x$
56. $|x + 1| = x^2 - 5$

57. Error Analysis Find the error(s) in the solution.

$$\sqrt{3x} = \sqrt{7x + 4}$$
$$3x^2 = 7x + 4$$
$$x = \frac{-7 \pm \sqrt{7^2 - 4(3)(4)}}{2(3)}$$
$$x = -1 \text{ and } x = -\frac{4}{3}$$

58. Error Analysis Find the error(s) in the solution.

$$\sqrt{6 - 2x} - 3 = 0$$
$$6 - 2x + 9 = 0$$
$$-2x = -15$$
$$x = \frac{15}{2}$$

In Exercises 59–62, use a calculator to find the real solutions of the equation. (Round your answers to three decimal places.)

59. $3.2x^4 - 1.5x^2 - 2.1 = 0$

60. $7.08x^6 + 4.15x^3 - 9.6 = 0$

61. $1.8x - 6\sqrt{x} - 5.6 = 0$

62. $4x + 8\sqrt{x} + 3.6 = 0$

63. Sharing the Cost A college charters a bus for $1700 to take a group of students to the Fiesta Bowl. When six more students join the trip, the cost per student decreases by $7.50. How many students were in the original group?

64. Sharing the Cost Three students plan to share equally in the rent of an apartment. By adding a fourth person, each person could save $125 a month. How much is the monthly rent of the apartment?

65. Compound Interest A deposit of $3000 reaches a balance of $4296.16 after 6 years. The interest on the account is compounded monthly. What is the annual interest rate for this investment?

66. Compound Interest A sales representative describes a "guaranteed investment fund" that is offered to new investors. You are told that if you deposit $15,000 in the fund you will be guaranteed to receive a total of at least $40,000 after 20 years. (a) If after 20 years you received the minimum guarantee, what annual interest rate did you receive? (b) If after 20 years you received $48,000, what annual interest rate did you receive? (Assume that the interest in the fund is compounded quarterly.)

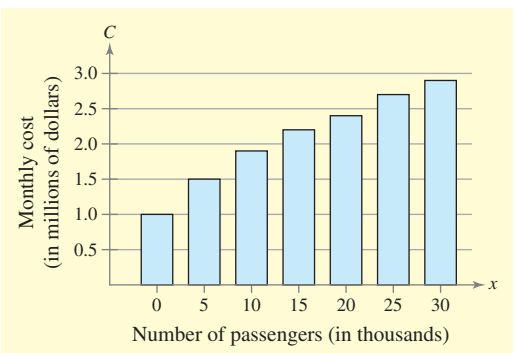
67. Borrowing Money You borrow $300 from a friend and agree to pay the money back, plus $20 in interest, after 6 months. Assuming that the interest is compounded monthly, what annual interest rate are you paying?

68. Cash Advance You take out a cash advance of $1000 on a credit card. After 2 months, you owe $1041.93. The interest is compounded monthly. What is the annual interest rate for this cash advance?

69. Airline Passengers An airline offers daily flights between Chicago and Denver. The total monthly cost C (in millions of dollars) of these flights is modeled by

$$C = \sqrt{0.25x + 1}$$

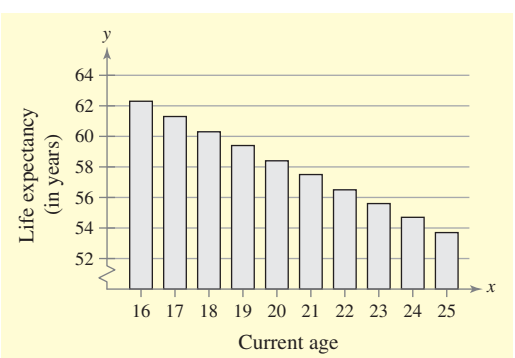
where x is the number of passengers flying that month in thousands (see figure). The total cost of the flights for a month is 3.5 million dollars. Use the model to determine how many passengers flew that month.



70. Life Expectancy The life expectancy of a person who is 16 to 25 years old can be modeled by

$$y = \sqrt{1.244x^2 - 161.16x + 6138.6}, \quad 16 \le x \le 25$$

where y represents the number of additional years the person is expected to live and x represents the person's current age. *(Source: U.S. National Center for Health Statistics)*



(a) Determine the life expectancies of persons who are 18, 20, and 22 years old.

(b) A person's life expectancy is 62 years. Use the model to determine the age of the person.

71. Life Expectancy The life expectancy of a person who is 48 to 65 years old can be modeled by

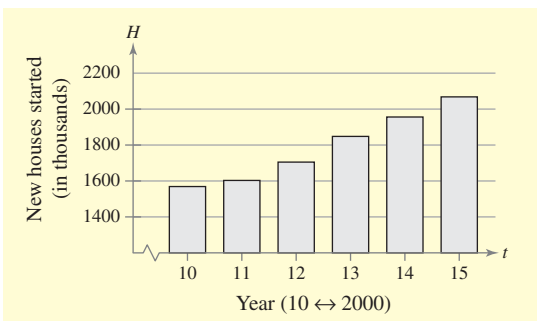
$$y = \sqrt{0.874x^2 - 140.07x + 5752.5}, \quad 48 \le x \le 65$$

where y represents the number of additional years the person is expected to live and x represents the person's current age. A person's life expectancy is 20 years. How old is the person? *(Source: U.S. National Center for Health Statistics)*

72. New Homes The number of new privately owned housing projects H (in thousands) started from 2000 to 2005 can be modeled by

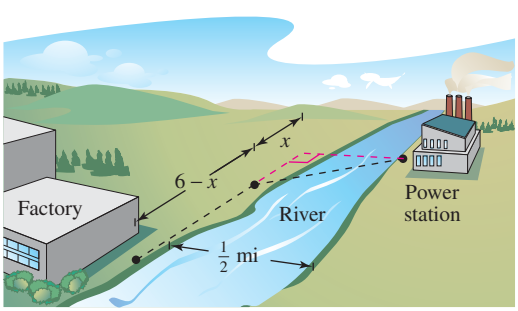
$$H = -1993 + 204.9t + \frac{15{,}005}{t}, \quad 10 \le t \le 15$$

where t represents the year, with $t = 10$ corresponding to 2000 (see figure). Use the model to predict the year in which about 2,500,000 new housing projects were started. *(Source: U.S. Census Bureau)*

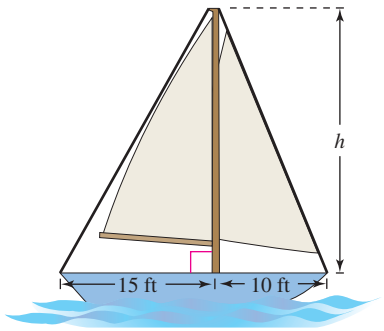


73. Market Research The demand equation for a product is modeled by $p = 40 - \sqrt{0.01x + 1}$, where x is the number of units demanded per day and p is the price per unit. Find the demand when the price is set at \$13.95. Explain why this model is only valid for $0 \le x \le 159{,}900$.

74. Power Line A power station is on one side of a river that is $\frac{1}{2}$ mile wide. A factory is 6 miles downstream on the other side of the river. It costs \$18 per foot to run power lines over land and \$24 per foot to run them under water. The project's cost is \$616,877.27. Find the length x as labeled in the figure.



75. Sailboat Stays Two stays for the mast on a sailboat are attached to the boat at two points, as shown in the figure. One point is 10 feet from the mast and the other point is 15 feet from the mast. The total length of the two stays is 35 feet. How high on the mast are the stays attached?



76. Flour Production A company weighs each 16-ounce bag of flour it produces. After production, any bag that does not weigh within 0.4 ounce of 16 ounces cannot be sold. Solve the equation $|x - 16| = 0.4$ to find the least and greatest acceptable weights of a 16-ounce bag of flour.

77. Sugar Production A company weighs each 80-ounce bag of sugar it produces. After production, any bag that does not weigh within 1.2 ounces of 80 ounces cannot be sold. Solve the equation $|x - 80| = 1.2$ to find the least and greatest acceptable weights of an 80-ounce bag of sugar.

78. Work Rate With only the cold water valve open, it takes 8 minutes to fill the tub of a washing machine. With both the hot and cold water valves open, it takes 5 minutes. The time it takes for the tub to fill with only the hot water valve open can be modeled by the equation

$$\frac{1}{8} + \frac{1}{t} = \frac{1}{5}$$

where t is the time (in minutes) for the tub to fill. How long does it take for the tub of the washing machine to fill with only the hot water valve open?

79. Community Service You and a friend volunteer to paint a small house as a community service project. Working alone, you can paint the house in 15 hours. Your friend can paint the house in 18 hours working alone. How long will it take both of you, working together, to paint the house?

80. Community Service You and a friend volunteer to paint a large house as a community service project. Working alone, you can paint the house in 28 hours. Your friend can paint the house in 25 hours working alone. How long will it take both of you, working together, to paint the house?

Section 1.6

Linear Inequalities

- Write bounded and unbounded intervals using inequalities or interval notation.
- Solve and graph a linear inequality.
- Construct and use a linear inequality to solve an application problem.

Introduction

Simple inequalities are used to *order* real numbers. The inequality symbols $<, \leq, >$, and $\geq$ are used to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality

$$x \geq 3$$

denotes all real numbers x that are greater than or equal to 3.

In this section you will expand your work with inequalities to include more involved statements such as

$$5x - 7 > 3x + 9 \quad \text{and} \quad -3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. Such values are **solutions** and are said to **satisfy** the inequality. The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality. For instance, the solution set of

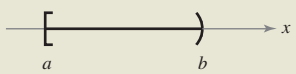
$$x + 3 > 4$$

is all real numbers that are greater than 1.

The set of all points on the real number line that represent the solution set of an inequality is the **graph** of the inequality. Graphs of many types of inequalities consist of intervals on the real number line. The four different types of **bounded** intervals are summarized below.

Bounded Intervals on the Real Number Line

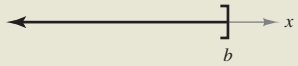
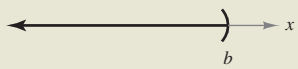
Let a and b be real numbers such that $a < b$. The following intervals on the real number line are **bounded.** The numbers a and b are the **endpoints** of each interval.

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, b]$	Closed	$a \leq x \leq b$	
(a, b)	Open	$a < x < b$	
$[a, b)$		$a \leq x < b$	
$(a, b]$		$a < x \leq b$	

Note that a closed interval contains both of its endpoints and an open interval does not contain either of its endpoints. Often, the solution of an inequality is an interval on the real line that is **unbounded.** For instance, the interval consisting of all positive numbers is unbounded. The symbols ∞, **positive infinity,** and $-\infty$, **negative infinity,** do not represent real numbers. They are simply convenient symbols used to describe the unboundedness of an interval such as $(1, \infty)$.

Unbounded Intervals on the Real Number Line

Let a and b be real numbers. The following intervals on the real number line are **unbounded.**

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, \infty)$		$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$		$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire real line	$-\infty < x < \infty$	

Example 1 Intervals and Inequalities

Write an inequality to represent each of the following intervals. Then state whether the interval is bounded or unbounded.

a. $(-3, 5]$

b. $(-3, \infty)$

c. $[0, 2]$

d. $(-\infty, 0)$

SOLUTION

a. $(-3, 5]$ corresponds to $-3 < x \leq 5$. Bounded

b. $(-3, \infty)$ corresponds to $x > -3$. Unbounded

c. $[0, 2]$ corresponds to $0 \leq x \leq 2$. Bounded

d. $(-\infty, 0)$ corresponds to $x < 0$. Unbounded

✓CHECKPOINT 1

Write an inequality to represent each of the following intervals. Then state whether the interval is bounded or unbounded.

a. $[2, 7)$

b. $(-\infty, 3)$ ■

Properties of Inequalities

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the **properties of inequalities.** These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, the direction of the inequality symbol must be reversed. Here is an example.

$$-2 < 5 \qquad \text{Original inequality}$$
$$(-3)(-2) > (-3)(5) \qquad \text{Multiply each side by } -3 \text{ and reverse the inequality symbol.}$$
$$6 > -15 \qquad \text{Simplify.}$$

Two inequalities that have the same solution set are **equivalent.** For instance, the inequalities

$$x + 2 < 5 \quad \text{and} \quad x < 3$$

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The following list describes operations that can be used to create equivalent inequalities.

Properties of Inequalities

Let a, b, c, and d be real numbers.

1. *Transitive Property*

$a < b$ and $b < c$ ⇨ $a < c$

2. *Addition of Inequalities*

$a < b$ and $c < d$ ⇨ $a + c < b + d$

3. *Addition of a Constant*

$a < b$ ⇨ $a + c < b + c$

4. *Multiplication by a Constant*

For $c > 0$, $a < b$ ⇨ $ac < bc$

For $c < 0$, $a < b$ ⇨ $ac > bc$ Reverse direction of inequality.

Each of the properties above is true if the symbol $<$ is replaced by $\leq$ and the symbol $>$ is replaced by $\geq$. For instance, another form of the multiplication property would be as follows.

For $c > 0$, $a \leq b$ ⇨ $ac \leq bc$.

For $c < 0$, $a \leq b$ ⇨ $ac \geq bc$.

On your own, try to verify each of the properties of inequalities by using several examples with real numbers.

Solving a Linear Inequality

The simplest type of inequality to solve is a **linear inequality** in a single variable. For instance, $2x + 3 > 4$ is a linear inequality in x.

As you read through the following examples, pay special attention to the steps in which the inequality symbol is reversed. Remember that when you multiply or divide by a negative number, you must reverse the inequality symbol.

Example 2 Solving a Linear Inequality

Solve $5x - 7 > 3x + 9$.

SOLUTION

$5x - 7 > 3x + 9$	Write original inequality.
$2x - 7 > 9$	Subtract $3x$ from each side.
$x > 8$	Add 7 to each side and then divide each side by 2.

The solution set is all real numbers that are greater than 8, which is denoted by $(8, \infty)$. The graph is shown in Figure 1.14.

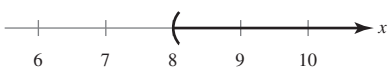


FIGURE 1.14 Solution Interval: $(8, \infty)$

✓CHECKPOINT 2

Solve $3x < 2x + 1$. ■

Checking the solution set of an inequality is not as simple as checking the solutions of an equation. You can, however, get an indication of the validity of a solution set by substituting a few convenient values of x to see whether the original inequality is satisfied.

Example 3 Solving a Linear Inequality

Solve $1 - \frac{3x}{2} \geq x - 4$.

SOLUTION

$1 - \frac{3x}{2} \geq x - 4$	Write original inequality.
$2 - 3x \geq 2x - 8$	Multiply each side by 2.
$2 - 5x \geq -8$	Subtract $2x$ from each side.
$-5x \geq -10$	Subtract 2 from each side.
$x \leq 2$	Divide each side by -5 and reverse inequality.

The solution set is all real numbers that are less than or equal to 2, which is denoted by $(-\infty, 2]$. The graph is shown in Figure 1.15.

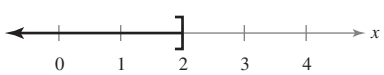


FIGURE 1.15 Solution Interval: $(-\infty, 2]$

✓CHECKPOINT 3

Solve $-\frac{4x}{3} \leq 2 - x$. Then graph the solution set on the real number line. ■

Sometimes it is convenient to write two inequalities as a **double inequality.** For instance, you can write the two inequalities $-4 \leq 5x - 2$ and $5x - 2 < 7$ more simply as

$$-4 \leq 5x - 2 < 7.$$

This enables you to solve the two inequalities together, as demonstrated in Example 4.

Example 4 Solving a Double Inequality

Solve $-3 \leq 6x - 1 < 3$.

SOLUTION To solve a double inequality, you can isolate x as the middle term.

$$-3 \leq 6x - 1 < 3 \qquad \text{Write original inequality.}$$

$$-3 + 1 \leq 6x - 1 + 1 < 3 + 1 \qquad \text{Add 1 to each part.}$$

$$-2 \leq 6x < 4 \qquad \text{Simplify.}$$

$$\frac{-2}{6} \leq \frac{6x}{6} < \frac{4}{6} \qquad \text{Divide each part by 6.}$$

$$-\frac{1}{3} \leq x < \frac{2}{3} \qquad \text{Simplify.}$$

The solution set is all real numbers that are greater than or equal to $-\frac{1}{3}$ and less than $\frac{2}{3}$. The interval notation for this solution set is

$$\left[-\tfrac{1}{3}, \tfrac{2}{3}\right). \qquad \text{Solution set}$$

The graph of this solution set is shown in Figure 1.16.

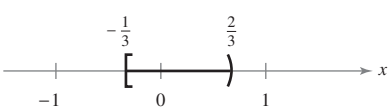


FIGURE 1.16 Solution Interval: $\left[-\frac{1}{3}, \frac{2}{3}\right)$.

✓CHECKPOINT 4

Solve $-1 < 3 - 2x \leq 5$. Then graph the solution set on the real number line. ■

The double inequality in Example 4 could have been solved in two parts as follows.

$$-3 \leq 6x - 1 \quad \text{and} \quad 6x - 1 < 3$$

$$-2 \leq 6x \qquad\qquad 6x < 4$$

$$-\frac{1}{3} \leq x \qquad\qquad x < \frac{2}{3}$$

The solution set consists of all real numbers that satisfy *both* inequalities. In other words, the solution set is the set of all values of x for which $-\frac{1}{3} \leq x < \frac{2}{3}$.

When combining two inequalities to form a double inequality, be sure that the inequalities satisfy the Transitive Property. For instance, it is *incorrect* to combine the inequalities $3 < x$ and $x \leq -1$ as $3 < x \leq -1$. This "inequality" is obviously wrong because 3 is not less than -1.

Inequalities Involving Absolute Value

STUDY TIP

When an absolute value inequality is solved, the solution may consist of a bounded interval on the real number line (Example 5) or two unbounded intervals on the real number line (Example 6).

Solving an Absolute Value Inequality

Let x be a variable or an algebraic expression and let a be a real number such that $a \geq 0$.

1. The solutions of $|x| < a$ are all values of x that lie between $-a$ and a.
 $|x| < a$ if and only if $-a < x < a$.
2. The solutions of $|x| > a$ are all values of x that are less than $-a$ or greater than a.
 $|x| > a$ if and only if $x < -a$ or $x > a$.

These rules are also valid if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Example 5 Solving an Absolute Value Inequality

Solve $|x - 5| < 2$.

SOLUTION

$\|x - 5\| < 2$	Write original inequality.
$-2 < x - 5 < 2$	Equivalent inequality
$-2 + 5 < x - 5 + 5 < 2 + 5$	Add 5 to each part.
$3 < x < 7$	Simplify.

The solution set consists of all real numbers that are greater than 3 and less than 7, which is denoted by $(3, 7)$. The graph is shown in Figure 1.17.

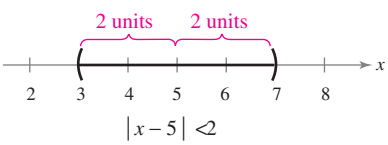


FIGURE 1.17

✓CHECKPOINT 5

Solve $|x + 2| \leq 7$. Then graph the solution set on the real number line. ■

Example 6 Solving an Absolute Value Inequality

Solve $|x + 3| \geq 7$.

SOLUTION

$\|x + 3\| \geq 7$			Write original inequality.
$x + 3 \leq -7$	or	$x + 3 \geq 7$	Equivalent inequalities
$x + 3 - 3 \leq -7 - 3$		$x + 3 - 3 \geq 7 - 3$	Subtract 3 from each side.
$x \leq -10$		$x \geq 4$	Simplify.

The solution set is all real numbers that are less than or equal to -10 *or* greater than or equal to 4, which is denoted by $(-\infty, -10] \cup [4, \infty)$ (see Figure 1.18). The symbol $\cup$ (union) means *or*.

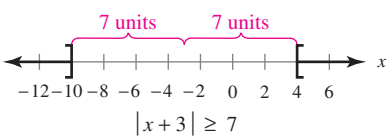


FIGURE 1.18

✓CHECKPOINT 6

Solve $|x + 1| > 3$. Then graph the solution set on the real number line. ■

Applications

Example 7 Comparative Shopping

The cost of renting a compact car from Company A is \$200 per week with no extra charge for mileage. The cost of renting a similar car from Company B is \$110 per week, plus \$0.25 for each mile driven. How many miles must you drive in a week to make the rental fee for Company B more than that for Company A?

SOLUTION

Verbal Model: Weekly cost for Company B $>$ Weekly cost for Company A

Labels:
Miles driven in one week $= m$ (miles)
Weekly cost for Company A $= 200$ (dollars)
Weekly cost for Company B $= 110 + 0.25m$ (dollars)

Inequality:
$$110 + 0.25m > 200$$
$$0.25m > 90$$
$$m > 360$$

When you drive more than 360 miles in a week, the rental fee for Company B is more than the rental fee for Company A.

✓CHECKPOINT 7

In Example 7, suppose the cost of renting a compact car from Company A is \$250 per week with no extra charge for mileage. How many miles must you drive in a week to make the rental fee for Company B more than that for Company A? ■

Example 8 Exercise Program

A 225-pound man begins an exercise and diet program that is designed to reduce his weight by at least 2 pounds per week. Find the maximum number of weeks before the man's weight will reach his goal of 192 pounds.

SOLUTION

Verbal Model: Desired weight $\leq$ Current weight $-$ 2 pounds per week $\cdot$ Number of weeks

Labels:
Desired weight $= 192$ (pounds)
Current weight $= 225$ (pounds)
Number of weeks $= x$ (weeks)

Inequality:
$$192 \leq 225 - 2x$$
$$-33 \leq -2x$$
$$16.5 \geq x$$

Losing at least 2 pounds per week, it will take at most $16\frac{1}{2}$ weeks for the man to reach his goal.



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Americans pay to be lean and fit. In 2005, Americans spent over \$5 billion on exercise equipment. *(Source: National Sporting Goods Association)*

✓CHECKPOINT 8

In Example 8, find the maximum number of weeks before the man's weight will reach 200 pounds. ■

Example 9 Accuracy of a Measurement

You go to a candy store to buy chocolates that cost \$9.89 per pound. The scale used in the store has a state seal of approval that indicates the scale is accurate to within half an ounce. According to the scale, your purchase weighs one-half pound and costs \$4.95. How much might you have been undercharged or over charged due to an error in the scale?

SOLUTION To solve this problem, let x represent the *true* weight of the candy. Because the scale is accurate to within one-half an ounce (or $\frac{1}{32}$ of a pound), you can conclude that the absolute value of the difference between the exact weight (x) and the scale weight $\left(\frac{1}{2}\text{ of a pound}\right)$ is less than or equal to $\frac{1}{32}$ of a pound. That is,

$$\left|x - \frac{1}{2}\right| \leq \frac{1}{32}.$$

You can solve this inequality as follows.

$$-\frac{1}{32} \leq x - \frac{1}{2} \leq \frac{1}{32}$$

$$\frac{15}{32} \leq x \leq \frac{17}{32}$$

$$0.46875 \leq x \leq 0.53125$$

In other words, your "one-half" pound of candy could have weighed as little as 0.46875 pound (which would have cost $0.46875 \cdot \$9.89 \approx \$4.64$) or as much as 0.53125 pound (which would have cost $0.53125 \cdot \$9.89 \approx \$5.25$). So, you could have been undercharged by as much as \$0.30 or overcharged by as much as \$0.31.

✓CHECKPOINT 9

You go to a grocery store to buy ground beef that costs \$3.96 per pound. The scale used in the store is accurate to within $\frac{1}{3}$ ounce $\left(\text{or } \frac{1}{48} \text{ pound}\right)$. According to the scale, your purchase weighs 7.5 pounds and costs \$29.70. How much might you have been undercharged or overcharged due to an error in the scale? ■

CONCEPT CHECK

1. Write an inequality for all values of x that lie between -6 and 8. Is the solution set bounded or unbounded? Explain.
2. Suppose $2x + 1 > 5$ and $y - 8 < 5$. Is it always true that $2x + 1 > y - 8$? Explain.
3. If $x < 12$, then $-x$ must be in what interval?
4. The solution set of an absolute value inequality is $(-\infty, -a] \cup [a, \infty)$. Is the inequality $|x| \leq a$ or $|x| \geq a$?

Skills Review 1.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.1.

In Exercises 1–4, determine which of the two numbers is larger.

1. $-\frac{1}{2}, -7$

2. $-\frac{1}{3}, -\frac{1}{6}$

3. $-\pi, -3$

4. $-6, -\frac{13}{2}$

In Exercises 5–8, use inequality notation to denote the statement.

5. x is nonnegative.

6. z is strictly between -3 and 10.

7. P is no more than 2.

8. W is at least 200.

In Exercises 9 and 10, evaluate the expression for the values of x.

9. $|x - 10|, x = 12, x = 3$

10. $|2x - 3|, x = \frac{3}{2}, x = 1$

Exercises 1.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, write an inequality that represents the interval. Then state whether the interval is bounded or unbounded.

1. $[-1, 5]$

2. $(2, 10]$

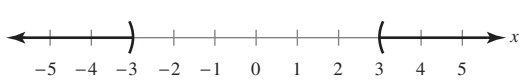
3. $(11, \infty)$

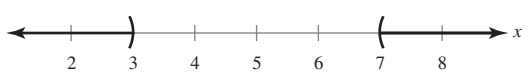
4. $[-5, \infty)$

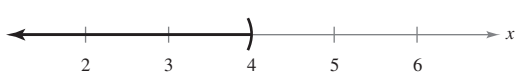
5. $(-\infty, -2)$

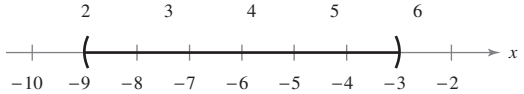
6. $(-\infty, 7]$

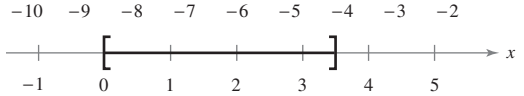
In Exercises 7–14, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

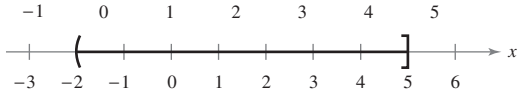
(a)


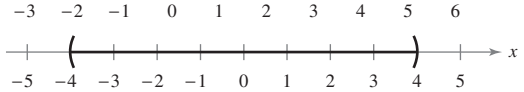
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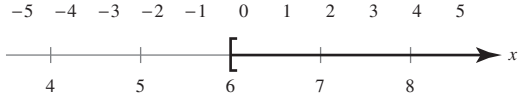
(c)


(d)


(e)


(f)


(g)


(h)


7. $x < 4$

8. $x \geq 6$

9. $-2 < x \leq 5$

10. $0 \leq x \leq \frac{7}{2}$

11. $|x| < 4$

12. $|x| > 3$

13. $|x - 5| > 2$

14. $|x + 6| < 3$

In Exercises 15–22, determine whether each value of x is a solution of the inequality.

15. $5x - 12 > 0$

(a) $x = 3$ (b) $x = -3$ (c) $x = \frac{5}{2}$ (d) $x = \frac{3}{2}$

16. $x + 1 < \frac{2x}{3}$

(a) $x = 0$ (b) $x = 4$ (c) $x = -4$ (d) $x = -3$

17. $0 < \frac{x - 2}{4} < 2$

(a) $x = 4$ (b) $x = 10$ (c) $x = 0$ (d) $x = \frac{7}{2}$

18. $-1 < \frac{3 - x}{2} \leq 1$

(a) $x = 0$ (b) $x = -5$ (c) $x = 1$ (d) $x = 5$

19. $|x - 10| \geq 3$

(a) $x = 13$ (b) $x = -1$ (c) $x = 14$ (d) $x = 9$

20. $|3x + 5| > 7$

(a) $x = -5$ (b) $x = -2$ (c) $x = \frac{1}{3}$ (d) $x = 10$

21. $|x + 2| \leq 10$

(a) $x = -15$ (b) $x = -4$ (c) $x = 1$ (d) $x = 8$

22. $|2x - 3| < 15$

(a) $x = -6$ (b) $x = 0$ (c) $x = 12$ (d) $x = 7$

In Exercises 23–30, copy and complete the statement using the correct inequality symbol.

23. If $2x > 6$, then x __?__ 3.

24. If $3x > 9$, then x __?__ 3.

25. If $2x \le -8$, then x __?__ -4.

26. If $3x \le -15$, then x __?__ -5.

27. If $2 - 4x > -10$, then x __?__ 3.

28. If $5 - 3x > -7$, then x __?__ 4.

29. If $-\frac{2}{3}x \ge -6$, then x __?__ 9.

30. If $-\frac{3}{4}x \ge -12$, then x __?__ 16.

In Exercises 31–70, solve the inequality. Then graph the solution set on the real number line.

31. $\frac{3}{2}x \ge 9$

32. $\frac{2}{5}x > 7$

33. $-10x < 40$

34. $-6x > 15$

35. $\frac{3}{5}x - 7 < 8$

36. $\frac{5}{4}x + 1 \le 11$

37. $2x + 7 < 3 + 4x$

38. $6x - 4 \le 2 + 8x$

39. $2x - 1 \ge 5x$

40. $3x + 1 \ge 2 + x$

41. $3(x + 2) + 7 < 2x - 5$

42. $2(x + 7) - 4 \ge 5(x - 3)$

43. $-3(x - 1) + 7 < 2x + 8$

44. $5 - 3x > -5(x + 4) + 6$

45. $3 \le 2x - 1 < 7$

46. $3 > 1 - \frac{x}{2} > -3$

47. $1 < 2x + 3 < 9$

48. $-8 \le 1 - 3(x - 2) < 13$

49. $-4 < \frac{2x - 3}{3} < 4$

50. $0 \le \frac{x + 3}{2} < 5$

51. $\frac{3}{4} > x + 1 > \frac{1}{4}$

52. $-1 < -\frac{x}{3} < 1$

53. $|x| < 6$

54. $|x| > 8$

55. $\left|\frac{x}{2}\right| > 3$

56. $|5x| > 10$

57. $|x + 3| < 5$

58. $\left|\frac{2x + 1}{2}\right| < 6$

59. $|x - 20| \le 4$

60. $|x - 7| < 6$

61. $|2x - 5| > 6$

62. $2|5 - 3x| + 7 < 21$

63. $\left|\frac{x - 3}{2}\right| \ge 5$

64. $\left|1 - \frac{2x}{3}\right| < 1$

65. $|9 - 2x| - 2 < -1$

66. $|x + 14| + 3 > 17$

67. $2|x + 10| \ge 9$

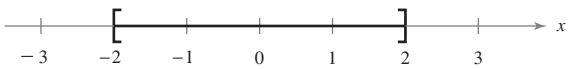
68. $3|4 - 5x| \le 9$

69. $|x - 5| < 0$

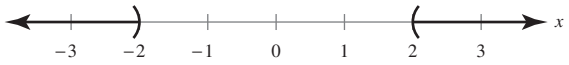
70. $|x - 5| \ge 0$

In Exercises 71–78, use absolute value notation to define the solution set.

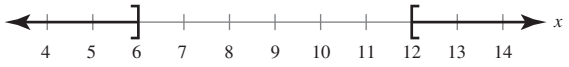
71.



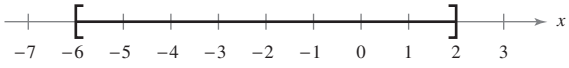
72.



73.



74.



75. All real numbers at most 10 units from 12

76. All real numbers at least 5 units from 8

77. All real numbers whose distances from -3 are more than 5

78. All real numbers whose distances from -6 are no more than 7

79. Comparative Shopping The cost of renting a midsize car from Company A is \$279 per week with no extra charge for mileage. The cost of renting a similar car from Company B is \$199 per week, plus 32 cents for each mile driven. How many miles must you drive in a week to make the rental fee for Company B greater than that for Company A?

80. Comparative Shopping Your department sends its copying to a photocopy center. The photocopy center bills your department \$0.08 per page. You are considering buying a departmental copier for \$2500. With your own copier the cost per page would be \$0.025. The expected life of the copier is 4 years. How many copies must you make in the four-year period to justify purchasing the copier?

81. Simple Interest For \$1500 to grow to more than \$1890 in 3 years, what must the simple interest rate be?

82. Simple Interest For \$2000 to grow to more than \$2500 in 2 years, what must the simple interest rate be?

83. Weight Loss Program A person enrolls in a diet program that guarantees a loss of at least $1\frac{1}{2}$ pounds per week. The person's weight at the beginning of the program is 180 pounds. Find the maximum number of weeks before the person attains a weight of 130 pounds.

84. Salary Increase You accept a new job with a starting salary of \$28,800. You are told that you will receive an annual raise of at least \$1500. What is the maximum number of years you must work before your annual salary will be \$40,000?

85. Maximum Width An overnight delivery service will not accept any package whose combined length and girth (perimeter of a cross section) exceeds 132 inches. Suppose that you are sending a rectangular package that has square cross sections. If the length of the package is 68 inches, what is the maximum width of the sides of its square cross sections?

86. Maximum Width An overnight delivery service will not accept any package whose combined length and girth (perimeter of a cross section) exceeds 126 inches. Suppose that you are sending a rectangular package that has square cross sections. If the length of the package is 66 inches, what is the maximum width of the sides of its square cross sections?

87. Break-Even Analysis The revenue R for selling x units of a product is

$R = 139.95x.$

The cost C of producing x units is

$C = 97x + 850.$

In order to obtain a profit, the revenue must be greater than the cost.

(a) Complete the table.

x	10	20	30	40	50	60
R						
C						

(b) For what values of x will this product return a profit?

88. Break-Even Analysis The revenue R for selling x units of a product is $R = 25.95x$. The cost C of producing x units is

$C = 13.95x + 125{,}000.$

In order to obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

89. Annual Operating Cost A utility company has a fleet of vans. The annual operating cost C per van is

$C = 0.32m + 2500$

where m is the number of miles traveled by a van in a year. What number of miles will yield an annual operating cost that is less than \$12,000?

90. Daily Sales A doughnut shop sells a dozen doughnuts for \$3.95. Beyond the fixed costs (rent, utilities, and insurance) of \$165 per day, it costs \$1.45 for enough materials (flour, sugar, and so on) and labor to produce a dozen doughnuts. The daily profit from doughnut sales varies between \$100 and \$400. Between what numbers of doughnuts (in dozens) do the daily sales vary?

91. IQ Scores The admissions office of a college wants to determine whether there is a relationship between IQ scores x and grade-point averages y after the first year of school. An equation that models the data obtained by the admissions office is

$y = 0.068x - 4.753.$

Estimate the values of x that predict a grade-point average of at least 3.0.

92. *MAKE A DECISION: WEIGHTLIFTING* You want to determine whether there is a relationship between an athlete's weight x (in pounds) and the athlete's maximum bench-press weight y (in pounds). An equation that models the data you obtained is

$y = 1.4x - 39.$

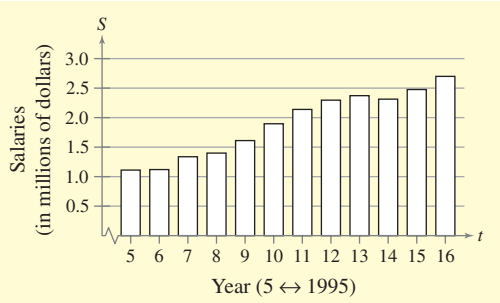
(a) Estimate the values of x that predict a maximum bench-press weight of at least 200 pounds.

(b) Do you think an athlete's weight is a good indicator of the athlete's maximum bench-press weight? What other factors might influence an individual's bench-press weight?

93. Baseball Salaries The average professional baseball player's salary S (in millions of dollars) from 1995 to 2006 can be modeled by

$S = 0.1527t + 0.294, \quad 5 \le t \le 16$

where t represents the year, with $t = 5$ corresponding to 1995 (see figure). Use the model to predict the year in which the average professional baseball player's salary exceeds \$3,000,000. *(Source: Major League Baseball)*



94. Public College Enrollment The projected public college enrollment E (in thousands) in the United States from 2005 to 2015 can be modeled by

$E = 180.3t + 12{,}312, \quad 5 \le t \le 15$

where t represents the year, with $t = 5$ corresponding to 2005 (see figure on next page). Use the model to predict the year in which public college enrollment will exceed 17,000,000. *(Source: U.S. National Center for Education Statistics)*

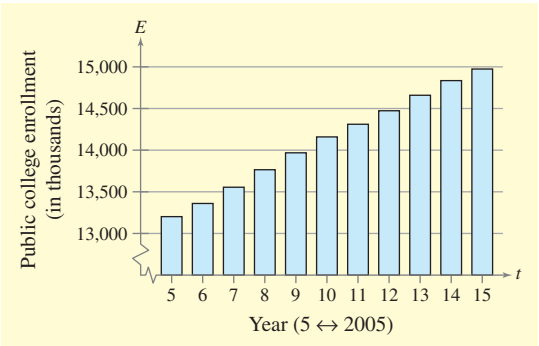


Figure for 94

95. Geometry You measure the side of a square as 10.4 inches with a possible error of $\frac{1}{16}$ inch. Using these measurements, determine the interval containing the possible areas of the square.

96. Geometry You measure the side of a square as 24.2 centimeters with a possible error of 0.25 centimeter. Using these measurements, determine the interval containing the possible areas of the square.

97. Accuracy of Measurement You buy six T-bone steaks that cost $7.99 per pound. The weight listed on the package is 5.72 pounds. The scale that weighed the package is accurate to within $\frac{1}{2}$ ounce. How much money might you have been undercharged or overcharged?

98. Accuracy of Measurement You stop at a self-service gas station to buy 15 gallons of 87-octane gasoline at $2.42 a gallon. The pump scale is accurate to within one-tenth of a gallon. How much money might you have been undercharged or overcharged?

99. Human Height The heights h of two-thirds of a population satisfy the inequality

$$|h - 68.5| \leq 2.7$$

where h is measured in inches. Determine the interval on the real number line in which these heights lie.

100. Time Study A time study was conducted to determine the length of time required to perform a particular task in a manufacturing process. The times required by approximately two-thirds of the workers in the study satisfied the inequality

$$\left|\frac{t - 15.6}{1.9}\right| < 1$$

where t is time in minutes. Determine the interval on the real number line in which these times lie.

101. Humidity Control The specifications for an electronic device state that it is to be operated in a room with relative humidity h defined by $|h - 50| \leq 30$. What are the minimum and maximum relative humidities for the operation of this device?

102. Body Temperature Physicians consider an adult's body temperature x (in degrees Fahrenheit) to be normal if it satisfies the inequality

$$|x - 98.6| \leq 1.$$

Determine the range of temperatures that are considered to be normal.

103. Brand Name Drugs The average price B (in dollars) of brand name prescription drugs from 1998 to 2005 can be modeled by

$$B = 6.928t - 3.45, \quad 8 \leq t \leq 15$$

where t represents the year, with $t = 8$ corresponding to 1998. Use the model to find the year in which the price of the average brand name drug prescription exceeded $75. *(Source: National Association of Chain Drug Stores)*

104. Generic Drugs The average price G (in dollars) of generic prescription drugs from 1998 to 2005 can be modeled by

$$G = 2.005t + 0.40, \quad 8 \leq t \leq 15$$

where t represents the year, with $t = 8$ corresponding to 1998. Use the model to find the year in which the price of the average generic drug prescription exceeded $19. *(Source: National Association of Chain Drug Stores)*

105. Domestic Oil Demand The daily demand D (in thousands of barrels) for refined oil in the United States from 1995 to 2005 can be modeled by

$$D = 276.4t + 16{,}656, \quad 5 \leq t \leq 15$$

where t represents the year, with $t = 5$ corresponding to 1995. *(Source: U.S. Energy Administration)*

(a) Use the model to find the year in which the demand for U.S. oil exceeded 18 million barrels a day.

(b) Use the model to predict the year in which the demand for U.S. oil will exceed 22 million barrels a day.

106. Imported Oil The daily amount I (in thousands of barrels) of crude oil imported to the United States from 1995 to 2005 can be modeled by

$$I = 428.2t + 6976, \quad 5 \leq t \leq 15$$

where t represents the year, with $t = 5$ corresponding to 1995. *(Source: U.S. Energy Administration)*

(a) Use the model to find the year in which the amount of crude oil imported to the United States exceeded 12 million barrels a day.

(b) Use the model to predict the year in which the amount of oil imported to the United States will exceed 14 million barrels a day.

Section 1.7

Other Types of Inequalities

- Use critical numbers to determine test intervals for a polynomial inequality.
- Solve and graph a polynomial inequality.
- Solve and graph a rational inequality.
- Determine the domain of an expression involving a square root.
- Construct and use a polynomial inequality to solve an application problem.

Polynomial Inequalities

To solve a polynomial inequality such as $x^2 - 2x - 3 < 0$, you can use the fact that a polynomial can change signs only at its **zeros** (the x-values that make the polynomial equal to zero). Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **critical numbers** of the inequality, and the resulting intervals are the **test intervals** for the inequality. For example, the polynomial above factors as

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

and has two zeros, $x = -1$ and $x = 3$. These zeros divide the real number line into three test intervals:

$$(-\infty, -1), \quad (-1, 3), \quad \text{and} \quad (3, \infty). \qquad \text{(See Figure 1.19.)}$$

So, to solve the inequality $x^2 - 2x - 3 < 0$, you need only test one value from each of these test intervals.

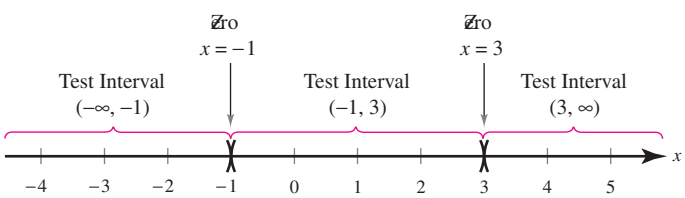


FIGURE 1.19 Three Test Intervals for $x^2 - 2x - 3 < 0$

STUDY TIP

If the value of the polynomial is negative at the representative x-value, the polynomial will have negative values for *every* x-value in the interval. If the value of the polynomial is positive, the polynomial will have positive values for *every* x-value in the interval.

Finding Test Intervals for a Polynomial

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order. These zeros are the **critical numbers** of the polynomial.
2. Use the critical numbers to determine the **test intervals.**
3. Choose one representative x-value in each test interval and evaluate the polynomial at that value.

Example 1 Solving a Polynomial Inequality

Solve $x^2 - x - 6 < 0$.

SOLUTION By factoring the quadratic as

$$x^2 - x - 6 = (x + 2)(x - 3)$$

you can see that the critical numbers are $x = -2$ and $x = 3$. The boundaries between the numbers that satisfy the inequality and the numbers that do not satisfy the inequality always occur at critical numbers. So, the polynomial's test intervals are

$$(-\infty, -2), \quad (-2, 3), \quad \text{and} \quad (3, \infty).$$ Test intervals

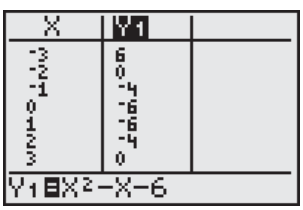
In each test interval, choose a representative x-value and evaluate the polynomial.

Test Interval	x-Value	Polynomial Value	Conclusion
$(-\infty, -2)$	$x = -3$	$(-3)^2 - (-3) - 6 = 6$	Positive
$(-2, 3)$	$x = 0$	$(0)^2 - (0) - 6 = -6$	Negative
$(3, \infty)$	$x = 4$	$(4)^2 - (4) - 6 = 6$	Positive

From this, you can conclude that the polynomial is positive for all x-values in $(-\infty, -2)$ and $(3, \infty)$, and is negative for all x-values in $(-2, 3)$. This implies that the solution of the inequality $x^2 - x - 6 < 0$ is the interval $(-2, 3)$, as shown in Figure 1.20.

TECHNOLOGY

You can use the *table* feature of your graphing utility to check the sign of the polynomial in each interval.



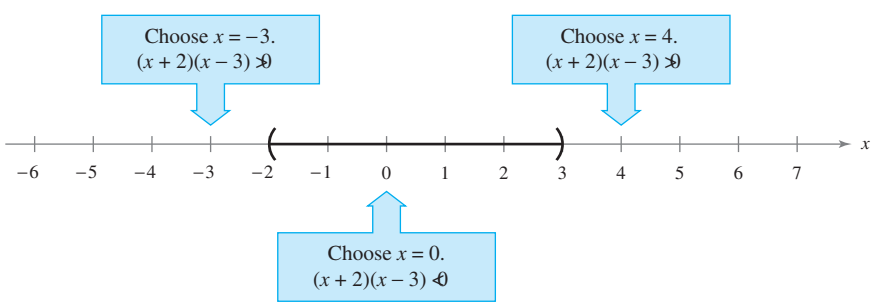


FIGURE 1.20

✓CHECKPOINT 1

Solve $x^2 + x - 2 < 0$. ■

As with linear inequalities, you can check a solution interval of a polynomial inequality by substituting x-values into the original inequality. For instance, to check the solution found in Example 1, try substituting several x-values from the interval $(-2, 3)$ into the inequality

$$x^2 - x - 6 < 0.$$

Regardless of which x-values you choose, the inequality will be satisfied.

In Example 1, the polynomial inequality was given in general form. Whenever this is not the case, begin the solution process by writing the inequality in general form with the polynomial on one side and zero on the other.

Example 2 Solving a Polynomial Inequality

Solve $x^3 - 3x^2 > 10x$.

SOLUTION

$$x^3 - 3x^2 > 10x \qquad \text{Write original inequality.}$$
$$x^3 - 3x^2 - 10x > 0 \qquad \text{Write in general form.}$$
$$x(x - 5)(x + 2) > 0 \qquad \text{Factor.}$$

You can see that the critical numbers are $x = -2$, $x = 0$, and $x = 5$, and the test intervals are $(-\infty, -2)$, $(-2, 0)$, $(0, 5)$, and $(5, \infty)$. In each test interval, choose a representative x-value and evaluate the polynomial.

Test Interval	x-Value	Polynomial Value	Conclusion
$(-\infty, -2)$	$x = -3$	$(-3)^3 - 3(-3)^2 - 10(-3) = -24$	Negative
$(-2, 0)$	$x = -1$	$(-1)^3 - 3(-1)^2 - 10(-1) = 6$	Positive
$(0, 5)$	$x = 2$	$2^3 - 3(2)^2 - 10(2) = -24$	Negative
$(5, \infty)$	$x = 6$	$6^3 - 3(6)^2 - 10(6) = 48$	Positive

From this, you can conclude that the inequality is satisfied on the open intervals $(-2, 0)$ and $(5, \infty)$. So, the solution set consists of all real numbers in the intervals $(-2, 0)$ and $(5, \infty)$, as shown in Figure 1.21.

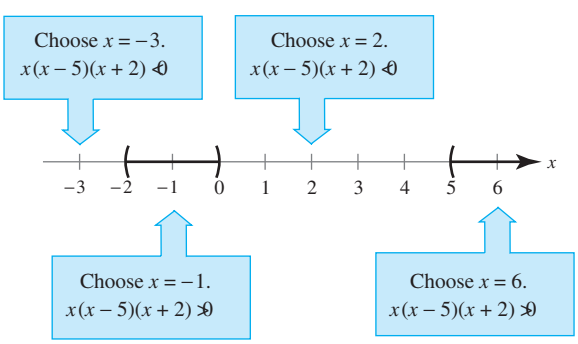


FIGURE 1.21

✓CHECKPOINT 2

Solve $x^2 - 3x > -2$. ■

When solving a polynomial inequality, be sure to account for the type of inequality symbol given in the inequality. For instance, in Example 2, note that the solution consisted of two *open* intervals because the original inequality contained a "greater than" symbol. If the original inequality had been $x^3 - 3x^2 \geq 10x$, the solution would have consisted of the *closed* interval $[-2, 0]$ and the interval $[5, \infty)$.

Each of the polynomial inequalities in Examples 1 and 2 has a solution set that consists of a single interval or the union of two intervals. When solving the exercises for this section, you should watch for some unusual solution sets, as illustrated in Example 3.

Example 3 Unusual Solution Sets

What is unusual about the solution set for each inequality?

a. $x^2 + 2x + 4 > 0$

The solution set for this inequality consists of the entire set of real numbers, $(-\infty, \infty)$. In other words, the value of the quadratic $x^2 + 2x + 4$ is positive for every real value of x.

b. $x^2 + 2x + 1 \leq 0$

The solution set for this inequality consists of the single real number $\{-1\}$, because the quadratic $x^2 + 2x + 1$ has one critical number, $x = -1$, and it is the only value that satisfies the inequality.

c. $x^2 + 3x + 5 < 0$

The solution set for this inequality is empty. In other words, the quadratic $x^2 + 3x + 5$ is *not* less than zero for any value of x.

d. $x^2 - 4x + 4 > 0$

The solution set for this inequality consists of all real numbers *except* the number 2. In interval notation, this solution can be written as $(-\infty, 2) \cup (2, \infty)$.

✓CHECKPOINT 3

What is unusual about the solution set for each inequality?

a. $x^2 + x + 3 \leq 0$

b. $x^2 - 2x + 1 > 0$ ■

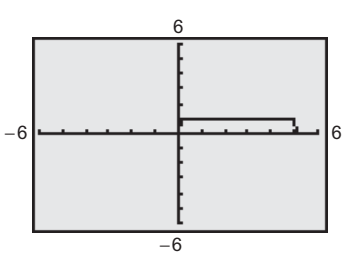


FIGURE 1.22

TECHNOLOGY

Graphs of Inequalities and Graphing Utilities Most graphing utilities can graph an inequality. Consult your user's guide for specific instructions. Once you know how to graph an inequality, you can check solutions by graphing. (Make sure you use an appropriate viewing window.) For example, the solution to

$$x^2 - 5x < 0$$

is the interval $(0, 5)$. When graphed, the solution occurs as an interval above the horizontal axis on the graphing utility, as shown in Figure 1.22. The graph does not indicate whether 0 and/or 5 are part of the solution. You must determine whether the endpoints are part of the solution based on the type of inequality.

Rational Inequalities

The concepts of critical numbers and test intervals can be extended to inequalities involving rational expressions. Use the fact that the value of a rational expression can change sign only at its *zeros* (the x-values for which its numerator is zero) and its *undefined values* (the x-values for which its denominator is zero). These two types of numbers make up the **critical numbers** of a rational inequality.

Example 4 Solving a Rational Inequality

Solve $\dfrac{2x-7}{x-5} \le 3$.

TECHNOLOGY

When using a graphing utility to check an inequality, always set your viewing window so that it includes all of the critical numbers.

SOLUTION

$$\frac{2x-7}{x-5} \le 3 \qquad \text{Write original inequality.}$$

$$\frac{2x-7}{x-5} - 3 \le 0 \qquad \text{Write in general form.}$$

$$\frac{2x-7-3x+15}{x-5} \le 0 \qquad \text{Add fractions.}$$

$$\frac{-x+8}{x-5} \le 0 \qquad \text{Simplify.}$$

Critical numbers: $x = 5, x = 8$

Test intervals: $(-\infty, 5), (5, 8), (8, \infty)$

Test: Is $\dfrac{-x+8}{x-5} \le 0$?

After testing these intervals, as shown in Figure 1.23, you can see that the inequality is satisfied on the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because $(-x+8)/(x-5) = 0$ when $x = 8$, you can conclude that the solution set consists of all real numbers in the intervals $(-\infty, 5) \cup [8, \infty)$.

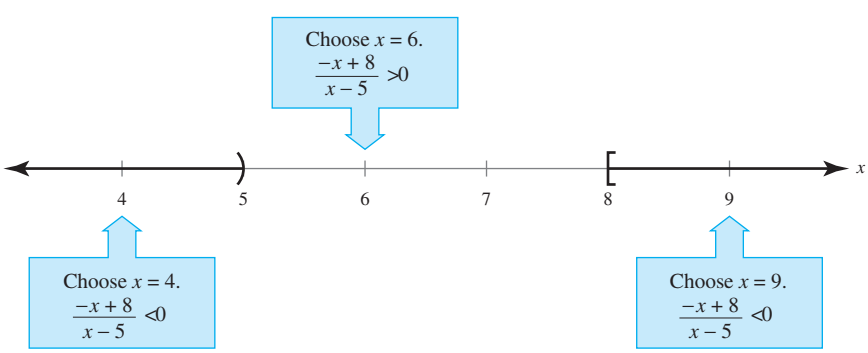


FIGURE 1.23

✓CHECKPOINT 4

Solve $\dfrac{x-1}{x-3} \ge -1$. ■

Applications

One common application of inequalities comes from business and involves profit, revenue, and cost. The formula that relates these three quantities is

$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$P = R - C.$$

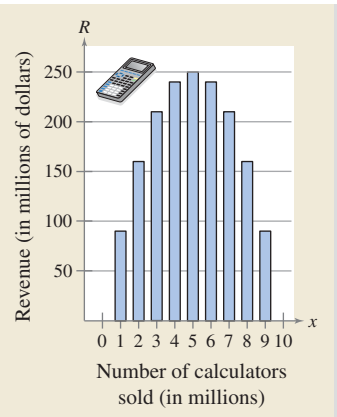


FIGURE 1.24

Example 5 Increasing the Profit for a Product

The marketing department of a calculator manufacturer has determined that the demand for a new model of calculator is given by

$$p = 100 - 10x, \quad 0 \le x \le 10$$ Demand equation

where p is the price per calculator in dollars and x represents the number of calculators sold, in millions. (If this model is accurate, no one would be willing to pay \$100 for the calculator. At the other extreme, the company couldn't *give* away more than 10 million calculators.) The revenue, in millions of dollars, for selling x million calculators is given by

$$R = xp = x(100 - 10x).$$ Revenue equation

See Figure 1.24. The total cost of producing x million calculators is \$10 per calculator plus a one-time development cost of \$2,500,000. So, the total cost, in millions of dollars, is

$$C = 10x + 2.5.$$ Cost equation

What prices can the company charge per calculator to obtain a profit of at least \$190,000,000?

SOLUTION

Verbal Model: $\text{Profit} = \text{Revenue} - \text{Cost}$

Equation: $P = R - C$

$$P = 100x - 10x^2 - (10x + 2.5)$$
$$P = -10x^2 + 90x - 2.5$$

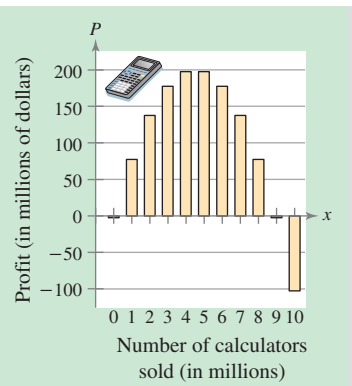


FIGURE 1.25

To answer the question, you must solve the inequality

$$-10x^2 + 90x - 2.5 \ge 190.$$

Using the techniques described in this section, you can find the solution set to be $3.5 \le x \le 5.5$, as shown in Figure 1.25. The prices that correspond to these x-values are given by

$$100 - 10(3.5) \ge p \ge 100 - 10(5.5)$$
$$45 \le p \le 65$$

The company can obtain a profit of \$190,000,000 or better by charging at least \$45 per calculator and at most \$65 per calculator.

✓CHECKPOINT 5

In Example 5, what prices can the company charge per calculator to obtain a profit of at least \$160,000,000? ■

Another common application of inequalities is finding the domain of an expression that involves a square root, as shown in Example 6.

Example 6 Finding the Domain of an Expression

Find the domain of the expression $\sqrt{64 - 4x^2}$.

SOLUTION Remember that the domain of an expression is the set of all x-values for which the expression is defined. Because $\sqrt{64 - 4x^2}$ is defined (has real values) only if $64 - 4x^2$ is nonnegative, the domain is given by $64 - 4x^2 \geq 0$.

$$64 - 4x^2 \geq 0 \qquad \text{Write in general form.}$$

$$16 - x^2 \geq 0 \qquad \text{Divide each side by 4.}$$

$$(4 - x)(4 + x) \geq 0 \qquad \text{Factor.}$$

So, the inequality has two critical numbers: $x = -4$ and $x = 4$. You can use these two numbers to test the inequality as follows.

Critical numbers: $x = -4, x = 4$

Test intervals: $(-\infty, -4), (-4, 4), (4, \infty)$

Test: Is $(4 - x)(4 + x) \geq 0$?

A test shows that $64 - 4x^2$ is greater than or equal to 0 in the *closed interval* $[-4, 4]$. So, the domain of the expression $\sqrt{64 - 4x^2}$ is the interval $[-4, 4]$, as shown in Figure 1.26.

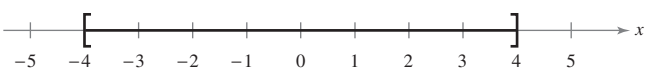


FIGURE 1.26

✓CHECKPOINT 6

Find the domain of each expression.

a. $\sqrt{12 - 3x^2}$

b. $\sqrt[3]{x^2 - 2x - 8}$ ■

CONCEPT CHECK

1. The test intervals for a polynomial inequality are $(-\infty, -2)$, $(-2, 0)$, $(0, 5)$, and $(5, \infty)$. What are the critical numbers of the polynomial?
2. Is -7 the only critical number of $\frac{x - 2}{x + 7} \geq 0$? Explain.
3. Describe and correct the error in the statement: The domain of the expression $\sqrt{(x - 3)(x + 3)}$ is all real numbers except -3 and 3.
4. Explain why the critical numbers of a polynomial inequality are not included in the test intervals.

Skills Review 1.7

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 1.6.

In Exercises 1–10, solve the inequality.

1. $-\frac{y}{3} > 2$

2. $-6z < 27$

3. $-3 \le 2x + 3 < 5$

4. $-3x + 5 \ge 20$

5. $10 > 4 - 3(x + 1)$

6. $3 < 1 + 2(x - 4) < 7$

7. $2|x| \le 7$

8. $|x - 3| > 1$

9. $|x + 4| > 2$

10. $|2 - x| \le 4$

Exercises 1.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the test intervals of the inequality.

1. $x^2 - 25 < 0$

2. $x^2 - 6x + 8 > 0$

3. $2x^2 + 7x + 16 \ge 20$

4. $3x^2 - 26x + 25 \le 9$

5. $\frac{x - 3}{x - 1} < 2$

6. $\frac{x - 4}{2x + 3} \ge 1$

In Exercises 7–36, solve the inequality. Then graph the solution set on the real number line.

7. $x^2 \le 9$

8. $x^2 < 5$

9. $x^2 > 4$

10. $(x - 3)^2 \ge 1$

11. $(x + 2)^2 < 25$

12. $(x + 6)^2 \le 8$

13. $x^2 + 4x + 4 \ge 9$

14. $x^2 - 6x + 9 < 16$

15. $x^2 + x < 6$

16. $x^2 + 2x > 3$

17. $3(x - 1)(x + 1) > 0$

18. $6(x + 2)(x - 1) < 0$

19. $x^2 + 2x - 3 < 0$

20. $x^2 - 4x - 1 > 0$

21. $4x^3 - 6x^2 < 0$

22. $4x^3 - 12x^2 > 0$

23. $x^3 - 4x \ge 0$

24. $2x^3 - x^4 \le 0$

25. $x^3 - 2x^2 - x + 2 \ge 0$

26. $x^3 + 5x^2 - 4x - 20 \le 0$

27. $\frac{1}{x} > x$

28. $\frac{1}{x} < 4$

29. $\frac{x + 6}{x + 1} < 2$

30. $\frac{x + 12}{x + 2} \ge 3$

31. $\frac{3x - 5}{x - 5} > 4$

32. $\frac{5 + 7x}{1 + 2x} < 4$

33. $\frac{4}{x + 5} > \frac{1}{2x + 3}$

34. $\frac{5}{x - 6} > \frac{3}{x + 2}$

35. $\frac{1}{x - 3} \le \frac{9}{4x + 3}$

36. $\frac{1}{x} \ge \frac{1}{x + 3}$

In Exercises 37–46, find the domain of the expression.

37. $\sqrt{x^2 - 9}$

38. $\sqrt{x^2 - 4}$

39. $\sqrt[4]{6 + x^2}$

40. $\sqrt{x^2 + 4}$

41. $\sqrt{81 - 4x^2}$

42. $\sqrt{147 - 3x^2}$

43. $\sqrt{x^2 - 7x + 10}$

44. $\sqrt{12 - x - x^2}$

45. $\sqrt{x^2 - 3x + 3}$

46. $\sqrt[4]{-x^2 + 2x - 2}$

In Exercises 47 and 48, consider the domains of the expressions $\sqrt[3]{x^2 - 7x + 12}$ and $\sqrt{x^2 - 7x + 12}$.

47. Explain why the domain of $\sqrt[3]{x^2 - 7x + 12}$ consists of all real numbers.

48. Explain why the domain of $\sqrt{x^2 - 7x + 12}$ is different from the domain of $\sqrt[3]{x^2 - 7x + 12}$.

In Exercises 49–54, solve the inequality and write the solution set in interval notation.

49. $6x^3 - 10x^2 > 0$

50. $25x^3 - 10x^2 < 0$

51. $x^3 - 9x \le 0$

52. $4x^3 - x^4 \ge 0$

53. $(x - 1)^2(x + 2)^3 \ge 0$

54. $x^4(x - 3) \le 0$

In Exercises 55–60, use a calculator to solve the inequality. (Round each number in your answer to two decimal places.)

55. $0.4x^2 + 5.26 < 10.2$

56. $-1.3x^2 + 3.78 > 2.12$

57. $-0.5x^2 + 12.5x + 1.6 > 0$

58. $1.2x^2 + 4.8x + 3.1 < 5.3$

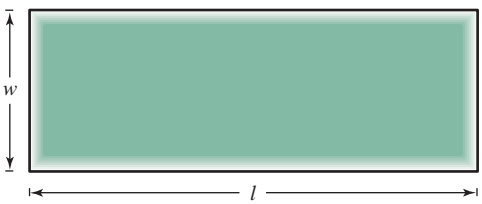
59. $\frac{1}{2.3x - 5.2} > 3.4$

60. $\frac{2}{3.1x - 3.7} > 5.8$

61. Height of a Projectile A projectile is fired straight upward from ground level with an initial velocity of 200 feet per second. During what time period will its height exceed 400 feet?

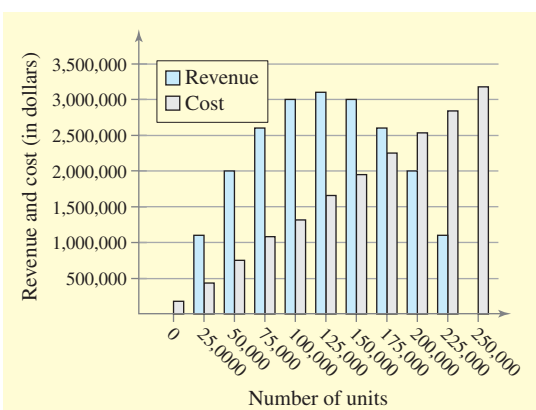
62. Height of a Projectile A projectile is fired straight upward from ground level with an initial velocity of 160 feet per second. During what time period will its height be less than 384 feet?

63. Geometry A rectangular playing field with a perimeter of 100 meters is to have an area of at least 500 square meters (see figure). Within what bounds must the length be?



64. Geometry A rectangular room with a perimeter of 50 feet is to have an area of at least 120 square feet. Within what bounds must the length be?

65. *MAKE A DECISION: COMPANY PROFITS* The revenue R and cost C for a product are given by $R = x(50 - 0.0002x)$ and $C = 12x + 150{,}000$, where R and C are measured in dollars and x represents the number of units sold (see figure).



(a) How many units must be sold to obtain a profit of at least \$1,650,000?

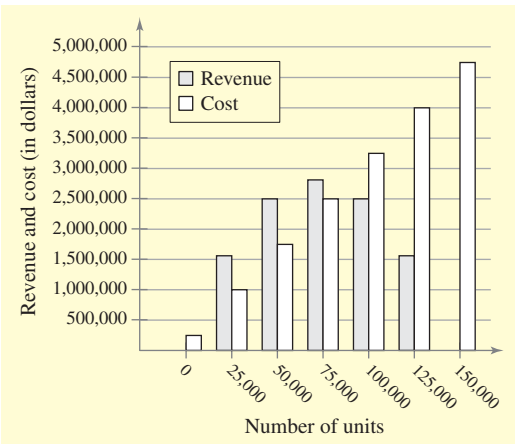
(b) The demand equation for the product is

$p = 50 - 0.0002x$

where p is the price per unit. What prices will produce a profit of at least \$1,650,000?

(c) As the number of units increases, the revenue eventually decreases. After this point, at what number of units is the revenue approximately equal to the cost? How should this affect the company's decision about the level of production?

66. *MAKE A DECISION: COMPANY PROFITS* The revenue R and cost C for a product are given by $R = x(75 - 0.0005x)$ and $C = 30x + 250{,}000$, where R and C are measured in dollars and x represents the number of units sold (see figure).



(a) How many units must be sold to obtain a profit of at least \$750,000?

(b) The demand equation for the product is

$p = 75 - 0.0005x$

where p is the price per unit. What prices will produce a profit of at least \$750,000?

(c) As the number of units increases, the revenue eventually decreases. After this point, at what number of units is the revenue approximately equal to the cost? How should this affect the company's decision about the level of production?

67. Compound Interest P dollars, invested at interest rate r compounded annually, increases to an amount

$A = P(1 + r)^3$

in 3 years. For an investment of \$1000 to increase to an amount greater than \$1500 in 3 years, the interest rate must be greater than what percent?

68. Compound Interest P dollars, invested at interest rate r compounded annually, increases to an amount

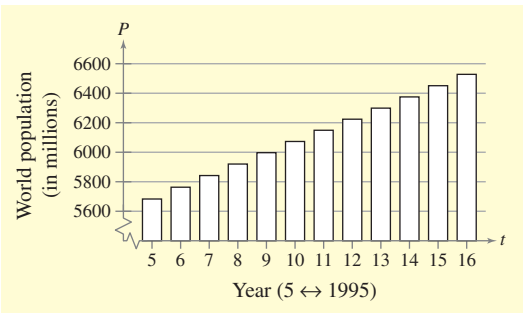
$A = P(1 + r)^2$

in 2 years. For an investment of \$2000 to increase to an amount greater than \$2350 in 2 years, the interest rate must be greater than what percent?

69. World Population The world population P (in millions) from 1995 to 2006 can be modeled by

$$P = -0.18t^2 + 80.30t + 5288, \quad 5 \le t \le 16$$

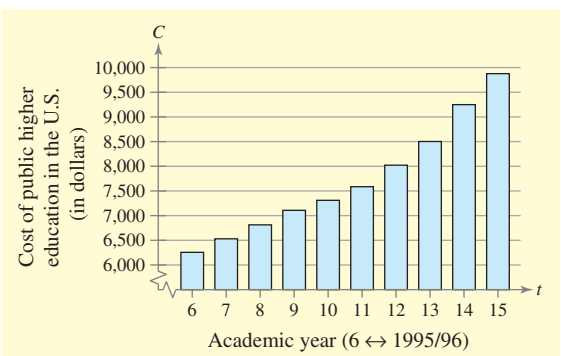
where t represents the year, with $t = 5$ corresponding to 1995 (see figure). Use the model to predict the year in which the world population will exceed 7,000,000,000. *(Source: U.S. Census Bureau)*



70. Higher Education The average yearly cost C of higher education at public institutions in the United States for the academic years 1995/1996 to 2004/2005 can be modeled by

$$C = 30.57t^2 - 259.6t + 6828, \quad 6 \le t \le 15$$

where t represents the year, with $t = 6$ corresponding to the 1995/1996 school year (see figure). Use the model to predict the academic year in which the average yearly cost of higher education at public institutions exceeds $12,000. *(Source: U.S. Department of Education)*



71. Higher Education The average yearly cost C of higher education at private institutions in the United States for the academic years 1995/1996 to 2004/2005 can be modeled by

$$C = 42.93t^2 + 68.0t + 15{,}309, \quad 6 \le t \le 15$$

where t represents the year, with $t = 6$ corresponding to the academic year 1995/1996 (see figure). Use the model to predict the academic year in which the average yearly cost of higher education at private institutions exceeds $32,000. *(Source: U.S. Department of Education.)*

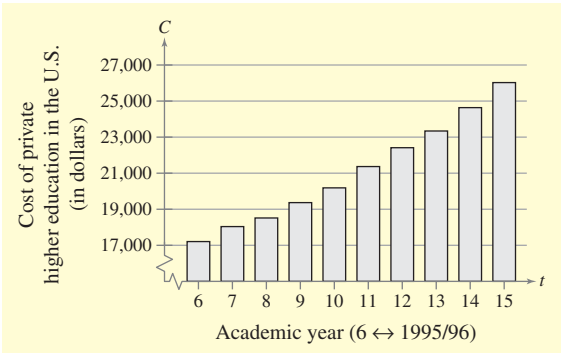
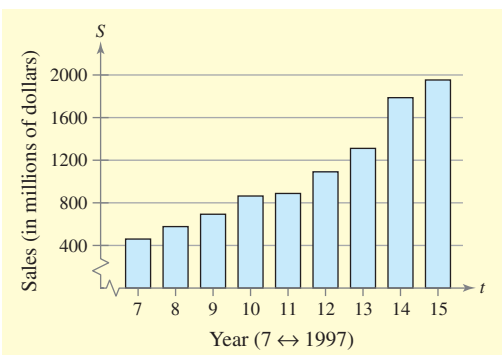


Figure for 71

72. Sales The total sales S (in millions of dollars) for Univision Communications from 1997 to 2005 can be modeled by

$$S = 18.471t^2 - 221.96t + 1152.6, \quad 7 \le t \le 15$$

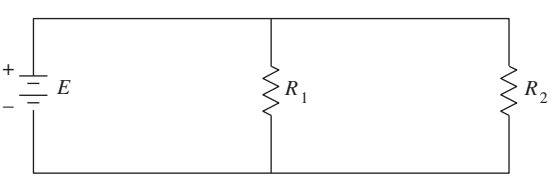
where t represents the year, with $t = 7$ corresponding to 1997 (see figure). Univision Communications predicts sales will exceed $2.7 billion between 2009 and 2011. Does the model support this prediction? Explain your reasoning. *(Source: Univision Communications)*



73. Resistors When two resistors of resistances R_1 and R_2 are connected in parallel (see figure), the total resistance R satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Find R_1 for a parallel circuit in which $R_2 = 2$ ohms and R must be at least 1 ohm.



Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 150.
Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 1.1	Review Exercises
■ Classify an equation as an identity or a conditional equation.	*1, 2*
■ Determine whether a given value is a solution.	*3, 4*
■ Solve a linear equation in one variable. Can be written in the general form: $ax + b = 0$.	*5–8*
■ Solve an equation involving fractions.	*9–12*
■ Use a calculator to solve an equation.	*13–16*

Section 1.2

- Use mathematical models to solve word problems. *17, 19, 27–30*
- Model and solve percent and mixture problems. *18, 20, 25, 26, 31, 32*
- Use common formulas to solve geometry and simple interest problems. *21–24*

Square: $A = s^2$, $P = 4s$

Rectangle: $A = lw$, $P = 2l + 2w$

Circle: $A = \pi r^2$, $C = 2\pi r$

Triangle: $A = \frac{1}{2}bh$, $P = a + b + c$

Cube: $V = s^3$

Rectangular Solid: $V = lwh$

Circular Cylinder: $V = \pi r^2 h$

Sphere: $V = \frac{4}{3}\pi r^3$

Temperature: $F = \frac{9}{5}C + 32$

Simple Interest: $I = Prt$

Distance: $d = rt$

Section 1.3

- Solve a quadratic equation by factoring. *33–36*
 Can be written in the general form: $ax^2 + bx + c = 0$.
 Zero-Factor Property: If $ab = 0$, then $a = 0$ or $b = 0$.
- Solve a quadratic equation by extracting square roots. *37–40*
- Analyze a quadratic equation. *41, 42*
- Construct and use a quadratic model to solve area problems, falling object problems, right triangle problems, and other applications. *43–46*

*Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

Section 1.4

	Review Exercises
■ Use the discriminant to determine the number of real solutions of a quadratic equation.	*47, 48*
If $b^2 - 4ac > 0$, the equation has two distinct real solutions.	
If $b^2 - 4ac = 0$, the equation has one repeated real solution.	
If $b^2 - 4ac < 0$, the equation has no real solutions.	
■ Solve a quadratic equation using the Quadratic Formula.	*49–58*
Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	
■ Use the Quadratic Formula to solve an application problem.	*59, 60*

Section 1.5

■ Solve a polynomial equation by factoring.	*61, 62*
■ Solve an equation of quadratic type.	*63, 64*
■ Rewrite and solve an equation involving radicals or rational exponents.	*65–70*
■ Rewrite and solve an equation involving fractions or absolute values.	*71–74*
■ Construct and use a nonquadratic model to solve an application problem.	*75, 76, 78*
■ Solve a compound interest problem.	*77*

Section 1.6

■ Solve and graph a linear inequality.	*79–82*
Transitive Property: $a < b$ and $b < c \implies a < c$	
Addition of Inequalities: $a < b$ and $c < d \implies a + c < b + d$	
Addition of a Constant: $a < b \implies a + c < b + c$	
Multiplication by a Constant: For $c > 0$, $a < b \implies ac < bc$	
For $c < 0$, $a < b \implies ac > bc$	
■ Solve and graph inequalities involving absolute value.	*83, 84*
$\lvert x \rvert < a$ if and only if $-a < x < a$	
$\lvert x \rvert > a$ if and only if $x < -a$ or $x > a$	
■ Construct and use a linear inequality to solve an application problem.	*85, 86*

Section 1.7

■ Solve and graph a polynomial inequality.	*87–89, 93, 94*
■ Solve and graph a rational inequality.	*90–92, 95, 96*
■ Determine the domain of an expression involving a radical.	*97–102*
■ Construct and use a polynomial inequality to solve an application problem.	*103–113*

Study Strategies

■ **Check Your Answers** Because of the number of steps involved in solving an equation or inequality, there are many ways to make mistakes. So, always check your answers. In some cases, you may even want to check your answers in more than one way, just to be sure.

■ **Using Test Intervals** Make sure that you understand how to use critical numbers to determine test intervals for inequalities. The logic and mathematical reasoning involved in this concept can be applied in many real-life situations.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, determine whether the equation is an identity or a conditional equation.

1. $5(x - 3) = 2x + 9$ **2.** $3(x + 2) = 3x + 6$

In Exercises 3 and 4, determine whether each value of x is a solution of the equation.

3. $3x^2 + 7x + 5 = x^2 + 9$

(a) $x = 0$ (b) $x = \frac{1}{2}$ (c) $x = -4$ (d) $x = -1$

4. $6 + \dfrac{3}{x - 4} = 5$

(a) $x = 5$ (b) $x = 0$ (c) $x = -2$ (d) $x = 7$

In Exercises 5–12, solve the equation (if possible) and check your solution.

5. $x + 7 = 20$

6. $2x + 15 = 43$

7. $4(x + 3) - 3 = 2(4 - 3x) - 4$

8. $(x + 3) + 2(x - 4) = 5(x + 3)$

9. $\dfrac{3x - 2}{5x - 1} = \dfrac{3}{4}$

10. $\dfrac{3}{x - 4} + \dfrac{8}{2x + 5} = \dfrac{11}{2x^2 - 3x - 20}$

11. $\dfrac{x}{x + 3} - \dfrac{4}{x + 3} + 2 = 0$

12. $7 - \dfrac{3}{x} = 8 + \dfrac{5}{x}$

In Exercises 13–16, use a calculator to solve the equation. (Round your solution to three decimal places.)

13. $0.375x - 0.75(300 - x) = 200$

14. $0.235x + 2.6(-x - 4) = 30$

15. $\dfrac{x}{0.055} + \dfrac{x}{0.085} = 1$

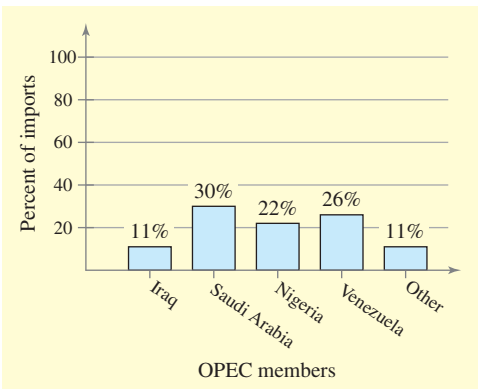
16. $\dfrac{x}{0.0645} + \dfrac{x}{0.098} = 2$

17. Three consecutive even integers have a sum of 42. Find the smallest of these integers.

18. Annual Salary Your annual salary is \$28,900. You receive a 7% raise. What is your new annual salary?

19. Fitness When using a pull-up weight machine, the amount you set is subtracted from your weight and you pull the remaining amount. Write a model that describes the weight x that must be set if a person weighing 150 pounds wishes to pull 120 pounds. Solve for x.

20. Oil Imports The United States imported 1738 million barrels of crude oil from members of OPEC (Organization of the Petroleum Exporting Countries) in 2005. Use the bar graph to determine the amount imported from each of the four top contributing countries. *(Source: U.S. Energy Information Administration)*



21. Geometry A volleyball court is twice as long as it is wide, and its perimeter is 177 feet. Find the dimensions of the volleyball court.

22. Geometry A room is 1.25 times as long as it is wide, and its perimeter is 90 feet. Find the dimensions of the room.

23. Simple Interest You deposit \$500 in a savings account earning 4% simple interest. How much interest will you have earned after 1 year?

24. Simple Interest You deposit \$800 in a money market account. One year later the account balance is \$862.40. What was the simple interest rate?

25. List Price The price of an outdoor barbeque grill has been discounted 15%. The sale price is \$139. Find the original price of the grill.

26. Discount Rate The price of a three-station home gym is discounted by \$300. The sale price is \$599.99. What percent of the original price is the discount?

27. Travel Time Two cars start at the same time at a given point and travel in the same direction at average speeds of 45 miles per hour and 50 miles per hour. After how long are the cars 10 miles apart?

28. Exercise Two bicyclists start at the same time at a given point and travel in the same direction at average speeds of 8 miles per hour and 10 miles per hour. After how long are the bicyclists 5 miles apart?

29. Projected Revenue From January through June, a company's revenues have totaled $375,832. If the monthly revenues continue at this rate, what will be the total revenue for the year?

30. Projected Profit From January through March, a company's profits have totaled $425,345. If the monthly profits continue at this rate, what will be the total profit for the year?

31. Mixture A car radiator contains 10 quarts of a 10% antifreeze solution. The car's owner wishes to create a 10-quart solution that is 30% antifreeze. How many quarts will have to be replaced with pure antifreeze?

32. Mixture A three-gallon acid solution contains 3% boric acid. How many gallons of 20% boric acid solution should be added to make a final solution that is 8% boric acid?

In Exercises 33–36, solve the quadratic equation by factoring. Check your solutions.

33. $6x^2 = 5x + 4$

34. $-x^2 = 15x + 36$

35. $x^2 - 11x + 24 = 0$

36. $4 - 4x + x^2 = 0$

In Exercises 37–40, solve the quadratic equation by extracting square roots. List both the exact answer and a decimal answer that has been rounded to two decimal places.

37. $x^2 = 11$

38. $16x^2 = 25$

39. $(x + 4)^2 = 18$

40. $(x - 1)^2 = 5$

41. Describe at least two ways you can use a graphing utility to check a solution of a quadratic equation.

42. Error Analysis A student solves Exercise 37 by extracting square roots and states that the exact and rounded solutions are $x = \sqrt{11}$ and $x \approx 3.32$. What error has the student made? Give an analytical argument to persuade the student that there are two *different* solutions to Exercise 37.

43. Geometry A billboard is 12 feet longer than it is high. The billboard has 405 square feet of advertising space. What are the dimensions of the billboard? Use a diagram to help answer the question.

44. Grand Canyon The Grand Canyon is 6000 feet deep at its deepest part. A rock is dropped over the deepest part of the canyon. How long does the rock take to hit the water in the Colorado River below?

45. Total Revenue The demand equation for a product is

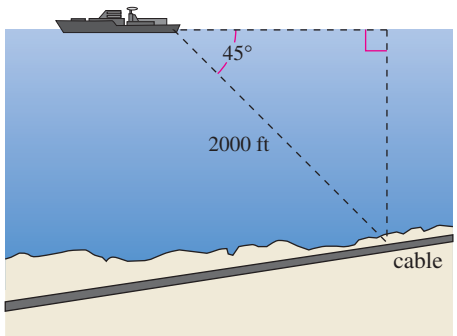
$$p = 60 - 0.0001x$$

where p is the price per unit and x is the number of units sold. The total revenue R for selling x units is given by

$$R = xp = x(60 - 0.0001x).$$

How many units must be sold to produce a revenue of $8,000,000?

46. Depth of an Underwater Cable A ship's sonar locates a cable 2000 feet from the ship (see figure). The angle between the surface of the water and a line from the ship to the cable is 45°. How deep is the cable?



In Exercises 47 and 48, use the discriminant to determine the number of real solutions of the quadratic equation.

47. $x^2 + 11x + 24 = 0$

48. $x^2 + 5x + 12 = 0$

In Exercises 49–54, use the Quadratic Formula to solve the quadratic equation. Check your solutions.

49. $x^2 - 12x + 30 = 0$

50. $5x^2 + 16x - 12 = 0$

51. $(y + 7)^2 = -5y$

52. $6x = 7 - 2x^2$

53. $x^2 + 6x - 3 = 0$

54. $10x^2 - 11x = 2$

In Exercises 55–58, use a calculator to solve the quadratic equation. (Round your answers to three decimal places.)

55. $3.6x^2 - 5.7x - 1.9 = 0$

56. $2.3x^2 + 6.6x - 3.9 = 0$

57. $34x^2 - 296x + 47 = 0$

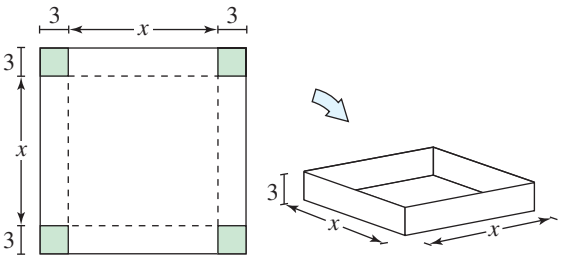
58. $39x^2 + 75x - 21 = 0$

59. On the Moon An astronaut standing on the edge of a cliff on the moon drops a rock over the cliff. The height s of the rock after t seconds is given by

$$s = -2.7t^2 + 200.$$

The rock's initial velocity is 0 feet per second and the initial height is 200 feet. Determine how long it will take the rock to hit the lunar surface. If the rock were dropped from a similar cliff on Earth, how long would it remain in the air?

60. Geometry An open box is to be made from a square piece of material by cutting three-inch squares from the corners and turning up the sides (see figure). The volume of the finished box is to be 363 cubic inches. Find the size of the original piece of material.



In Exercises 61–74, find the real solutions of the equation. Check your solutions.

61. $3x^3 - 9x^2 - 12x = 0$

62. $x^4 + 3x^3 - 5x - 15 = 0$

63. $x^4 - 5x^2 + 4 = 0$

64. $x^6 + 26x^3 - 27 = 0$

65. $2\sqrt{x} - 5 = 0$

66. $\sqrt{3x - 2} + x = 4$

67. $2\sqrt{x - 3} - 4 = 3x$

68. $\sqrt[3]{3x + 5} = 5$

69. $(x^2 - 5)^{2/3} = 9$

70. $(x^2 - 5x - 6)^{4/3} = 16$

71. $|5x + 4| = 11$

72. $|x^2 + 4x| - 2x = 8$

73. $\dfrac{5}{x + 1} + \dfrac{3}{x + 3} = 1$

74. $x + \dfrac{3}{x + 2} = 2$

75. Sharing the Cost Three students are planning to share the expense of renting a condominium at a resort for 1 week. By adding a fourth person to the group, each person could save $75 in rental fees. How much is the rent for the week?

76. Sharing the Cost A college charters a bus for $1800 to take a group to a museum. When four more students join the trip, the cost per student decreases by $5. How many students were in the original group?

77. Cash Advance You take out a cash advance of $500 on a credit card. After 3 months, the amount you owe is $535.76. What is the annual percentage rate for this cash advance? (Assume that the interest is compounded monthly and that you made no payments yet.)

78. Market Research The demand equation for a product is given by

$$p = 45 - \sqrt{0.002x + 1}$$

where x is the number of units demanded per day and p is the price per unit. Find the demand when the price is set at $19.95.

In Exercises 79–84, solve the inequality and graph the solution set on the real number line.

79. $3(x - 1) < 2x + 8$

80. $-5 \le 2 - 4(x + 2) \le 6$

81. $-3 < \dfrac{2x + 1}{4} < 3$

82. $-1 \le -5 - 3x < 4$

83. $|x + 10| + 3 < 5$

84. $|2x - 3| - 4 > 2$

85. *MAKE A DECISION: BREAK-EVEN ANALYSIS* The revenue R for selling x units of a product is

$$R = 89.95x.$$

The cost C of producing x units is

$$C = 35x + 2500.$$

In order to obtain a profit, the revenue must be greater than the cost. What are the numbers of units the company can produce in order to return a profit?

86. Accuracy of Measurement You buy a 16-inch gold chain that costs $9.95 per inch. If the chain is measured accurately to within $\frac{1}{16}$ of an inch, how much money might you have been undercharged or overcharged?

In Exercises 87–92, solve the inequality and graph the solution set on the real number line.

87. $5(x + 1)(x - 3) < 0$

88. $(x + 4)^2 \le 4$

89. $x^3 - 9x < 0$

90. $\dfrac{x + 5}{x + 8} \ge 2$

91. $\dfrac{2 + 3x}{4 - x} < 2$

92. $\dfrac{1}{x + 1} \ge \dfrac{1}{x + 5}$

In Exercises 93–96, use a calculator to solve the inequality. (Round each number in your answer to two decimal places.)

93. $-1.2x^2 + 4.76 > 1.32$

94. $3.5x^2 + 4.9x - 6.1 < 2.4$

95. $\dfrac{1}{3.7x - 6.1} > 2.9$

96. $\dfrac{3}{5.4x - 2.7} < 8.9$

In Exercises 97–102, find the domain of the expression.

97. $\sqrt{x - 10}$

98. $\sqrt[4]{2x + 5}$

99. $\sqrt[3]{2x - 1}$

100. $\sqrt[5]{x^2 - 4}$

101. $\sqrt{x^2 - 15x + 54}$

102. $\sqrt{81 - 4x^2}$

103. Height of a Projectile A projectile is fired straight upward from ground level with an initial velocity of 134 feet per second. During what time period will its height exceed 276 feet?

104. Height of a Flare A flare is fired straight upward from ground level with an initial velocity of 100 feet per second. During what time period will its height exceed 150 feet?

105. Path of a Soccer Ball The path of a soccer ball kicked from the ground can be modeled by

$$y = -0.054x^2 + 1.43x$$

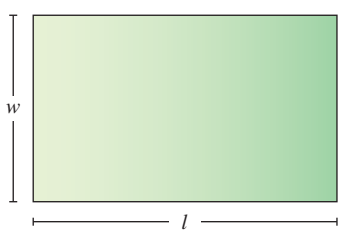
where x is the horizontal distance (in feet) from where the ball was kicked and y is the corresponding height (in feet).

(a) A soccer goal is 8 feet high. Write an inequality to determine for what values of x the ball is low enough to go into the goal.

(b) Solve the inequality from part (a).

(c) A soccer player kicks the ball toward the goal from a distance of 15 feet. No one is blocking the goal. Will the player score a goal? Explain your reasoning.

106. Geometry A rectangular field with a perimeter of 80 meters is to have an area of at least 380 square meters (see figure). Within what bounds must the length be?



107. Geometry A rectangular room with a perimeter of 60 feet is to have an area of at least 150 square feet. Within what bounds must the length be?

108. Compound Interest P dollars, invested at interest rate r compounded annually, increases to an amount

$$A = P(1 + r)^5$$

in 5 years. An investment of \$1000 increases to an amount greater than \$1400 in 5 years. The interest rate must be greater than what percent?

109. Compound Interest P dollars, invested at an interest rate r compounded semiannually, increases to an amount

$$A = P(1 + r/2)^{2 \cdot 8}$$

in 8 years. An investment of \$2000 increases to an amount greater than \$4200 in 8 years. The interest rate must be greater than what percent?

110. Company Profits The revenue R and cost C for a product are given by

$$R = x(75 - 0.0005x) \quad \text{and} \quad C = 25x + 100{,}000$$

where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$500,000?

111. Price of a Product In Exercise 110, the revenue equation is

$$R = x(75 - 0.0005x)$$

which implies that the demand equation is

$$p = 75 - 0.0005x$$

where p is the price per unit. What prices per unit can the company set to obtain a profit of at least \$1,000,000?

112. Mail Order Sales The total sales S (in billions of dollars) of prescription drugs by mail order in the United States from 1998 to 2005 can be approximated by the model

$$S = 4.37t - 21.4, \quad 8 \le t \le 15$$

where t represents the year, with $t = 8$ corresponding to 1998. *(Source: National Center for Health Statistics)*

(a) Complete the table.

t	8	11	13	15
S				

(b) Use the model to predict the year in which mail order drug sales will be at least \$60 billion

113. Revenue The revenue per share R (in dollars) for the Sonic Corporation from 1996 to 2005 can be approximated by the model

$$R = 0.0399t^2 - 0.244t + 1.61, \quad 6 \le t \le 15$$

where t represents the year, with $t = 6$ corresponding to 1996. *(Source: Sonic Corporation)*

(a) Complete the table. Round each value of R to the nearest cent.

t	6	10	13	15
R				

(b) In 2006, Sonic predicted that their revenue per share would be at least \$8.80 in 2007. Does the model support this prediction? Explain.

(c) Sonic also predicted their revenue per share will be at least \$11.10 sometime between 2009 and 2011. Does the model support this prediction? Explain.

Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

1. Solve the equation $3(x + 2) - 8 = 4(2 - 5x) + 7$.

2. Find the domain of (a) $\sqrt[3]{2x + 3}$ and (b) $\sqrt{9 - x^2}$.

3. In May, the total profit for a company was 8% less than it was in April. The total profit for the 2 months was \$625,509.12. Find the profit for each month.

In Exercises 4–13, solve the equation. Check your solution(s).

4. *Factoring:* $6x^2 + 7x = 5$

5. *Factoring:* $12 + 5x - 2x^2 = 0$

6. *Extracting roots:* $x^2 - 5 = 10$

7. *Quadratic Formula:* $(x + 5)^2 = -3x$

8. *Quadratic Formula:* $3x^2 - 11x = 2$

9. *Quadratic Formula:* $5.4x^2 - 3.2x - 2.5 = 0$

10. $|2x - 3| = 10$

11. $\sqrt{x - 3} + x = 5$

12. $x^4 - 10x^2 + 9 = 0$

13. $(x^2 - 9)^{2/3} = 9$

14. The demand equation for a product is $p = 40 - 0.0001x$, where p is the price per unit and x is the number of units sold. The total revenue R for selling x units is given by $R = xp$. How many units must be sold to produce a revenue of \$2,000,000? Explain your reasoning.

In Exercises 15–18, solve the inequality and graph the solution set on the real number line.

15. $\dfrac{3x + 1}{5} < 2$

16. $|4 - 5x| \geq 24$

17. $\dfrac{x + 3}{x + 7} > 2$

18. $3x^3 - 12x \leq 0$

19. The revenue R and cost C for a product are given by

$R = x(90 - 0.0004x)$ and $C = 25x + 300{,}000$

where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$800,000?

20. The average annual cost C (in dollars) to stay in a college dormitory from 2000 to 2005 can be approximated by the model

$C = 7.71t^2 + 136.9t + 2433, \quad 0 \leq t \leq 5$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to predict the year in which the average dormitory cost exceeds \$4000. *(Source: U.S. National Center for Education Statistics)*

Cumulative Test: Chapters 0–1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, simplify the expression.

1. $4(-2x^2)^3$ **2.** $\sqrt{18x^5}$ **3.** $\dfrac{2}{3-\sqrt{5}}$

4. Factor completely: $x^3 - 6x^2 - 3x + 18$.

5. Simplify: $\dfrac{x^2-16}{5x-20}$. **6.** Simplify: $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{y}+\frac{1}{x}}$.

7. The average monthly retail sales C (in billions of dollars) in the United States from 2000 to 2005 can be approximated by the model

$C = 11.9t + 243, \quad 0 \le t \le 5$

where t represents the year, with $t = 0$ corresponding to 2000. *(Source: U.S. Council of Economic Advisors)*

(a) Estimate the average monthly retail sales in 2005.

(b) Use the model to predict the first year in which the average monthly retail sales will exceed \$360,000,000,000.

In Exercises 8–13, solve the equation.

8. *Factoring:* $2x^2 - 11x = -5$

9. *Quadratic Formula:* $5.2x^2 + 1.5x - 3.9 = 0$

10. $|3x + 1| = 9$

11. $\sqrt{2x-1} + x = 4$

12. $x^4 - 17x^2 = -16$

13. $(x^2 - 14)^{3/2} = 8$

In Exercises 14–16, solve the inequality and graph the solution set on the real number line.

14. $-2 < \dfrac{1-3x}{5} < 2$

15. $2x^3 - 16x \ge 0$

16. $|5 - 3x| \le 21$

17. The revenue R and cost C for a product are given by

$R = x(120 - 0.0002x) \quad \text{and} \quad C = 40x + 200{,}000$

where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$600,000?

18. The per capita gross domestic product D (in dollars) in the United States from 2000 to 2005 can be approximated by the model

$D = 228.57t^2 + 323.3t + 34{,}808, \quad 0 \le t \le 5$

where t represents the year, with $t = 0$ corresponding to 2000. According to this model, in what year will per capita gross domestic product first exceed \$50,000? *(Source: U.S. Bureau of Economic Analysis)*

2 Functions and Graphs



altrendo images/Getty Images

The first Ferris wheel stood about 264 feet tall. It was designed by George Washington Gale Ferris Jr. for the World's Columbian Exposition in Chicago, Illinois, in 1893. You can use the standard form of the equation of a circle to model the shape of a Ferris wheel. (See Section 2.1, Exercises 111 and 112.)

2.1 Graphs of Equations
2.2 Lines in the Plane
2.3 Linear Modeling and Direct Variation
2.4 Functions
2.5 Graphs of Functions
2.6 Transformations of Functions
2.7 The Algebra of Functions
2.8 Inverse Functions

Applications

Functions and graphs are used to model and solve many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Make a Decision: Yahoo! Inc. Revenue, Exercise 101, page 181
- Path of a Salmon, Exercise 74, page 205
- Earnings-Dividend Ratio, Exercise 73, page 247

Section 2.1

Graphs of Equations

- Plot points in the Cartesian plane.
- Use the Distance Formula to find the distance between two points in the coordinate plane.
- Use the Midpoint Formula to find the midpoint of a line segment joining two points.
- Determine whether a point is a solution of an equation.
- Sketch the graph of an equation using a table of values.
- Find the x- and y-intercepts of the graph of an equation.
- Determine the symmetry of a graph.
- Write the equation of a circle in standard form.

The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane. This plane is called the **rectangular coordinate system,** or the **Cartesian plane,** named after the French mathematician René Descartes (1596–1650).

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure 2.1. The horizontal real number line is usually called the ***x*-axis,** and the vertical real number line is usually called the ***y*-axis.** The point of intersection of these two axes is the **origin,** and the two axes divide the plane into four parts called **quadrants.**

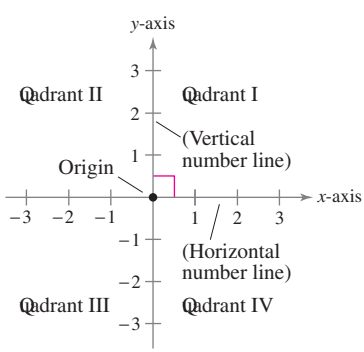


FIGURE 2.1

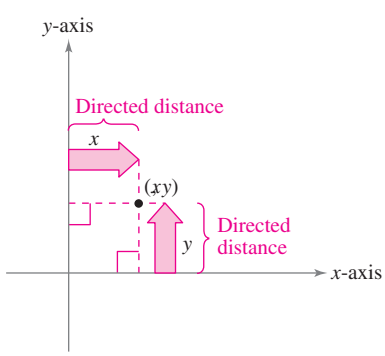


FIGURE 2.2

Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y, called the **coordinates** of the point. The ***x*-coordinate** represents the directed distance from the y-axis to the point, and the ***y*-coordinate** represents the directed distance from the x-axis to the point, as shown in Figure 2.2.

Directed distance from y-axis (x, y) Directed distance from x-axis

The notation (x, y) denotes both a point in the plane and an open interval on the real number line. The context will tell you which meaning is intended.

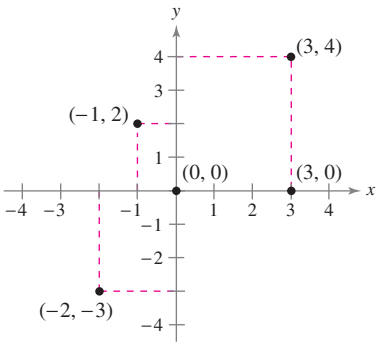


FIGURE 2.3

Example 1 **Plotting Points in the Cartesian Plane**

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

SOLUTION To plot the point $(-1, 2)$, imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. The other four points can be plotted in a similar way, as shown in Figure 2.3.

✓CHECKPOINT 1

Plot the points $(2, 3)$ and $(-4, 1)$. ■

The Distance and Midpoint Formulas

Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse of length c and legs of lengths a and b, you have $a^2 + b^2 = c^2$, as shown in Figure 2.4. (The converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.)

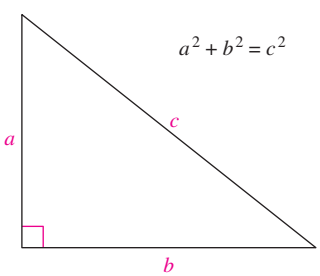


FIGURE 2.4

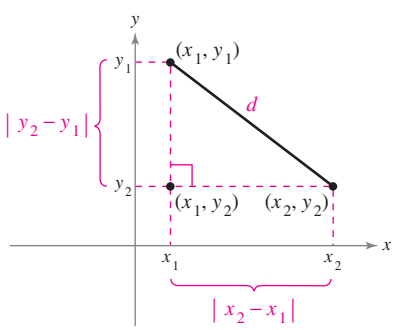


FIGURE 2.5

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) that do not lie on the same horizontal or vertical line. With these two points, a right triangle can be formed, as shown in Figure 2.5. The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem, you can write

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$ Choose positive square root.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The result is the **Distance Formula.**

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the coordinate plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The following formula shows how to find the *midpoint* of the line segment that joins two points.

The Midpoint Formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) in the coordinate plane is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 2 Using the Distance and Midpoint Formulas

Find (a) the distance between, and (b) the midpoint of the line segment joining, the points $(-2, 1)$ and $(3, 4)$.

SOLUTION

a. Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$, and apply the Distance Formula.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} && \text{Substitute for } x_1, x_2, y_1, \text{ and } y_2. \\ &= \sqrt{5^2 + 3^2} && \text{Simplify.} \\ &= \sqrt{34} \approx 5.83 && \text{Simplify.} \end{aligned}$$

See Figure 2.6.

b. By the Midpoint Formula, you have

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) && \text{Midpoint Formula} \\ &= \left(\frac{-2 + 3}{2}, \frac{1 + 4}{2}\right) && \text{Substitute for } x_1, x_2, y_1, \text{ and } y_2. \\ &= \left(\frac{1}{2}, \frac{5}{2}\right). && \text{Simplify.} \end{aligned}$$

See Figure 2.7.

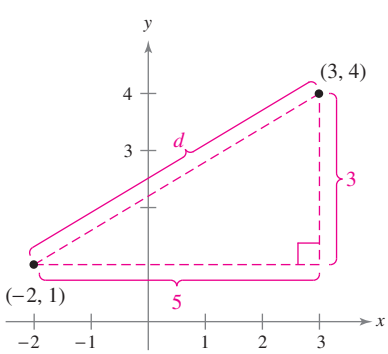


FIGURE 2.6 $d = \sqrt{34} \approx 5.83$

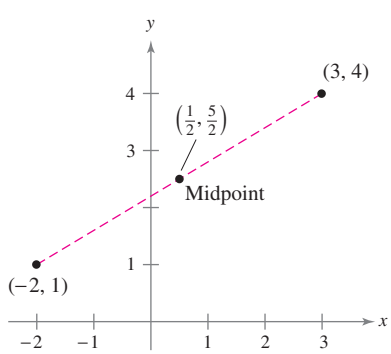


FIGURE 2.7

✓CHECKPOINT 2

Find (a) the distance between, and (b) the midpoint of the line segment joining, the points $(-2, 2)$ and $(6, -4)$. ■

The Graph of an Equation

Frequently, a relationship between two quantities is written in the form of an equation. In the remainder of this section, you will study a procedure for sketching the graph of an equation. For an equation in the variables x and y, a point (a, b) is a **solution** if the substitution $x = a$ and $y = b$ satisfies the equation.

Example 3 Solution of an Equation

Determine whether $(-1, 0)$ is a solution of the equation $y = 2x^2 - 4x - 6$.

SOLUTION

$$y = 2x^2 - 4x - 6 \qquad \text{Write original equation.}$$
$$0 \stackrel{?}{=} 2(-1)^2 - 4(-1) - 6 \qquad \text{Substitute } -1 \text{ for } x \text{ and } 0 \text{ for } y.$$
$$0 = 0 \qquad \text{Simplify.}$$

Both sides of the equation are equivalent, so the point $(-1, 0)$ is a solution.

✓CHECKPOINT 3

Determine whether $(-1, 3)$ is a solution of the equation $y = x + 4$. ■

Most equations have *infinitely* many solutions. The **graph of an equation** is the set of all points that are solutions of the equation.

Example 4 Sketching the Graph of an Equation

Sketch the graph of $3x + y = 5$.

SOLUTION First rewrite the equation as $y = 5 - 3x$ with y isolated on the left. Next, construct a table of values by choosing several values of x and calculating the corresponding values of y.

x	-1	0	1	2	3
$y = 5 - 3x$	8	5	2	-1	-4

From the table, it follows that $(-1, 8)$, $(0, 5)$, $(1, 2)$, $(2, -1)$, and $(3, -4)$ are solution points of the equation. After plotting these points and connecting them, you can see that they appear to lie on a line, as shown in Figure 2.8.

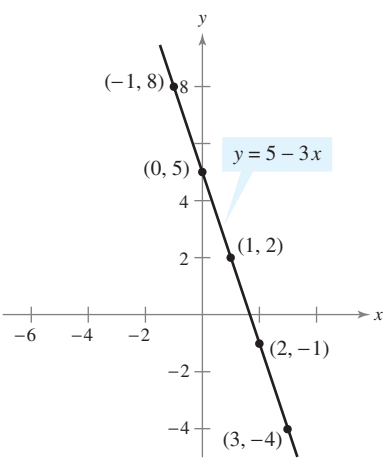


FIGURE 2.8

✓CHECKPOINT 4

Sketch the graph of $y - x = 3$. ■

The Point-Plotting Method of Graphing

1. If possible, isolate one of the variables.
2. Construct a table of values showing several solution points.
3. Plot these points on a rectangular coordinate system.
4. Connect the points with a smooth curve or line.

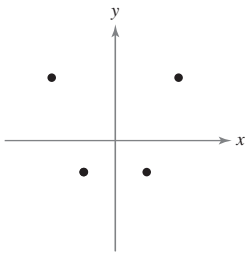


FIGURE 2.9

Step 4 of the point-plotting method can be difficult. For instance, how would you connect the four points in Figure 2.9? Without further information about the equation, any one of the three graphs in Figure 2.10 would be reasonable. These graphs show that with too few solution points, you can misrepresent the graph of an equation. Throughout this course, you will study many ways to improve your graphing techniques. For now, you should plot enough points to reveal the essential behavior of the graph. It is important to use negative values, zero, and positive values for x when constructing a table.

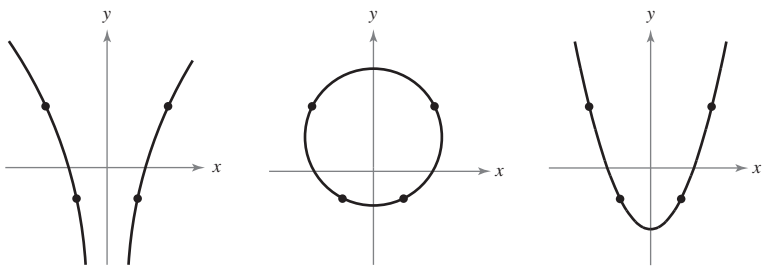


FIGURE 2.10

Example 5 Sketching the Graph of an Equation

Sketch the graph of $y = x^2 - 2$.

SOLUTION First, construct a table of values by choosing several convenient values of x and calculating the corresponding values of y.

x	-3	-2	-1	0	1	2	3
$y = x^2 - 2$	7	2	-1	-2	-1	2	7

Next, plot the corresponding solution points. Finally, connect the points with a smooth curve, as shown in Figure 2.11.

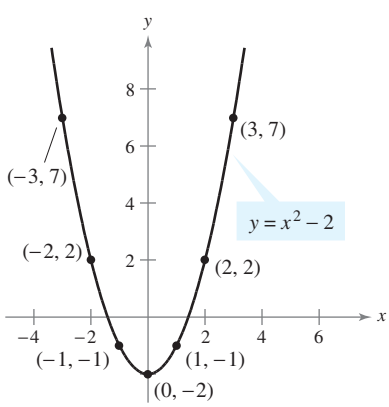


FIGURE 2.11

✓CHECKPOINT 5

Sketch the graph of $y = x^2 + 3$. ■

Intercepts of a Graph

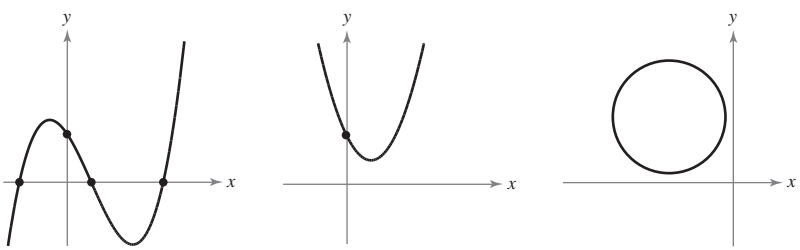
When you are sketching a graph, points for which either the y-coordinate or the x-coordinate is zero are especially useful.

Definition of Intercepts

1. The ***x*-intercepts** of a graph are the points at which the graph intersects the x-axis. To find the x-intercepts, let y equal zero and solve for x.
2. The ***y*-intercepts** of a graph are the points at which the graph intersects the y-axis. To find the y-intercepts, let x equal zero and solve for y.

Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, we will use the term *intercept* to mean either the point or the coordinate.

A graph may have no intercepts, one intercept, or several intercepts. For instance, consider the three graphs in Figure 2.12.



Three x-intercepts
One y-intercept

No x-intercept
One y-intercept

No intercepts

FIGURE 2.12

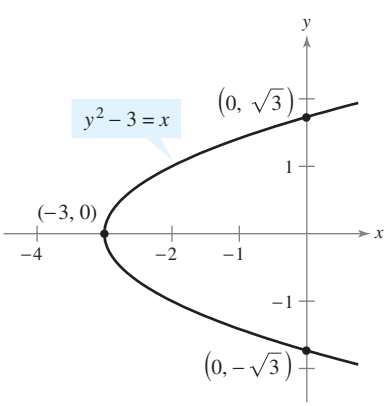


FIGURE 2.13

Example 6 Finding *x*- and *y*-Intercepts

Find the x- and y-intercepts of the graph of

$$y^2 - 3 = x.$$

SOLUTION To find the x-intercept, let $y = 0$. This produces $-3 = x$, which implies that the graph has one x-intercept, which occurs at

$$(-3, 0).$$ x-intercept

To find the y-intercept, let $x = 0$. This produces $y^2 - 3 = 0$, which has two solutions: $y = \pm\sqrt{3}$. So, the graph has two y-intercepts, which occur at

$$(0, \sqrt{3}) \quad \text{and} \quad (0, -\sqrt{3}).$$ y-intercepts

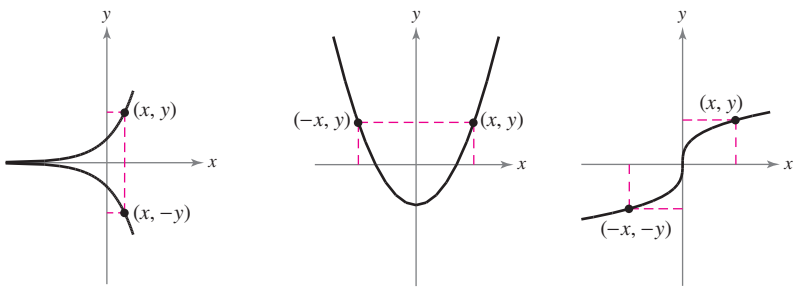
See Figure 2.13.

✓CHECKPOINT 6

Find the x- and y-intercepts of the graph of $y = 3x + 2$. ■

Symmetry

Symmetry with respect to the x-axis means that if the Cartesian plane were folded along the x-axis, the portion of the graph above the x-axis would coincide with the portion below the x-axis. Symmetry with respect to the y-axis can be described in a similar manner. Symmetry with respect to the origin means that when the graph rotates 180° about the origin, it looks the same. (See Figure 2.14.)



x-Axis symmetry y-Axis symmetry Origin symmetry

FIGURE 2.14

Knowing the symmetry of a graph *before* attempting to sketch it is helpful, because then you need only half as many solution points to sketch the graph. The three basic types of symmetry are described as follows.

Definition of Symmetry

1. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is on the graph, $(x, -y)$ is also on the graph.
2. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is on the graph, $(-x, y)$ is also on the graph.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is on the graph, $(-x, -y)$ is also on the graph.

You can apply this definition of symmetry to the graph of the equation $y = x^2 - 1$. Replacing x with $-x$ produces the following.

$y = x^2 - 1$ Write original equation.

$y = (-x)^2 - 1$ Replace x with $-x$.

$y = x^2 - 1$ Replacement yields equivalent equation.

Because the substitution did not change the equation, it follows that if (x, y) is a solution of the equation, then $(-x, y)$ must also be a solution. So, the graph of $y = x^2 - 1$ is symmetric with respect to the y-axis. By plotting the points in the table below, you can confirm that the graph is symmetric with respect to the y-axis, as shown in Figure 2.15.

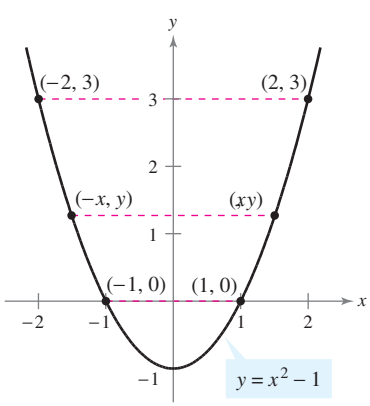


FIGURE 2.15 y-Axis Symmetry

x	-2	-1	1	2
y	3	0	0	3

Tests for Symmetry

1. The graph of an equation is symmetric with respect to the x-axis if replacing y with $-y$ yields an equivalent equation.
2. The graph of an equation is symmetric with respect to the y-axis if replacing x with $-x$ yields an equivalent equation.
3. The graph of an equation is symmetric with respect to the *origin* if replacing x with $-x$ *and* y with $-y$ yields an equivalent equation.

Example 7 Using Symmetry as a Sketching Aid

Describe the symmetry of the graph of $x - y^2 = 1$.

SOLUTION Of the three tests for symmetry, the only one that is satisfied by this equation is the test for x-axis symmetry.

$$x - y^2 = 1 \qquad \text{Write original equation.}$$
$$x - (-y)^2 = 1 \qquad \text{Replace } y \text{ with } -y.$$
$$x - y^2 = 1 \qquad \text{Replacement yields equivalent equation.}$$

So, the graph is symmetric with respect to the x-axis. To sketch the graph, plot the points above the x-axis and use symmetry to complete the graph, as shown in Figure 2.16.

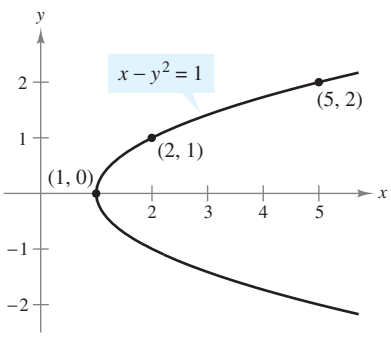


FIGURE 2.16

✓CHECKPOINT 7

Describe the symmetry of the graph of $y = |x|$. ■

The Equation of a Circle

In this section, you have studied the point-plotting method and two additional concepts (intercepts and symmetry) that can be used to streamline the graphing procedure. Another graphing aid is *equation recognition,* which is the ability to recognize the general shape of a graph simply by looking at its equation.

Figure 2.17 shows a circle of *radius* r with *center* at the point (h, k). The point (x, y) is on this circle if and only if its distance from the center (h, k) is r. This means that a **circle** in the plane consists of all points (x, y) that are a given positive distance r from a fixed point (h, k). Using the Distance Formula, you can conclude that the point (x, y) lies on the circle if and only if

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

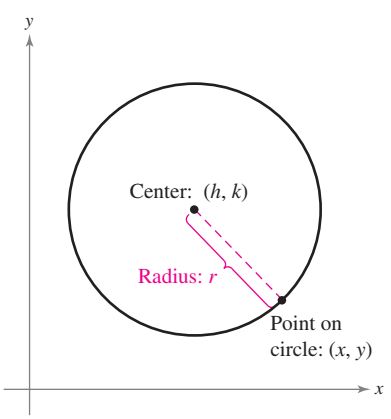


FIGURE 2.17

By squaring each side of this equation, you obtain the **standard form of the equation of a circle.** For example, a circle with its center at the origin, $(h, k) = (0, 0)$, and radius $r = 4$ is given by

$$\sqrt{(x - 0)^2 + (y - 0)^2} = 4 \qquad \text{Substitute for } h, k, \text{ and } r.$$
$$\sqrt{x^2 + y^2} = 4 \qquad \text{Simplify.}$$
$$x^2 + y^2 = 16. \qquad \text{Square each side.}$$

Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** is

$$(x - h)^2 + (y - k)^2 = r^2.$$

The point (h, k) is called the **center** of the circle, and the positive number r is called the **radius** of the circle. The standard form of the equation of a circle whose center is the *origin* is $x^2 + y^2 = r^2$.

Example 8 Finding the Equation of a Circle

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure 2.18. Write the standard form of the equation of this circle.

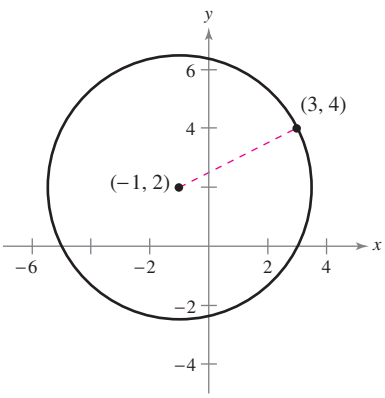


FIGURE 2.18

SOLUTION The radius of the circle is the distance between the center $(-1, 2)$ and the point $(3, 4)$.

$$r = \sqrt{(x - h)^2 + (y - k)^2} \qquad \text{Distance Formula}$$
$$r = \sqrt{[3 - (-1)]^2 + (4 - 2)^2} \qquad \text{Substitute for } x, y, h, \text{ and } k.$$
$$= \sqrt{4^2 + 2^2} \qquad \text{Simplify.}$$
$$= \sqrt{16 + 4} \qquad \text{Simplify.}$$
$$= \sqrt{20} \qquad \text{Radius}$$

Using $(h, k) = (-1, 2)$ and $r = \sqrt{20}$, the equation of the circle is

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Equation of circle}$$
$$[x - (-1)]^2 + (y - 2)^2 = \left(\sqrt{20}\right)^2 \qquad \text{Substitute for } h, k, \text{ and } r.$$
$$(x + 1)^2 + (y - 2)^2 = 20. \qquad \text{Standard form}$$

✓CHECKPOINT 8

The point $(4, 4)$ lies on a circle whose center is at $(0, 1)$. Write the standard form of the equation of this circle. ■

If you expand the standard equation in Example 8, you obtain the following.

$$(x + 1)^2 + (y - 2)^2 = 20 \qquad \text{Standard form}$$
$$x^2 + 2x + 1 + y^2 - 4y + 4 = 20 \qquad \text{Expand terms.}$$
$$x^2 + y^2 + 2x - 4y - 15 = 0 \qquad \text{General form}$$

The last equation is in the **general form of the equation of a circle,**

$$Ax^2 + Ay^2 + Dx + Ey + F = 0, \quad A \neq 0.$$

The general form of the equation of a circle is less useful than the standard form. For instance, it is not immediately apparent from the general equation shown above that the center is $(-1, 2)$ and the radius is $\sqrt{20}$. To graph the equation of a circle, it is best to write the equation in standard form. You can do this by **completing the square,** as demonstrated in Example 9.

Example 9 Completing the Square to Sketch a Circle

Sketch the circle given by $4x^2 + 4y^2 + 20x - 16y + 37 = 0$.

SOLUTION Begin by writing the original equation in standard form by completing the square for both the x-terms *and* the y-terms.

$$4x^2 + 4y^2 + 20x - 16y + 37 = 0 \qquad \text{Write original equation.}$$

$$x^2 + y^2 + 5x - 4y + \frac{37}{4} = 0 \qquad \text{Divide by 4.}$$

$$\left(x^2 + 5x + \square\right) + \left(y^2 - 4y + \square\right) = -\frac{37}{4} \qquad \text{Group terms.}$$

$$\left[x^2 + 5x + \left(\frac{5}{2}\right)^2\right] + (y^2 - 4y + 2^2) = -\frac{37}{4} + \frac{25}{4} + 4 \qquad \text{Complete the square.}$$

$$\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = 1 \qquad \text{Standard form}$$

So, the center of the circle is $\left(-\frac{5}{2}, 2\right)$ and the radius of the circle is 1. Using this information, you can sketch the circle, as shown in Figure 2.19.

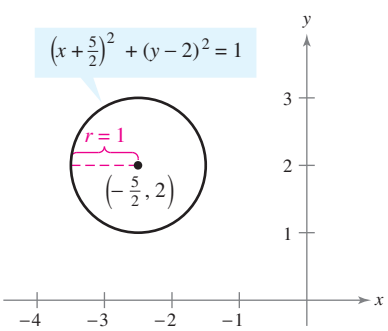


FIGURE 2.19

STUDY TIP

Recall that to complete the square, you add the square of half the coefficient of the linear term to each side.

✓CHECKPOINT 9

Sketch the circle given by $x^2 + y^2 - 2x - 4y + 1 = 0$. ■

CONCEPT CHECK

1. Determine whether the following statement is true or false. Explain your reasoning.

 The points (3, 4) and (−4, 3) both lie on the same circle whose center is the origin.

2. Explain how to find the x- and y-intercepts of the graph of an equation.
3. For every point (x, y) on a graph, the point $(-x, y)$ is also on the graph. What type of symmetry must the graph have? Explain.
4. Is the point (0, 0) on the circle whose equation in standard form is $(x - 0)^2 + (y - 0)^2 = 4$? Explain.

Skills Review 2.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.4, and 1.5.

In Exercises 1–6, simplify the expression.

1. $\sqrt{(2-6)^2+[1-(-2)]^2}$

2. $\sqrt{(1-4)^2+(-2-1)^2}$

3. $\dfrac{4+(-2)}{2}$

4. $\dfrac{-1+(-3)}{2}$

5. $\sqrt{18}+\sqrt{45}$

6. $\sqrt{12}+\sqrt{44}$

In Exercises 7–10, solve the equation.

7. $\sqrt{(4-x)^2+(5-2)^2}=\sqrt{58}$

8. $\sqrt{(8-6)^2+(y-5)^2}=2\sqrt{5}$

9. $x^3-9x=0$

10. $x^4-8x^2+16=0$

Exercises 2.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

1. $(2,-5),(-6,1)$

2. $(1,12),(6,0)$

3. $(3,-11),(-12,-3)$

4. $(-7,3),(2,-9)$

5. $(-1,2),(5,4)$

6. $(2,10),(10,2)$

7. $\left(\frac{1}{2},1\right),\left(-\frac{5}{2},\frac{4}{3}\right)$

8. $\left(-\frac{1}{3},-\frac{1}{3}\right),\left(-\frac{1}{6},-\frac{1}{2}\right)$

9. $(1.8,7.5),(-2.5,2.1)$

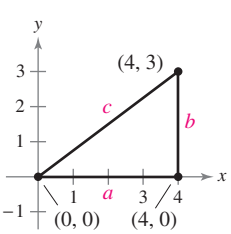
10. $(37.5,-12.3),(-6.2,5.9)$

11. $(-36,-18),(48,-72)$

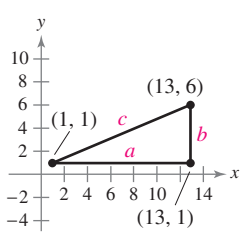
12. $(1.451,3.051),(5.906,11.360)$

In Exercises 13–16, find the length of the hypotenuse in two ways: (a) using the Pythagorean Theorem and (b) using the Distance Formula.

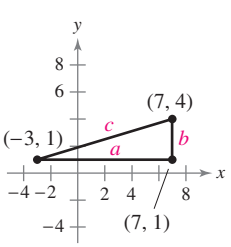
13.



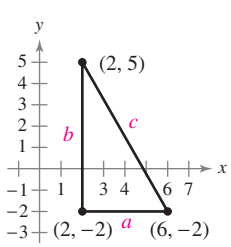
14.



15.



16.



In Exercises 17 and 18, find x such that the distance between the points is 15.

17. $(3,-4),(x,5)$

18. $(x,8),(-9,-4)$

In Exercises 19 and 20, find y such that the distance between the points is 20.

19. $(-15,y),(-3,-7)$

20. $(6,-1),(-10,y)$

In Exercises 21–24, determine whether each point is a solution of the equation.

	Equation	*Points*	
21.	$2x-3y+11=0$	(a) $(2,5)$	(b) $(3,2)$
22.	$y=2x^2-7x+3$	(a) $(1,-1)$	(b) $(3,0)$
23.	$y=\sqrt{x-5}$	(a) $(9,2)$	(b) $(21,4)$
24.	$y=\dfrac{x+1}{5-x}$	(a) $\left(1,\frac{1}{2}\right)$	(b) $(0,1)$

In Exercises 25 and 26, complete the table. Use the resulting solution points to sketch the graph of the equation.

25. $y=\frac{3}{4}x-1$

x	-2	0	1	$\frac{4}{3}$	2
y					

26. $y=5-x^2$

x	-2	-1	0	1	2
y					

In Exercises 27–34, find the x- and y-intercepts of the graph of the equation.

27. $y = 2x - 1$

28. $y = (x - 4)(x + 2)$

29. $y = x^2 + x - 2$

30. $y = 4 - x^2$

31. $y = x\sqrt{x + 2}$

32. $y = x\sqrt{x + 5}$

33. $2y - xy + 3x = 4$

34. $x^2y - x^2 + 4y = 0$

35. Use your knowledge of the Cartesian plane and intercepts to explain why you let y equal zero when you are finding the x-intercepts of the graph of an equation, and why you let x equal zero when you are finding the y-intercepts of the graph of an equation.

36. Is it possible for a graph to have no x-intercepts? no y-intercepts? no x-intercepts and no y-intercepts? Give examples to support your answers.

In Exercises 37–48, check for symmetry with respect to both axes and the origin.

37. $x^4 - 2y = 0$

38. $y = x^4 - x^2 + 3$

39. $x - y^2 = 0$

40. $y^2 = x + 2$

41. $y = \sqrt{16 - x^2}$

42. $y = \sqrt{4 - x^2}$

43. $xy = 2$

44. $x^3y = 1$

45. $y = \dfrac{x}{x^2 - 4}$

46. $y = \dfrac{x}{x^2 + 1}$

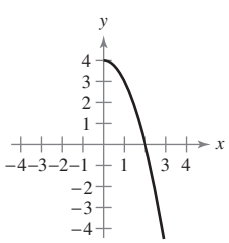
47. $x^2 + y^2 = 25$

48. $x^2 + y^2 = 9$

In Exercises 49–52, use symmetry to complete the graph of the equation.

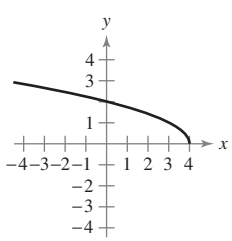
49. y-axis symmetry

$y = -x^2 + 4$



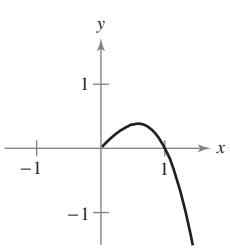
50. x-axis symmetry

$y^2 = -x + 4$



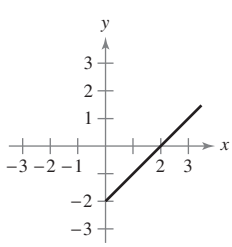
51. Origin symmetry

$y = -x^3 + x$



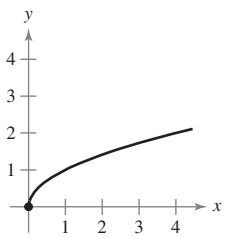
52. y-axis symmetry

$y = |x| - 2$

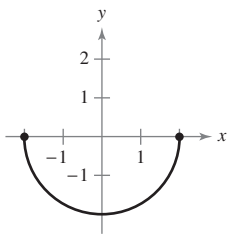


In Exercises 53–58, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

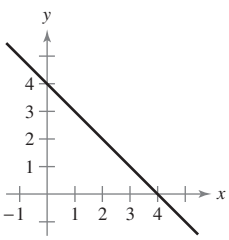
(a)



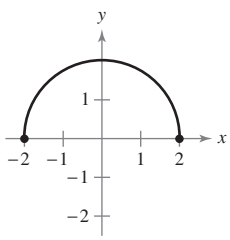
(b)



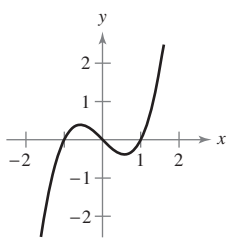
(c)



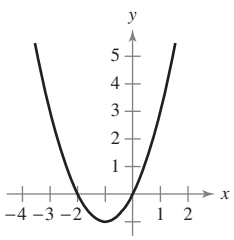
(d)



(e)



(f)



53. $y = 4 - x$

54. $y = \sqrt{4 - x^2}$

55. $y = x^2 + 2x$

56. $y = \sqrt{x}$

57. $y = x^3 - x$

58. $y = -\sqrt{4 - x^2}$

In Exercises 59–78, sketch the graph of the equation. Identify any intercepts and test for symmetry.

59. $y = 5 - 3x$

60. $y = 2x - 3$

61. $y = 1 - x^2$

62. $y = x^2 - 1$

63. $y = x^2 - 4x + 3$

64. $y = -x^2 - 4x$

65. $y = x^3 + 2$

66. $y = x^3 - 1$

67. $y = \dfrac{8}{x^2 + 4}$

68. $y = \dfrac{4}{x^2 + 1}$

69. $y = \sqrt{x + 1}$

70. $y = \sqrt{1 - x}$

71. $y = \sqrt[3]{x}$

72. $y = \sqrt[3]{x + 1}$

73. $y = |x - 4|$

74. $y = |x| - 3$

75. $x = y^2 - 1$

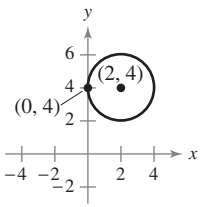
76. $x = y^2 - 4$

77. $x^2 + y^2 = 4$

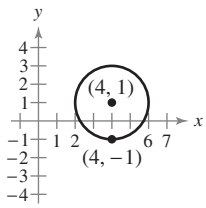
78. $x^2 + y^2 = 16$

In Exercises 79–82, find the radius of the circle given the center and a point on the circle.

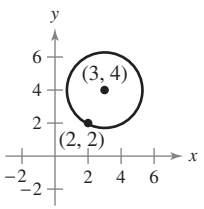
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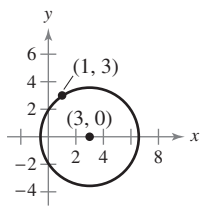
80.



81.

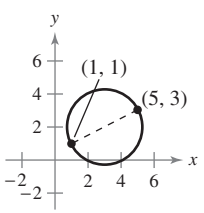


82.

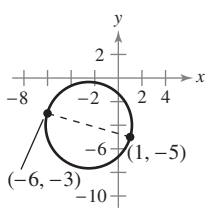


In Exercises 83 and 84, find the center and radius of the circle given the endpoints of the diameter of the circle.

83.



84.



In Exercises 85–92, find the standard form of the equation of the specified circle.

85. Center: $(0, 0)$; radius: 3

86. Center: $(0, 0)$; radius: 5

87. Center: $(-4, 1)$; radius: $\sqrt{2}$

88. Center: $\left(0, \frac{1}{2}\right)$; radius: $\frac{2}{3}$

89. Center: $(-1, 2)$; point on circle: $(0, 0)$

90. Center: $(3, -2)$; point on circle: $(-1, 1)$

91. Endpoints of a diameter: $(-3, 4)$, $(5, -2)$

92. Endpoints of a diameter: $(-4, -1)$, $(4, 1)$

In Exercises 93–100, write the equation of the circle in standard form. Then sketch the circle.

93. $x^2 + y^2 - 6x + 4y - 3 = 0$

94. $x^2 + y^2 - 2x + 6y - 15 = 0$

95. $x^2 + y^2 - 4x + 6y + 9 = 0$

96. $5x^2 + 5y^2 + 10x + 1 = 0$

97. $2x^2 + 2y^2 - 2x - 2y - 3 = 0$

98. $4x^2 + 4y^2 - 4x + 2y - 1 = 0$

99. $16x^2 + 16y^2 + 16x + 40y - 7 = 0$

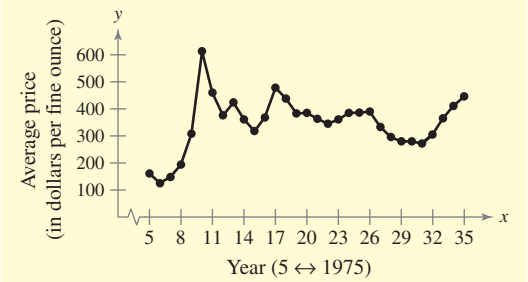
100. $x^2 + y^2 - 4x + 2y + 3 = 0$

In Exercises 101 and 102, an equation of a circle is written in standard form. Indicate the coordinates of the center of the circle and determine the radius of the circle. Rewrite the equation of the circle in general form.

101. $(x - 3)^2 + (y + 1)^2 = 25$

102. $\left(x - \frac{1}{2}\right)^2 + (y - 2)^2 = 7$

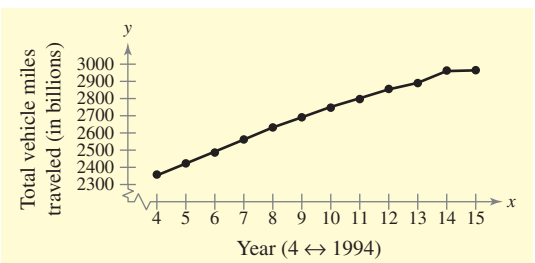
Gold Prices In Exercises 103 and 104, use the figure below, which shows the average prices of gold for the years 1975 to 2005. *(Sources: U.S. Bureau of Mines; U.S. Geological Survey)*



103. What is the highest price of gold shown in the graph? When did this occur?

104. What is the lowest price of gold shown in the graph? When did this occur?

Vehicle Distance In Exercises 105 and 106, use the figure below, which shows the total number of miles traveled by vehicles in the United States each year from 1994 to 2005. *(Source: National Highway Traffic Safety Administration)*



105. Estimate the percent increase in miles traveled by vehicles from 1994 to 2000.

106. Estimate the percent increase in miles traveled by vehicles from 2000 to 2005.

107. Population The population y (in millions of people) of North America from 1980 to 2050 can be modeled by

$$y = 5.3x + 377, \quad -20 \le x \le 50$$

where x represents the year, with $x = 50$ corresponding to 2050. *(Source: U.S. Census Bureau)*

(a) Find the y-intercept of the graph of the model. What does it represent in the given situation?

(b) Construct a table of values for $x = -20, -10, 0, 10, 20, 30, 40$, and 50.

(c) Plot the solution points given by the table in part (b) and use the points to sketch the graph of the model.

108. Profit The annual profits y (in millions of dollars) of UnitedHealth Group from 1997 to 2006 can be modeled by

$$y = 58.86x^2 + 228.0x + 677, \quad -3 \le x \le 6$$

where x represents the year, with $x = 6$ corresponding to 2006. *(Source: UnitedHealth Group)*

(a) Find the y-intercept of the graph of the model. What does it represent in the given situation?

(b) Use the model to complete the table of values for $x = -3, -2, -1, 0, 1, 2, 3, 4, 5$, and 6.

(c) Plot the solution points given by the table in part (b) and use the points to sketch the graph of the model.

109. Earnings per Share The earnings per share y (in dollars) for Dollar Tree Stores from 1996 to 2005 can be modeled by

$$y = -0.0082t^2 + 0.318t - 1.28, \quad 6 \le t \le 15$$

where t represents the year, with $t = 6$ corresponding to 1996. *(Source: Dollar Tree Stores)*

(a) Sketch a graph of the equation.

(b) In 2005, Dollar Tree predicted that its earnings per share would be \$1.75 in 2006 and \$1.95 in 2007. Use the model to predict the earnings per share for these years. How well does the model support Dollar Tree's predictions?

(c) Dollar Tree also predicted its earnings per share to reach \$3.00 sometime in 2009, 2010, or 2011. How well does the model support Dollar Tree's prediction?

110. Earnings per Share The earnings per share y (in dollars) for Paychex, Inc. from 1996 to 2005 can be modeled by

$$y = -0.0014t^2 + 0.123t - 0.57, \quad 6 \le t \le 15$$

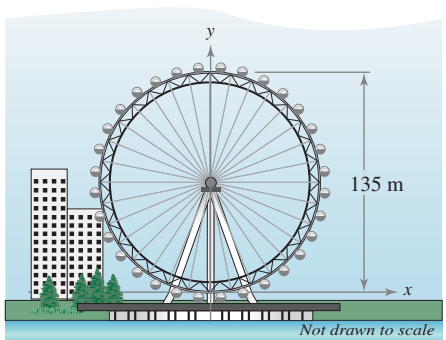
where t represents the year, with $t = 6$ corresponding to 1996. *(Source: Paychex, Inc.)*

(a) Sketch a graph of the equation.

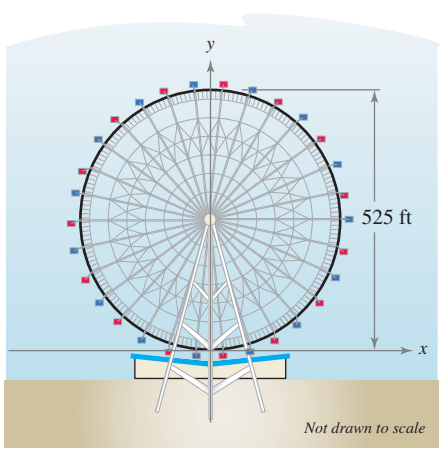
(b) In 2005, Paychex predicted that its earnings per share would be \$1.22 in 2006 and \$1.40 in 2007. Use the model to predict the earnings per share for these years. How well does the model support Paychex's predictions?

(c) Paychex also predicted its earnings per share to reach \$1.90 sometime in 2009, 2010, or 2011. How well does the model support Paychex's prediction?

111. The London Eye The London Eye is a Ferris wheel that opened in 1999 in London, England as the tallest in the world. It remained the tallest Ferris wheel in the world until the Star of Nanchang was opened in 2006. The London Eye stands 135 meters tall. Use the diagram below to write an equation that models the circular shape of the London Eye wheel.



112. Star of Nanchang In 2006, a Ferris wheel called the Star of Nanchang was opened in the Jiangxi province of China to replace the London Eye as the world's largest Ferris wheel. The Star of Nanchang stands 525 feet tall. Use the diagram below to write an equation that models the circular shape of the Star of Nanchang wheel.



Section 2.2
Lines in the Plane

- Find the slope of a line passing through two points.
- Use the point-slope form to find the equation of a line.
- Use the slope-intercept form to sketch a line.
- Use slope to determine if lines are parallel or perpendicular, and write the equation of a line parallel or perpendicular to a given line.

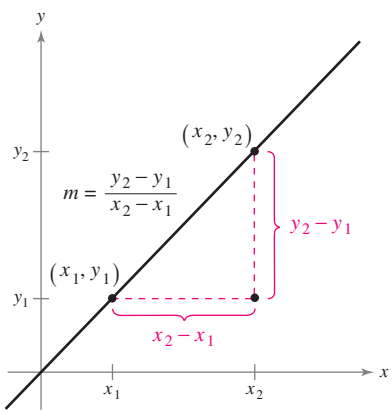


FIGURE 2.20

The Slope of a Line

The **slope** of a nonvertical line is a measure of the steepness of the line. The slope represents the number of units the line rises or falls vertically for each unit of horizontal change from left to right. For instance, consider the two points (x_1, y_1) and (x_2, y_2) on the line shown in Figure 2.20. As you move from left to right along this line, a change of $y_2 - y_1$ units in the vertical direction corresponds to a change of $x_2 - x_1$ units in the horizontal direction. That is,

$$y_2 - y_1 = \text{the change in } y \quad \text{and} \quad x_2 - x_1 = \text{the change in } x.$$

The slope of the line is defined as the quotient of these two changes.

Definition of the Slope of a Line

The **slope** m of the nonvertical line passing through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

where $x_1 \neq x_2$.

The change in x is sometimes called the *run* and the change in y is sometimes called the *rise*.

When this formula is used for slope, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as (x_1, y_1) and the other as (x_2, y_2). However, once this is done, you must form the numerator and denominator using the same order of subtraction.

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_1 - y_2}{x_1 - x_2}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_2 - y_1}{x_1 - x_2}}_{\text{Incorrect}}$$

In real-life problems, such as finding the steepness of a ramp or the increase in the value of a product, the slope of a line can be interpreted as either a *ratio* or a *rate*. If the x-axis and the y-axis have the same units of measure, then the slope has no units and is a *ratio*. If the x-axis and the y-axis have different units of measure, then the slope is a *rate* or *rate of change*. You will learn more about rates of change in Section 2.3.

Example 1 Finding the Slope of a Line Through Two Points

Find the slope of the line passing through each pair of points.

a. $(-2, 0)$ and $(3, 1)$ **b.** $(-1, 2)$ and $(2, 2)$

c. $(0, 4)$ and $(1, -1)$ **d.** $(3, 4)$ and $(3, 1)$

SOLUTION

a. Letting $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (3, 1)$, you obtain a slope of

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{3 - (-2)} = \frac{1}{5}.$$

Difference in y-values / Difference in x-values

b. The slope of the line passing through $(-1, 2)$ and $(2, 2)$ is

$$m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0.$$

c. The slope of the line passing through $(0, 4)$ and $(1, -1)$ is

$$m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5.$$

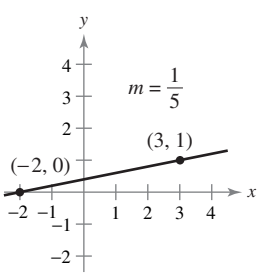
d. The slope of the line passing through $(3, 4)$ and $(3, 1)$ is undefined. Applying the formula for slope, you have

$$m = \frac{1 - 4}{3 - 3} = \frac{-3}{0}.$$

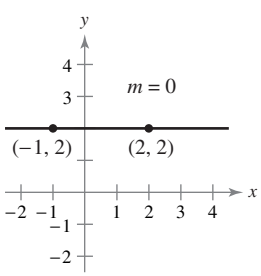
Division by zero is undefined.

Because division by zero is not defined, the slope of a vertical line is not defined.

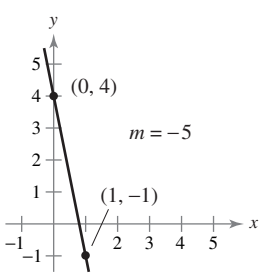
The graphs of the four lines are shown in Figure 2.21.



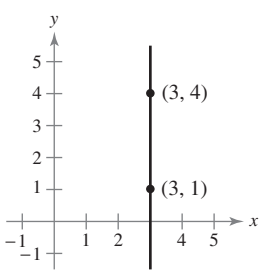
(a) If m is positive, the line rises from left to right.



(b) If m is zero, the line is horizontal.



(c) If m is negative, the line falls from left to right.



(d) If the line is vertical, the slope is undefined.

FIGURE 2.21

✓CHECKPOINT 1

Find the slope of the line passing through the points $(-1, 2)$ and $(3, 4)$. ■

From the slopes of the lines shown in Example 1, you can make the following generalizations about the slope of a line.

Slope of a Line

1. A line with positive slope ($m > 0$) *rises* from left to right.
2. A line with negative slope ($m < 0$) *falls* from left to right.
3. A line with zero slope ($m = 0$) is *horizontal*.
4. A line with undefined slope is *vertical*.

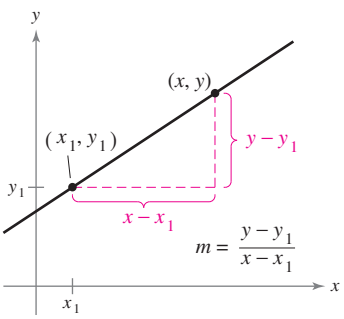


FIGURE 2.22 Any two points on a line can be used to determine the slope of the line.

The Point-Slope Form

If you know the slope of a line and the coordinates of one point on the line, you can find an equation for the line. For instance, in Figure 2.22, let (x_1, y_1) be a given point on the line whose slope is m. If (x, y) is *any other* point on the line, it follows that

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten to produce the following **point-slope form** of the equation of a line.

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

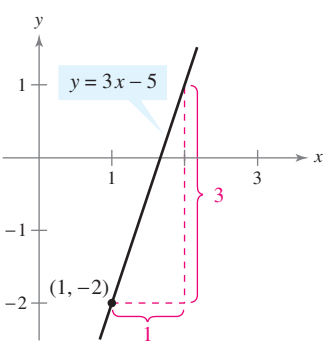


FIGURE 2.23

Example 2 **The Point-Slope Form of the Equation of a Line**

Find an equation of the line that passes through $(1, -2)$ and has a slope of 3.

SOLUTION Use the point-slope form with $(x_1, y_1) = (1, -2)$ and $m = 3$.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute $y_1 = -2$, $x_1 = 1$, and $m = 3$.
$y + 2 = 3x - 3$	Simplify.
$y = 3x - 5$	Equation of line

The graph of this line is shown in Figure 2.23.

✓CHECKPOINT 2

Find an equation of the line that passes through the given point and has the given slope.

a. $(2, 4)$, $m = -2$

b. $(-8, -3)$, $m = \frac{3}{2}$

TECHNOLOGY

You will find programs that use the two-point form to find an equation of a line for several models of graphing utilities on the website for this text at *college.hmco.com/info/larsonapplied.* After you enter the coordinates of two points, the program outputs the slope and y-intercept of the line that passes through the points.

The point-slope form can be used to find the equation of a line passing through two points (x_1, y_1) and (x_2, y_2). First, use the formula for the slope of a line passing through two points. Then, use the point-slope form to obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This is sometimes called the **two-point form** of the equation of a line.

Example 3 **A Linear Model for Sales Prediction**

During the first two quarters of the year, a jewelry company had sales of \$3.4 million and \$3.7 million, respectively. (a) Write a linear equation giving the sales y in terms of the quarter x. (b) Use the equation to predict the sales during the fourth quarter. Can you assume that sales will follow this linear pattern?

SOLUTION

a. Let $(1, 3.4)$ and $(2, 3.7)$ be two points on the line representing the total sales. Use the two-point form to find an equation of the line.

$$y - 3.4 = \frac{3.7 - 3.4}{2 - 1}(x - 1) \qquad \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2 \text{ in two-point form.}$$

$$y - 3.4 = 0.3(x - 1) \qquad \text{Simplify quotient.}$$

$$y = 0.3x + 3.1 \qquad \text{Equation of line}$$

b. Using the equation from part (a), the fourth-quarter sales $(x = 4)$ should be $y = 0.3(4) + 3.1 = \$4.3$ million. See Figure 2.24. Without more data, you cannot assume that the sales pattern will be linear. Many factors, such as seasonal demand and past sales history, help to determine the sales pattern.

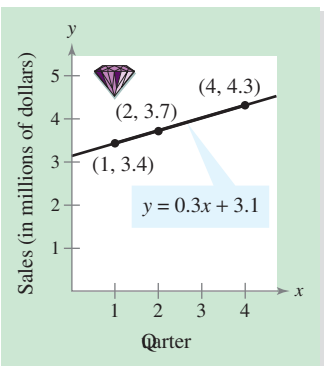


FIGURE 2.24

✓CHECKPOINT 3

A company has sales of \$1.2 million and \$1.4 million in its first two years. Write a linear equation giving the sales y in terms of the year x. ■

DISCOVERY

Use a graphing utility to graph each equation in the same viewing window.

$y_1 = x + 1$

$y_2 = \frac{1}{4}x + 1$

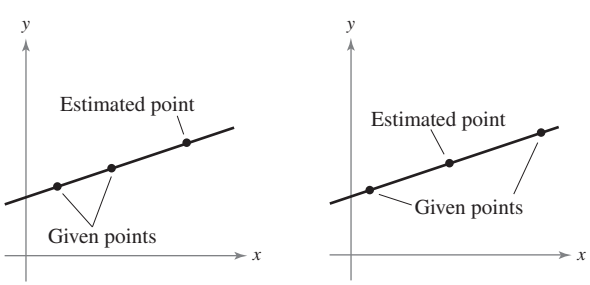
$y_3 = -x + 1$

$y_4 = 3x + 1$

$y_5 = -3x + 1$

What effect does the coefficient of x have on the graph? What is the y-intercept of each graph?

The estimation method illustrated in Example 3 is called **linear extrapolation.** Note in Figure 2.25(a) that for linear extrapolation, the estimated point lies to the *right* of the given points. When the estimated point lies *between* two given points, the procedure is called **linear interpolation,** as shown in Figure 2.25(b).



(a) Linear extrapolation (b) Linear interpolation

FIGURE 2.25

Sketching Graphs of Lines

You have seen that to *find the equation of a line* it is convenient to use the point-slope form. This formula, however, is not particularly useful for *sketching the graph of a line.* The form that is better suited to graphing linear equations is the **slope-intercept form** of the equation of a line. To derive the slope-intercept form, write the following.

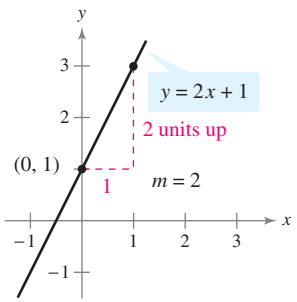
$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$
$$y = mx - mx_1 + y_1 \qquad \text{Solve for } y.$$
$$y = mx + (y_1 - mx_1) \qquad \text{Commutative Property of Addition}$$
$$y = mx + b \qquad \text{Slope-intercept form}$$

Slope-Intercept Form of the Equation of a Line

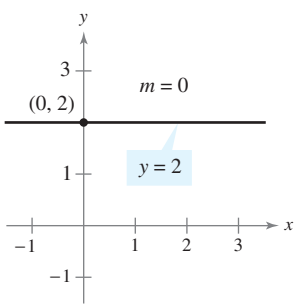
The graph of the equation

$$y = mx + b$$

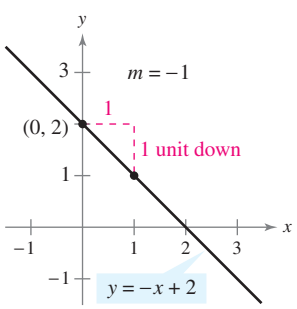
is a line whose slope is m and whose y-intercept is $(0, b)$.



(a) When m is positive, the line rises from left to right.



(b) When m is zero, the line is horizontal.



(c) When m is negative, the line falls from left to right.

FIGURE 2.26

Example 4 Sketching the Graphs of Linear Equations

Sketch the graph of each linear equation.

a. $y = 2x + 1$

b. $y = 2$

c. $x + y = 2$

SOLUTION

a. Because $b = 1$, the y-intercept is $(0, 1)$. Moreover, because the slope is $m = 2$, this line *rises* two units for each unit it moves to the right, as shown in Figure 2.26(a).

b. By writing the equation $y = 2$ in the form

$$y = (0)x + 2$$

you can see that the y-intercept is $(0, 2)$ and the slope is zero. A zero slope implies that the line is horizontal, as shown in Figure 2.26(b).

c. By writing the equation $x + y = 2$ in slope-intercept form

$$y = -x + 2$$

you can see that the y-intercept is $(0, 2)$. Moreover, because the slope is $m = -1$, this line *falls* one unit for each unit it moves to the right, as shown in Figure 2.26(c).

✓CHECKPOINT 4

Sketch the graph of the linear equation $y - 2x = -3$. ■

From the slope-intercept form of the equation of a line, you can see that a horizontal line ($m = 0$) has an equation of the form

$$y = (0)x + b \quad \text{or} \quad y = b. \qquad \text{Horizontal line}$$

This is consistent with the fact that each point on a horizontal line through $(0, b)$ has a y-coordinate of b, as shown in Figure 2.27.

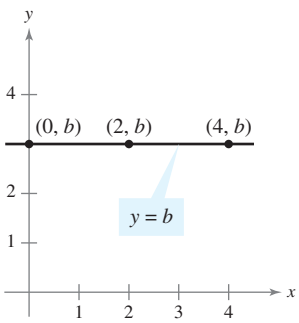


FIGURE 2.27 Horizontal Line

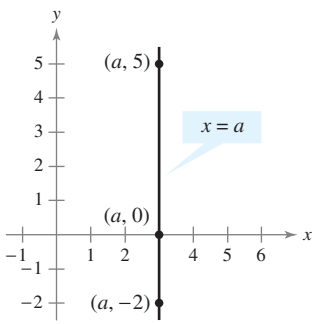


FIGURE 2.28 Vertical Line

Similarly, each point on a vertical line through $(a, 0)$ has an x-coordinate of a, as shown in Figure 2.28. So, a vertical line has an equation of the form

$$x = a. \qquad \text{Vertical line}$$

This equation cannot be written in slope-intercept form because the slope of a vertical line is undefined. However, *every* line has an equation that can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form}$$

where A and B are not *both* zero. If $A = 0$ (and $B \neq 0$), the general equation can be reduced to the form $y = b$, which represents a horizontal line. If $B = 0$ (and $A \neq 0$), the general equation can be reduced to the form $x = a$, which represents a vertical line.

Summary of Equations of Lines

1.	General form:	$Ax + By + C = 0$
2.	Vertical line:	$x = a$
3.	Horizontal line:	$y = b$
4.	Slope-intercept form:	$y = mx + b$
5.	Point-slope form:	$y - y_1 = m(x - x_1)$

DISCOVERY

Use a graphing utility to graph each equation in the same viewing window.

$$y_1 = \tfrac{3}{2}x - 1 \qquad y_2 = \tfrac{3}{2}x \qquad y_3 = \tfrac{3}{2}x + 2$$

What is true about the graphs? What do you notice about the slopes of the equations?

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel, perpendicular, or neither.

Parallel Lines

Two distinct nonvertical lines are **parallel** if and only if their slopes are equal.

Example 5 Equations of Parallel Lines

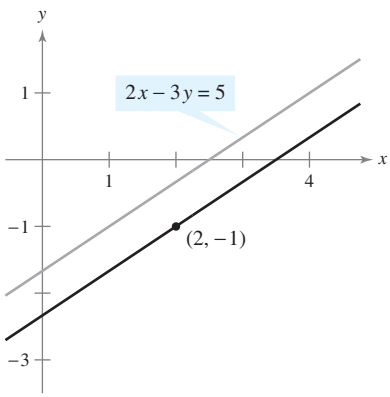


FIGURE 2.29

Find an equation of the line that passes through the point $(2, -1)$ and is parallel to the line $2x - 3y = 5$, as shown in Figure 2.29.

SOLUTION Start by rewriting the equation in slope-intercept form.

$$2x - 3y = 5 \qquad \text{Write original equation.}$$

$$-3y = -2x + 5 \qquad \text{Subtract } 2x \text{ from each side.}$$

$$y = \frac{2}{3}x - \frac{5}{3} \qquad \text{Write in slope-intercept form.}$$

So, the given line has a slope of $m = \frac{2}{3}$. Because any line parallel to the given line must also have a slope of $\frac{2}{3}$, the required line through $(2, -1)$ has the following equation.

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - (-1) = \frac{2}{3}(x - 2) \qquad \text{Substitute for } y_1, x_1, \text{ and } m.$$

$$y + 1 = \frac{2}{3}x - \frac{4}{3} \qquad \text{Simplify.}$$

$$y = \frac{2}{3}x - \frac{4}{3} - 1 \qquad \text{Solve for } y.$$

$$y = \frac{2}{3}x - \frac{7}{3} \qquad \text{Write in slope-intercept form.}$$

Notice the similarity between the slope-intercept form of the original equation and the slope-intercept form of the parallel equation.

✓CHECKPOINT 5

Find an equation that passes through the point $(2, 4)$ and is parallel to the line $2y - 6x = 2$. ■

You have seen that two nonvertical lines are parallel if and only if they have the same slope. Two nonvertical lines are *perpendicular* if and only if their slopes are negative reciprocals of each other. For instance, the lines $y = 2x$ and $y = -\frac{1}{2}x$ are perpendicular because one has a slope of $2 = \frac{2}{1}$ and the other has a slope of $-\frac{1}{2}$.

DISCOVERY

Use a graphing utility to graph each equation in the same viewing window.

$y_1 = \frac{2}{3}x + \frac{5}{2}$

$y_2 = -\frac{3}{2}x + 2$

When you examine the graphs with a square setting, what do you observe? What do you notice about the slopes of the two lines?

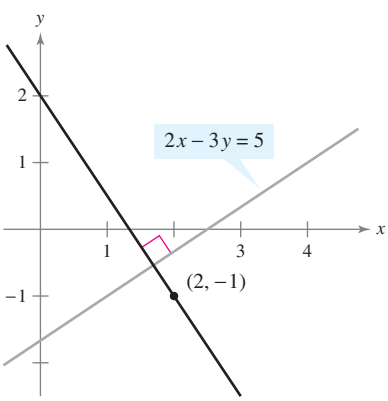


FIGURE 2.30

Perpendicular Lines

Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is,

$$m_1 = -\frac{1}{m_2}.$$

Example 6 **Equations of Perpendicular Lines**

Find an equation of the line that passes through the point $(2, -1)$ and is perpendicular to the line $2x - 3y = 5$, as shown in Figure 2.30.

SOLUTION By writing the equation of the original line in slope-intercept form

$$y = \frac{2}{3}x - \frac{5}{3}$$

you can see that the line has a slope of $\frac{2}{3}$. So, any line that is perpendicular to this line must have a slope of $-\frac{3}{2}$ (because $-\frac{3}{2}$ is the negative reciprocal of $\frac{2}{3}$). The required line through the point $(2, -1)$ has the following equation.

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - (-1) = -\frac{3}{2}(x - 2) \qquad \text{Substitute for } y_1, x_1, \text{ and } m.$$

$$y + 1 = -\frac{3}{2}x + 3 \qquad \text{Simplify.}$$

$$y = -\frac{3}{2}x + 3 - 1 \qquad \text{Solve for } y.$$

$$y = -\frac{3}{2}x + 2 \qquad \text{Write in slope-intercept form.}$$

✓CHECKPOINT 6

Find an equation of the line that passes through the point $(-2, 12)$ and is perpendicular to the line $y = \frac{1}{4}x - 2$. ■

CONCEPT CHECK

1. What is the slope of a line that falls five units for each two units it moves to the right?
2. What is an equation of a horizontal line that passes through the point (a, b)?
3. Why is it convenient to use the slope-intercept form when sketching the graph of a linear equation?
4. Line A and line B are perpendicular to each other and the slope of line A is 1/2. What is the slope of line B?

Skills Review 2.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2 and 1.1.

In Exercises 1–4, simplify the expression.

1. $\dfrac{4-(-5)}{-3-(-1)}$

2. $\dfrac{-5-8}{0-(-3)}$

3. Find $-1/m$ for $m = 4/5$.

4. Find $-1/m$ for $m = -2$.

In Exercises 5–10, solve for y in terms of x.

5. $2x - 3y = 5$

6. $4x + 2y = 0$

7. $y - (-4) = 3[x - (-1)]$

8. $y - 7 = \dfrac{2}{3}(x - 3)$

9. $y - (-1) = \dfrac{3-(-1)}{2-4}(x - 4)$

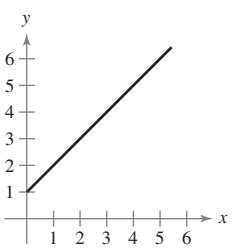
10. $y - 5 = \dfrac{3-5}{0-2}(x - 2)$

Exercises 2.2

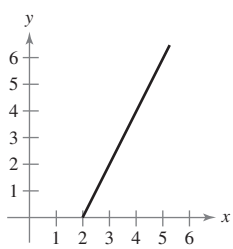
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, estimate the slope of the line.

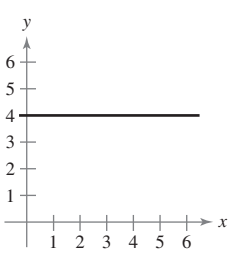
1.



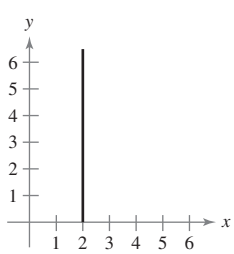
2.



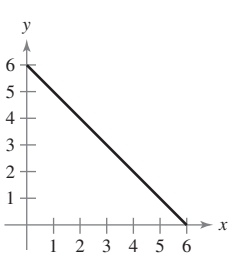
3.



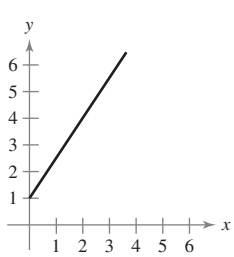
4.



5.



6.



In Exercises 7 and 8, determine if a line with the following description has a positive slope, a negative slope, or an undefined slope.

7. Line rises from left to right

8. Vertical line

In Exercises 9 and 10, sketch the lines through the point with the indicated slopes on the same set of coordinate axes.

	Point	*Slopes*	
9.	$(-3, 4)$	(a) -2	(b) $\frac{2}{3}$
		(c) 0	(d) Undefined
10.	$(-2, -5)$	(a) -1	(b) $\frac{3}{4}$
		(c) 0	(d) Undefined

In Exercises 11–16, plot the points and find the slope of the line passing through the points.

11. $(6, 9), (-4, -1)$

12. $(2, 4), (4, -4)$

13. $(-6, -1), (-6, 4)$

14. $(0, -10), (-4, 0)$

15. $\left(-\frac{1}{3}, 1\right), \left(-\frac{2}{3}, \frac{5}{6}\right)$

16. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

In Exercises 17–24, use the point on the line and the slope of the line to find three additional points through which the line passes. (There are many correct answers.)

	Point	*Slope*
17.	$(5, -2)$	$m = 0$
18.	$(-3, 4)$	$m = 0$
19.	$(2, -5)$	m is undefined.
20.	$(-1, 3)$	m is undefined.
21.	$(5, -6)$	$m = 1$
22.	$(10, -6)$	$m = -1$
23.	$(-6, -1)$	$m = \frac{1}{2}$
24.	$(7, -5)$	$m = -\frac{2}{3}$

In Exercises 25–38, find an equation of the line that passes through the point and has the indicated slope. Then sketch the line.

	Point	*Slope*
25.	$(7, 0)$	$m = 1$
26.	$(0, -4)$	$m = -1$
27.	$(-2, 0)$	$m = -4$
28.	$(1, 3)$	$m = 3$
29.	$(-3, 6)$	$m = -2$
30.	$(-8, 3)$	$m = -\frac{1}{2}$
31.	$(4, 0)$	$m = -\frac{1}{3}$
32.	$(-2, -5)$	$m = \frac{3}{4}$
33.	$(6, -1)$	m is undefined.
34.	$(3, -2)$	m is undefined.
35.	$(-2, -7)$	$m = 0$
36.	$(-10, 4)$	$m = 0$
37.	$\left(4, \frac{5}{2}\right)$	$m = \frac{4}{3}$
38.	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$m = -3$

In Exercises 39–48, find the slope and y-intercept (if possible) of the line specified by the equation. Then sketch the line.

39. $y = 2x - 1$

40. $y = 3 - x$

41. $4x - y - 6 = 0$

42. $2x + 3y - 9 = 0$

43. $8 - 3x = 0$

44. $2x + 5 = 0$

45. $7x + 6y - 30 = 0$

46. $x - y - 10 = 0$

47. $2y - 7 = 0$

48. $8 - 5y = 0$

In Exercises 49–60, find an equation of the line passing through the points.

49. $(2, 5), (-1, -4)$

50. $(6, -1), (-2, 1)$

51. $(7, -4), (-7, 3)$

52. $(4, 3), (-4, -4)$

53. $(-9, 11), (-9, 14)$

54. $(3, 5), (3, -2)$

55. $(-1, 7), (3, 7)$

56. $(3, -2), (-8, -2)$

57. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$

58. $(1, 1), \left(6, -\frac{2}{3}\right)$

59. $(1, 0.6), (-2, -0.6)$

60. $(-8, 0.6), (2, -2.4)$

61. A fellow student does not understand why the slope of a vertical line is undefined. Describe how you would help this student understand the concept of undefined slope.

62. Another student overhears your conversation in Exercise 61 and states, "I do not understand why a horizontal line has zero slope and how that is different from undefined or no slope." Describe how you would explain the concepts of zero slope and undefined slope and how they are different from each other.

In Exercises 63–68, use the *intercept form* to find the equation of the line with the given intercepts. The intercept form of the equation of a line with intercepts $(a, 0)$ and $(0, b)$ is

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, \quad b \neq 0.$$

63. x-intercept: $(1, 0)$
y-intercept: $(0, -4)$

64. x-intercept: $(-3, 0)$
y-intercept: $(0, 4)$

65. x-intercept: $(-2, 0)$
y-intercept: $(0, -2)$

66. x-intercept: $(5, 0)$
y-intercept: $(0, 1)$

67. x-intercept: $\left(-\frac{1}{6}, 0\right)$
y-intercept: $\left(0, -\frac{2}{3}\right)$

68. x-intercept: $\left(-\frac{2}{3}, 0\right)$
y-intercept: $\left(0, \frac{1}{2}\right)$

In Exercises 69–76, the equations of two lines are given. Determine if lines L_1 and L_2 are parallel, perpendicular, or neither.

69. $L_1: y = 3x + 4;\ L_2: y = x - \frac{1}{4}$

70. $L_1: y = \frac{3}{4}x + 1;\ L_2: y = -\frac{4}{3}x + 3$

71. $L_1: 2x - y = 1;\ L_2: x + 2y = -1$

72. $L_1: x - 5y = -2;\ L_2: -3x + 15y = 6$

73. $L_1: x - 3y = -3;\ L_2: 2x - 6y = 6$

74. $L_1: 4x - y = -2;\ L_2: 8x - 2y = 6$

75. $L_1: 2x - 3y - 15 = 0;\ L_2: 3x + 2y + 8 = 0$

76. $L_1: x - 4y - 12 = 0;\ L_2: 3x - 4y - 8 = 0$

In Exercises 77–84, determine if the lines L_1 and L_2 passing through the indicated pairs of points are parallel, perpendicular, or neither.

77. $L_1: (-5, 0), (-2, 1);\ L_2: (0, 1), (3, 2)$

78. $L_1: (-1, 6), (1, 4);\ L_2: (3, -3), (6, -9)$

79. $L_1: (0, -1), (5, 9);\ L_2: (0, 3), (4, 1)$

80. $L_1: (3, 6), (-6, 0);\ L_2: (0, -1), \left(5, \frac{7}{3}\right)$

81. $L_1: (-2, -1), (1, 5);\ L_2: (1, 3), (5, -5)$

82. $L_1: (4, 8), (-4, 2);\ L_2: (3, -5), \left(-1, \frac{1}{3}\right)$

83. $L_1: (-1, 7), (-6, 4);\ L_2: (0, 1), (5, 4)$

84. $L_1: (-1, 3), (2, -5);\ L_2: (3, 0), (2, -7)$

In Exercises 85–90, write equations of the lines through the point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Line*
85.	$(6, 2)$	$y - 2x = -1$
86.	$(-5, 4)$	$x + y = 8$
87.	$\left(\frac{1}{4}, -\frac{2}{3}\right)$	$2x - 3y = 5$

88. $\left(\frac{7}{8}, \frac{3}{4}\right)$ $5x + 3y = 0$

89. $(-1, 0)$ $y = -3$

90. $(2, 5)$ $x = 4$

91. Temperature Find an equation of the line that gives the relationship between the temperature in degrees Celsius C and the temperature in degrees Fahrenheit F. Remember that water freezes at 0° Celsius (32° Fahrenheit) and boils at 100° Celsius (212° Fahrenheit).

92. Temperature Use the result of Exercise 91 to complete the table. Is there a temperature for which the Fahrenheit reading is the same as the Celsius reading? If so, what is it?

C		$-10°$	$10°$			$177°$
F	$0°$			$68°$	$90°$	

93. Simple Interest A person deposits P dollars in an account that pays simple interest. After 2 months, the balance in the account is \$813 and after 3 months, the balance in the account is \$819.50. Find an equation that gives the relationship between the balance A and the time t in months.

94. Simple Interest Use the result of Exercise 93 to complete the table.

A			\$813.00	\$819.50			
t	0	1			4	5	6

95. Wheelchair Ramp The maximum recommended slope of a wheelchair ramp is $\frac{1}{12}$. A business is installing a wheelchair ramp that rises 34 inches over a horizontal length of 30 feet. Is the ramp steeper than recommended? *(Source: Americans with Disabilities Act Handbook)*

96. Revenue A line representing daily revenues y in terms of time x in days has a slope of $m = 100$. Interpret the change in daily revenues for a one-day increase in time.

97. College Enrollment A small college had 3125 students in 2005 and 3582 students in 2008. The enrollment follows a linear growth pattern. How many students will the college have in 2012?

98. Annual Salary Your salary was \$30,200 in 2007 and \$33,500 in 2009. Your salary follows a linear growth pattern. What salary will you be making in 2012?

99. *MAKE A DECISION: FOURTH-QUARTER SALES* During the first and second quarters of the year, a business had sales of \$158,000 and \$165,000. From these data, can you assume that the sales follow a linear growth pattern? If the pattern is linear, what will the sales be during the fourth quarter?

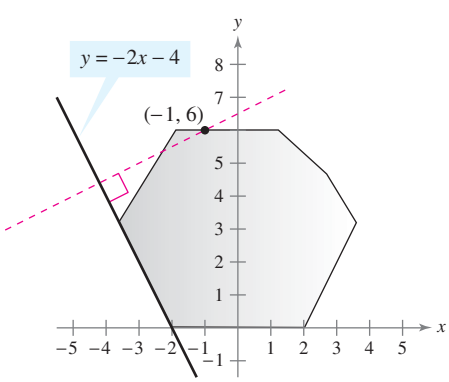
100. Fatal Crashes In 1998, there were 37,107 motor vehicle traffic crashes involving fatalities in the United States. In 2005, there were 39,189 such crashes. Assume that the trend is linear. Predict the number of crashes with fatalities in 2007. *(Source: National Highway Traffic Safety Administration)*

101. *MAKE A DECISION: YAHOO! INC. REVENUE* In 2000, Yahoo! Inc. had revenues of \$1110.2 million. In 2003, their revenues were \$1625.1 million. Assume the revenue followed a linear trend. What would the approximate revenue have been in 2005? The actual revenue in 2005 was \$5257.7 million. Do you think the yearly revenue followed a linear trend? Explain your reasoning. *(Source: Yahoo! Inc.)*

102. Applebee's Revenue Applebee's is one of the largest casual dining chains in the United States. In 2000, Applebee's had revenues of \$690.2 million. In 2004, their revenues were \$1111.6 million. Assume the yearly revenue followed a linear trend. What would the approximate revenue have been in 2005? The actual revenue in 2005 was \$1216.6 million. From these data, is it possible that Applebee's yearly revenue followed a linear trend? Explain your reasoning. *(Source: Applebee's International, Inc.)*

103. Scuba Diving The pressure (in atmospheres) exerted on a scuba diver's body has a linear relationship with the diver's depth. At sea level (or a depth of 0 feet), the pressure exerted on a diver is 1 atmosphere. At a depth of 99 feet, the pressure exerted on a diver is 4 atmospheres. Write a linear equation to describe the pressure p (in atmospheres) in terms of the depth d (in feet) below the surface of the sea. What is the rate of change of pressure with respect to depth? *(Source: PADI Open Water Diver Manual)*

104. Stone Cutting A stone cutter is making a 6-foot tall memorial stone. The diagram shows coordinates labeled in feet. The stone cutter plans to make the cut indicated by the dashed line. This cut follows a line perpendicular to one side of the stone that passes through the point labeled $(-1, 6)$. Find an equation of the line of the cut.



Section 2.3

Linear Modeling and Direct Variation

- Use a mathematical model to approximate a set of data points.
- Construct a linear model to relate quantities that vary directly.
- Construct and use a linear model with slope as the rate of change.
- Use a scatter plot to find a linear model that fits a set of data.

Introduction

The primary objective of applied mathematics is to find equations or **mathematical models** that describe real-world situations. In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals—accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results.

You have already studied some techniques for fitting models to data. For instance, in Section 2.2, you learned how to find the equation of a line that passes through two points. In this section, you will study other techniques for fitting models to data: *direct variation, rates of change,* and *linear regression.*



Image Source Pink/Getty Images

For most breeds, the body weight of a dog increases at an approximately constant rate through the first several months of life.

Example 1 A Mathematical Model

The weight of a puppy recorded every two months is shown in the table.

Age (in months)	2	4	6	8	10	12
Weight (in pounds)	24	45	67	93	117	130

A linear model that approximates the puppy's weight w (in pounds) in month t is

$$w = 11.03t + 2.1, \quad 2 \le t \le 12.$$

How closely does the model represent the data?

SOLUTION By graphing the data points with the linear model (see Figure 2.31), you can see that the model is a "good fit" for the actual data. The table shows how each actual weight w compares with the weight w^* given by the model.

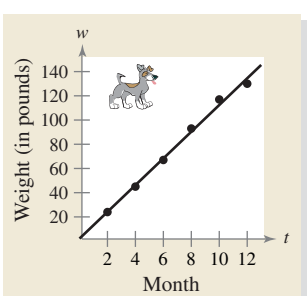


FIGURE 2.31

t	w	w^*
2	24	24.16
4	45	46.22
6	67	68.28
8	93	90.34
10	117	112.4
12	130	134.46

✓CHECKPOINT 1

In Example 1, what are the best and worst approximations given by the model? ■

Direct Variation

There are two basic types of linear models in x and y. The more general model has a y-intercept that is nonzero: $y = mx + b$, $b \neq 0$. The simpler model, $y = mx$, has a y-intercept that is zero. In the simpler model, y is said to **vary directly** as x, or to be **directly proportional** to x.

Direct Variation

The following statements are equivalent.

1. y **varies directly** as x.
2. y is **directly proportional** to x.
3. $y = mx$ for some nonzero constant m, where m is the **constant of variation** or the **constant of proportionality.**

Example 2 State Income Tax

In Colorado, state income tax is directly proportional to *taxable income.* For a taxable income of \$30,000, the Colorado state income tax is \$1389. Find a mathematical model that gives the Colorado state income tax in terms of taxable income.

SOLUTION

Verbal Model: State income tax $=$ m $\cdot$ Taxable income

Labels:
State income tax $= y$ (dollars)
Taxable income $= x$ (dollars)
Income tax rate $= m$ (percent in decimal form)

Equation: $y = mx$

Find m by substituting the given information into the equation $y = mx$.

$y = mx$ Direct variation model

$1389 = m(30{,}000)$ Substitute $y = 1389$ and $x = 30{,}000$.

$0.0463 = m$ Income tax rate

An equation (or model) for state income tax in Colorado is

$y = 0.0463x.$

So, Colorado has a state income tax rate of 4.63% of taxable income. The graph of this equation is shown in Figure 2.32.

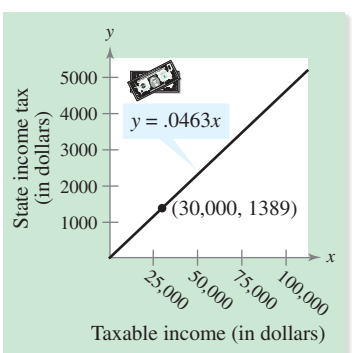


FIGURE 2.32

✓CHECKPOINT 2

You buy a flash drive for \$14.50 and pay sales tax of \$0.87. The sales tax is directly proportional to the price. Find a mathematical model that gives the sales tax in terms of the price. ■

Most measurements in the English system and metric system are directly proportional. The next example shows how to use a direct proportion to convert between miles per hour and kilometers per hour.

Example 3 The English and Metric Systems

While driving, your speedometer indicates that your speed is 64 miles per hour or 103 kilometers per hour. Use this information to find a mathematical model that relates miles per hour to kilometers per hour.

SOLUTION Let y represent the speed in miles per hour and let x represent the speed in kilometers per hour. Then y and x are related by the equation

$$y = mx.$$

Use the fact that $y = 64$ when $x = 103$ to find the value of m.

$$y = mx \qquad \text{Direct variation model}$$
$$64 = m(103) \qquad \text{Substitute } y = 64 \text{ and } x = 103.$$
$$\frac{64}{103} = m \qquad \text{Divide each side by 103.}$$
$$0.62136 \approx m \qquad \text{Use a calculator.}$$

So, the conversion factor from kilometers per hour to miles per hour is approximately 0.62136, and the model is

$$y = 0.62136x.$$

The graph of this equation is shown in Figure 2.33.

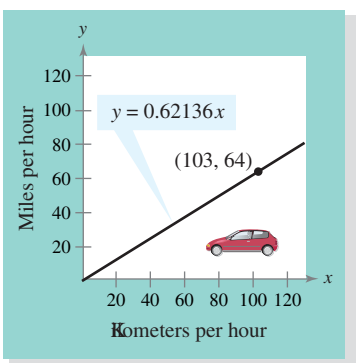


FIGURE 2.33

✓CHECKPOINT 3

You buy an ice bucket with a capacity of 44 ounces, or 1.3 liters. Write a mathematical model that relates ounces to liters. ■

You can use the model from Example 3 to convert any speed in kilometers per hour to miles per hour, as shown in the table.

Kometers per hour	Miles per hour
20	12.4
40	24.9
60	37.3
80	49.7
100	62.1
120	74.6

The conversion equation $y = 0.62136x$ can be approximated by the simpler equation $y = \frac{5}{8}x$ because $\frac{5}{8} = 0.625$.

Rates of Change

A second common type of linear model is one that involves a known rate of change. In the linear equation

$$y = mx + b$$

you know that m represents the slope of the line. In real-life problems, the slope can often be interpreted as the **rate of change** of y with respect to x. Rates of change should always be listed in appropriate units of measure.

Example 4 Mountain Climbing

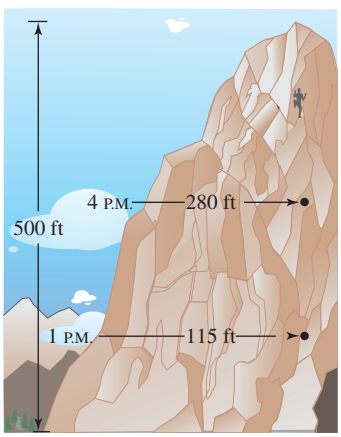


FIGURE 2.34

A mountain climber is climbing up a 500-foot cliff. At 1 P.M., the climber is 115 feet up the cliff. By 4 P.M., the climber has reached a height of 280 feet, as shown in Figure 2.34.

a. Find the average rate of change of the climber. Use this rate of change to find an equation that relates the height of the climber to the time.

b. Use the equation to estimate the time when the climber reaches the top of the cliff.

SOLUTION

a. Let y represent the climber's height on the cliff and let t represent the time. Then the two points that represent the climber's two positions are

$$(t_1, y_1) = (1, 115) \quad \text{and} \quad (t_2, y_2) = (4, 280).$$

So, the average rate of change of the climber is

$$\text{Average rate of change} = \frac{y_2 - y_1}{t_2 - t_1} = \frac{280 - 115}{4 - 1} = 55 \text{ feet per hour.}$$

An equation that relates the height of the climber to the time is

$$y - y_1 = m(t - t_1) \qquad \text{Point-slope form}$$
$$y - 115 = 55(t - 1) \qquad \text{Substitute } y_1 = 115,\ t_1 = 1, \text{ and } m = 55.$$
$$y = 55t + 60. \qquad \text{Linear model}$$

If you had chosen to use the point (t_2, y_2) to determine the equation, you would have obtained a different equation initially: $y - 280 = 55(t - 4)$. However, simplifying this equation yields the same linear model $y = 55t + 60$.

b. To estimate the time when the climber reaches the top of the cliff, let $y = 500$ and solve for t to obtain $t = 8$. Because $t = 8$ corresponds to 8 P.M., at the average rate of change, the climber will reach the top at 8 P.M.

✓CHECKPOINT 4

How long does it take the climber in Example 4 to climb 275 feet? ■

Example 5 Population of Orlando, Florida

Between 1990 and 2005, the population of Orlando, Florida increased at an average rate of approximately 3233 people per year. In 1990, the population was about 164,700. Find a mathematical model that gives the population of Orlando in terms of the year, and use the model to estimate the population in 2010. *(Source: U.S. Census Bureau)*

SOLUTION Let y represent the population of Orlando, and let t represent the calendar year, with $t = 0$ corresponding to 1990. It is convenient to let $t = 0$ correspond to 1990 because you were given the population in 1990. Now, using the rate of change of 3233 people per year, you have

Rate of change: 3233 — 1990 population: 164,700

$$y = mt + b$$

$$y = 3233t + 164{,}700.$$

Using this model, you can predict the 2010 population to be

$$\text{2010 population} = 3233(20) + 164{,}700$$
$$= 229{,}360.$$

The graph is shown in Figure 2.35.

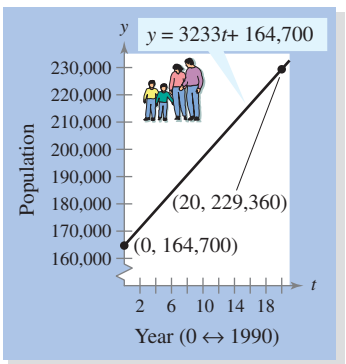


FIGURE 2.35

✓CHECKPOINT 5

Use the model in Example 5 to predict the population of Orlando in 2012. ■

Example 6 Straight-Line Depreciation

A racing team buys a $4750 welder that has a useful life of 10 years. The salvage value of the welder at the end of the 10 years is $400. Write a linear equation that describes the value of the welder throughout its useable life.

SOLUTION Let V represent the value of the welder (in dollars) at the end of the year t. You can represent the initial value of the welder by the ordered pair (0, 4750) and the salvage value by the ordered pair (10, 400). The slope of the line is

$$m = \frac{400 - 4750}{10 - 0}$$
$$= -435$$

which represents the annual depreciation in *dollars per year*. Using the slope-intercept form, you can write the equation of the line as follows.

$$V = -435t + 4750 \qquad \text{Slope-intercept form}$$

The graph of the equation is shown in Figure 2.36.

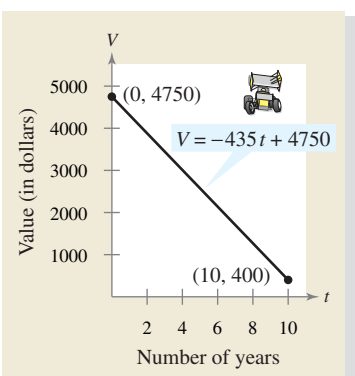


FIGURE 2.36

✓CHECKPOINT 6

Write a linear equation to model the value of a new machine that costs $2300 and is worth $350 after 10 years. ■

TECHNOLOGY

When you use the *regression* feature of your graphing utility, you may obtain an "r value," which gives a measure of how well the model fits the data (see figure).

```
LinReg
 y=ax+b
 a=18.48351648
 b=500.4175824
 r²=.9987189141
 r=.9993592518
```

The closer the value of $|r|$ is to 1, the better the fit. For the data in Example 7, $r \approx 0.999$, which implies that the model is a good fit. For instructions on how to use the *regression* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

Scatter Plots and Regression Analysis

Another type of linear modeling is a graphical approach that is commonly used in statistics. To find a mathematical model that approximates a set of actual data points, plot the points on a rectangular coordinate system. This collection of points is called a **scatter plot.** You can use the statistical features of a graphing utility to calculate the equation of the best-fitting line for the data in your scatter plot. The statistical method of fitting a line to a collection of points is called **linear regression.** A discussion of linear regression is beyond the scope of this text, but the program in most graphing utilities is easy to use and allows you to analyze linear data that may not be convenient to graph by hand.

Example 7 **Dentistry**

The table shows the numbers of employees y (in thousands) in dentist offices and clinics in the United States in the years 1993 to 2005. *(Source: U.S. Bureau of Labor Statistics)*

Year	x	Employees, y
1993	3	556
1994	4	574
1995	5	592
1996	6	611
1997	7	629
1998	8	646
1999	9	667

Year	x	Employees, y
2000	10	688
2001	11	705
2002	12	725
2003	13	744
2004	14	760
2005	15	771

a. Use the *regression* feature of a graphing utility to find a linear model for the data. Let $x = 3$ represent 1993.

b. Use a graphing utility to graph the linear model along with a scatter plot of the data.

c. Use the linear model to estimate the number of employees in 2007.

SOLUTION

a. Enter the data into a graphing utility. Then, using the *regression* feature of the graphing utility, you should obtain a linear model for the data that can be rounded to the following.

$$y = 18.48x + 500.4, \quad 3 \le x \le 15$$

b. The graph of the equation and the scatter plot are shown in Figure 2.37.

c. Substituting $x = 17$ into the equation $y = 18.48x + 500.4$, you get $y = 814.56$. So, according to the model, there will be about 815,000 employees in dentist offices and clinics in the United States in 2007.

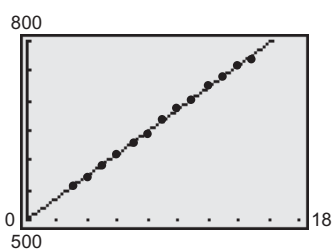


FIGURE 2.37

✓CHECKPOINT 7

Redo Example 7 using only the data for the years 2000–2005. ■

Example 8 **Prize Money at the Indianapolis 500**

The total prize money p (in millions of dollars) awarded at the Indianapolis 500 in each year from 1995 to 2006 is shown in the table. Construct a scatter plot that represents the data and find a linear model that approximates the data. *(Source: Indianapolis 500)*

Year	1995	1996	1997	1998	1999	2000
p	8.06	8.11	8.61	8.72	9.05	9.48

Year	2001	2002	2003	2004	2005	2006
p	9.61	10.03	10.15	10.25	10.30	10.52

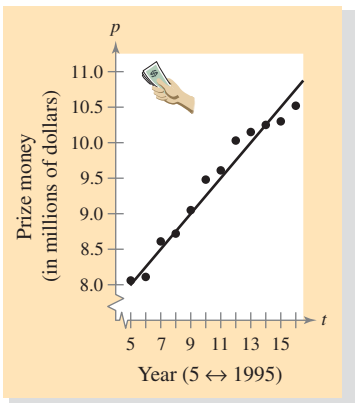


FIGURE 2.38

SOLUTION Let $t = 5$ represent 1995. The scatter plot of the data is shown in Figure 2.38. Draw a line on the scatter plot that approximates the data. To find an equation of the line, approximate two points on the line: (5, 8) and (9, 9). So, the slope of the line is

$$m \approx \frac{p_2 - p_1}{t_2 - t_1} = \frac{9 - 8}{9 - 5} = 0.25.$$

Using the point-slope form, you can determine that an equation of the line is

$p - 8 = 0.25(t - 5)$ Point-slope form

$p = 0.25t + 6.75.$ Slope-intercept form

To check this model, compare the actual p-values with the p-values given by the model (these values are labeled as p^* in the table at the left).

t	p	p^*
5	8.06	8.00
6	8.11	8.25
7	8.61	8.50
8	8.72	8.75
9	9.05	9.00
10	9.48	9.25
11	9.61	9.50
12	10.03	9.75
13	10.15	10.00
14	10.25	10.25
15	10.30	10.5
16	10.52	10.75

✓CHECKPOINT 8

Redo Example 8 using only the data for 2001 to 2006. ■

CONCEPT CHECK

1. Name a point that is on the graph of any direct variation equation.
2. What does the constant of variation tell you about the graph of a direct variation equation?
3. The cost y (in dollars) of producing x units of a product is modeled by
 $y = 30x + 240.$
 Explain what the rate of change represents in this situation.
4. A girl grows at a rate of 2 inches per year from the time she is 2 years old until she is 10 years old. What other information do you need to write an equation that models the girl's height during this time period? Explain.

Skills Review 2.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.1 and 2.2.

In Exercises 1–4, sketch the line.

1. $y = 2x$

2. $y = \frac{1}{2}x$

3. $y = 2x + 1$

4. $y = \frac{1}{2}x + 1$

In Exercises 5 and 6, find an equation of the line that has the given slope and y-intercept.

5. Slope: 1; y-intercept: $(0, 2)$

6. Slope: $\frac{3}{2}$; y-intercept: $(0, 3)$

In Exercises 7–10, find an equation of the line that passes through the two points.

7. $(1, 3)$ and $(6, 8)$

8. $(0, 4)$ and $(7, 10)$

9. $(1, 5.2)$ and $(5, 4.7)$

10. $(2, 6.5)$ and $(8, 3.6)$

Exercises 2.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Dog Growth The weight of a puppy recorded every two months is shown in the table.

Age (in months)	2	4	6
Weight (in pounds)	21	44	63

Age (in months)	8	10	12
Weight (in pounds)	82	92	101

A linear model that approximates the puppy's weight w (in pounds) in month t is $w = 8.0t + 11$, $2 \le t \le 12$. Plot the actual data with the model. How closely does the model represent the data?

2. Non-Wage Earners The numbers of working-age civilians (in millions) in the United States that were not involved in the labor force from 1995 to 2005 are given by the following ordered pairs.

(1995, 66.3) (1996, 66.6) (1997, 66.8)
(1998, 67.5) (1999, 68.4) (2000, 70.0)
(2001, 71.4) (2002, 72.7) (2003, 74.7)
(2004, 76.0) (2005, 76.8)

A linear model that approximates the data is $y = 1.16t + 59.1$, $5 \le t \le 15$, where y is the number of civilians (in millions) and $t = 5$ represents 1995. Plot the actual data with the model. How closely does the model represent the data? *(Source: U.S. Bureau of Labor Statistics)*

3. UPS Revenue The yearly revenues (in billions of dollars) of UPS from 1997 to 2005 are given by the following ordered pairs.

(1997, 22.5) (1998, 24.8) (1999, 27.1)
(2000, 29.8) (2001, 30.6) (2002, 31.3)
(2003, 33.5) (2004, 36.6) (2005, 42.6)

Use a graphing utility to create a scatter plot of the data. Let $x = 7$ represent 1997. Then use the *regression* feature of the graphing utility to find a best-fitting line for the data. Graph the model and the data together. How closely does the model represent the data? *(Source: United Parcel Service)*

4. Consumer Price Index For urban consumers of educational and communication materials, the Consumer Price Index giving the dollar amount equal to the buying power of $100 in December 1997 is given for each year from 1994 to 2005 by the following ordered pairs.

(1994, 88.8) (1995, 92.2) (1996, 95.3)
(1997, 98.4) (1998, 100.3) (1999, 101.2)
(2000, 102.5) (2001, 105.2) (2002, 107.9)
(2003, 109.8) (2004, 111.6) (2005, 113.7)

Use a graphing utility to create a scatter plot of the data. Let $x = 4$ represent 1994. Then use the *regression* feature of the graphing utility to find a best-fitting line for the data. Graph the model and the data together. How closely does the model represent the data? *(Source: U.S. Bureau of Labor Statistics)*

Direct Variation In Exercises 5–10, y is proportional to x. Use the x- and y-values to find a linear model that relates y and x.

5. $x = 8,\ y = 3$

6. $x = 5,\ y = 9$

7. $x = 15,\ y = 300$

8. $x = 12,\ y = 204$

9. $x = 7,\ y = 3.2$

10. $x = 11,\ y = 1.5$

Direct Variation In Exercises 11–14, write a linear model that relates the variables.

11. H varies directly as p; $H = 27$ when $p = 9$

12. s is proportional to t; $s = 32$ when $t = 4$

13. c is proportional to d; $c = 12$ when $d = 20$

14. r varies directly as s; $r = 25$ when $s = 40$

15. Simple Interest The simple interest received from an investment is directly proportional to the amount of the investment. By investing $2500 in a bond issue, you obtain an interest payment of $187.50 at the end of 1 year. Find a mathematical model that gives the interest I at the end of 1 year in terms of the amount invested P.

16. Simple Interest The simple interest received from an investment is directly proportional to the amount of the investment. By investing $5000 in a municipal bond, you obtain interest of $337.50 at the end of 1 year. Find a mathematical model that gives the interest I at the end of 1 year in terms of the amount invested P.

17. Property Tax Your property tax is based on the assessed value of your property. (The assessed value is often lower than the actual value of the property.) A house that has an assessed value of $150,000 has a property tax of $5520.

(a) Find a mathematical model that gives the amount of property tax y in terms of the assessed value x of the property.

(b) Use the model to find the property tax on a house that has an assessed value of $185,000.

18. State Sales Tax An item that sells for $145.99 has a sales tax of $10.22.

(a) Find a mathematical model that gives the amount of sales tax y in terms of the retail price x.

(b) Use the model to find the sales tax on a purchase that has a retail price of $540.50.

19. Centimeters and Inches On a yardstick, you notice that 13 inches is the same length as 33 centimeters.

(a) Use this information to find a mathematical model that relates centimeters to inches.

(b) Use the model to complete the table.

Inches	5	10	20	25	30
Centimeters					

20. Liters and Gallons You are buying gasoline and notice that 14 gallons of gasoline is the same as 53 liters.

(a) Use this information to find a mathematical model that relates gallons to liters.

(b) Use the model to complete the table.

Gallons	5	10	20	25	30
Liters					

In Exercises 21–26, you are given the 2005 value of a product *and* the rate at which the value is expected to change during the next 5 years. Use this information to write a linear equation that gives the dollar value of the product in terms of the year. (Let $t = 5$ represent 2005.)

	2005 Value	*Rate*
21.	$2540	$125 increase per year
22.	$156	$4.50 increase per year
23.	$20,400	$2000 decrease per year
24.	$45,000	$2800 decrease per year
25.	$154,000	$12,500 increase per year
26.	$245,000	$5600 increase per year

27. Parachuting After opening the parachute, the descent of a parachutist follows a linear model. At 2:08 P.M., the height of the parachutist is 7000 feet. At 2:10 P.M., the height is 4600 feet.

(a) Write a linear equation that gives the height of the parachutist in terms of the time t. (Let $t = 0$ represent 2:08 P.M. and let t be measured in seconds.)

(b) Use the equation in part (a) to find the time when the parachutist will reach the ground.

28. Distance Traveled by a Car You are driving at a constant speed. At 4:30 P.M., you drive by a sign that gives the distance to Montgomery, Alabama as 84 miles. At 4:59 P.M., you drive by another sign that gives the distance to Montgomery as 56 miles.

(a) Write a linear equation that gives your distance from Montgomery in terms of time t. (Let $t = 0$ represent 4:30 P.M. and let t be measured in minutes.)

(b) Use the equation in part (a) to find the time when you will reach Montgomery.

29. Straight-Line Depreciation A business purchases a piece of equipment for $875. After 5 years the equipment will have no value. Write a linear equation giving the value V of the equipment during the 5 years.

30. Straight-Line Depreciation A business purchases a piece of equipment for $25,000. The equipment will be replaced in 10 years, at which time its salvage value is expected to be $2000. Write a linear equation giving the value V of the equipment during the 10 years.

31. Sale Price and List Price A store is offering a 15% discount on all items. Write a linear equation giving the sale price S for an item with a list price L.

32. Sale Price and List Price A store is offering a 25% discount on all shirts. Write a linear equation giving the sale price S for a shirt with a list price L.

33. Hourly Wages A manufacturer pays its assembly line workers $11.50 per hour. In addition, workers receive a piecework rate of $0.75 per unit produced. Write a linear equation for the hourly wages W in terms of the number of units x produced per hour.

34. Sales Commission A salesperson receives a monthly salary of $2500 plus a commission of 7% of sales. Write a linear equation for the salesperson's monthly wage W in terms of the person's monthly sales S.

35. Deer Population A forest region had a population of 1300 deer in the year 2000. During the next 8 years, the deer population increased by about 60 deer per year.

(a) Write a linear equation giving the deer population P in terms of the year t. Let $t = 0$ represent 2000.

(b) The deer population keeps growing at this constant rate. Predict the number of deer in 2012.

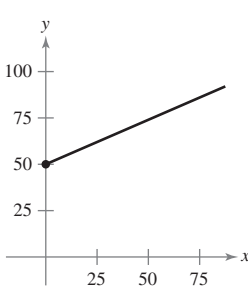
36. Pest Management The cost of implementing an invasive species management system in a forest is related to the area of the forest. It costs $630 to implement the system in a forest area of 10 acres. It costs $1070 in a forest area of 18 acres.

(a) Write a linear equation giving the cost of the invasive species management system in terms of the number of acres x of forest.

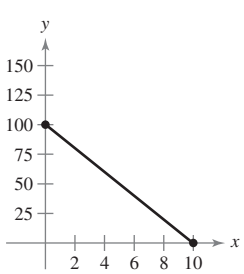
(b) Use the equation in part (a) to find the cost of implementing the system in a forest area of 30 acres.

In Exercises 37–40, match the description with one of the graphs. Also find the slope of the graph and describe how it is interpreted in the real-life situation. [The graphs are labeled (a), (b), (c), and (d).]

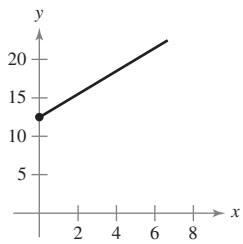
(a)



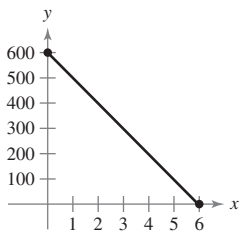
(b)



(c)



(d)



37. A person is paying $10 per week to a friend to repay a $100 loan.

38. An employee is paid $12.50 per hour plus $1.50 for each unit produced per hour.

39. A sales representative receives $50 per day for food, plus $0.48 for each mile traveled.

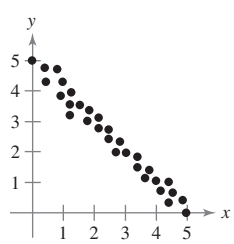
40. You purchased a digital camera for $600 that depreciates $100 per year.

41. Think About It You begin a video game with 100 points and earn 10 points for each coin you collect. Does this description match graph (b) in Exercises 3740? Explain.

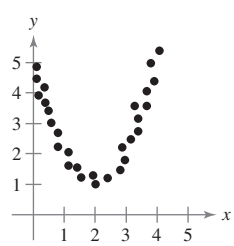
42. Think About It You start with $1.50 and save $12.50 per week. Does this description match graph (c) in Exercises 3740? Explain.

In Exercises 43–48, can the data be approximated by a linear model? If so, sketch the line that best approximates the data. Then find an equation of the line.

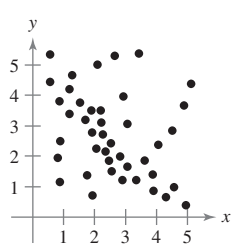
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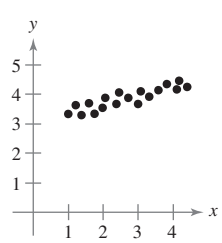
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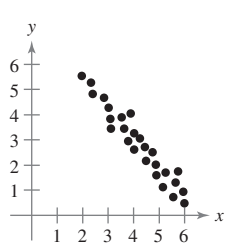
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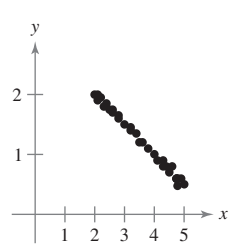
46.



47.



48.



49. Advertising The estimated annual amounts A (in millions of dollars) spent on cable TV advertising for the years 1996 to 2005 are shown in the table. *(Source: Universal McCann)*

Year	1996	1997	1998	1999
Advertising, A	7778	8750	10,340	12,570

Year	2000	2001	2002
Advertising, A	15,455	15,536	16,297

Year	2003	2004	2005
Advertising, A	18,814	21,527	24,501

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 6$ represent 1996. Do the data appear linear?

(b) Use the *regression* feature of a graphing utility to find a linear model for the data.

(c) State the slope of the graph of the linear model from part (b) and interpret its meaning in the context of the problem.

(d) Use the linear model to estimate the amounts spent on cable TV advertising in 2006 and 2007. Are your estimates reasonable?

50. Japan The population of Japan is expected to drop by 30% over the next 50 years as the percent of its citizens that are elderly increases. Projections for Japan's population through 2050 are shown in the table.

Year, t	2005	2010	2020
Population, P (in millions)	127.8	127.5	124.1

Year, t	2030	2040	2050
Population, P (in millions)	117.6	109.3	100.6

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 5$ represent 2005. Do the data appear linear?

(b) Use the *regression* feature of a graphing utility to find a linear model for the data.

(c) Identify the slope of the model from part (b) and interpret its meaning in the context of the problem.

(d) Use the linear model to predict the populations in 2015, 2035, and 2060. Are these predictions reasonable?

51. Yearly Revenue The yearly revenues (in millions of dollars) for Sonic Corporation for the years 1996 to 2005 are given by the following ordered pairs. *(Source: Sonic Corporation)*

(1996, 151.1) (1997, 184.0) (1998, 219.1)
(1999, 257.6) (2000, 280.1) (2001, 330.6)
(2002, 400.2) (2003, 446.6) (2004, 536.4)
(2005, 623.1)

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 6$ represent 1996.

(b) Use two points on the scatter plot to find an equation of a line that approximates the data.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data. Use this model and the model from part (b) to predict the revenues in 2006 and 2007.

(d) Sonic Corporation projected its revenues in 2006 and 2007 to be \$695 million and \$765 million. How close are these projections to the predictions from the models?

(e) Sonic Corporation also expected their yearly revenue to reach \$965 million in 2009, 2010, or 2011. Do the models from parts (b) and (c) support this? Explain your reasoning.

52. Revenue per Share The revenues per share of stock (in dollars) for Sonic Corporation for the years 1996 to 2005 are given by the following ordered pairs. *(Source: Sonic Corporation)*

(1996, 1.48) (1997, 1.90) (1998, 2.29)
(1999, 2.74) (2000, 3.15) (2001, 3.64)
(2002, 4.48) (2003, 5.06) (2004, 6.01)
(2005, 7.00)

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 6$ represent 1996.

(b) Use two points on the scatter plot to find an equation of a line that approximates the data.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data. Use this model and the model from part (b) to predict the revenues per share in 2006 and 2007.

(d) Sonic projected the revenues per share in 2006 and 2007 to be \$8.00 and \$8.80. How close are these projections to the predictions from the models?

(e) Sonic also expected the revenue per share to reach \$11.10 in 2009, 2010, or 2011. Do the models from parts (b) and (c) support this? Explain your reasoning.

53. Purchasing Power The value (in 1982 dollars) of each dollar received by producers in each of the years from 1991 to 2005 in the United States is represented by the following ordered pairs. *(Source: U.S. Bureau of Labor Statistics)*

(1991, 0.822) (1992, 0.812) (1993, 0.802)
(1994, 0.797) (1995, 0.782) (1996, 0.762)
(1997, 0.759) (1998, 0.765) (1999, 0.752)
(2000, 0.725) (2001, 0.711) (2002, 0.720)
(2003, 0.698) (2004, 0.673) (2005, 0.642)

(a) Use a spreadsheet software program to generate a scatter plot of the data. Let $t = 1$ represent 1991. Do the data appear to be linear?

(b) Use the *regression* feature of a spreadsheet software program to find a linear model for the data.

(c) Use the model to estimate the value (in 1982 dollars) of 1 dollar received by producers in 2007 and in 2008. Discuss the reliability of your estimates based on your scatter plot and the graph of your linear model for the data.

54. Purchasing Power The value (in 1982–1984 dollars) of each dollar paid by consumers in each of the years from 1991 to 2005 in the United States is represented by the following ordered pairs. *(Source: U.S. Bureau of Labor Statistics)*

(1991, 0.734) (1992, 0.713) (1993, 0.692)
(1994, 0.675) (1995, 0.656) (1996, 0.638)
(1997, 0.623) (1998, 0.614) (1999, 0.600)
(2000, 0.581) (2001, 0.565) (2002, 0.556)
(2003, 0.544) (2004, 0.530) (2005, 0.512)

(a) Use a spreadsheet software program to generate a scatter plot of the data. Let $t = 1$ represent 1991. Do the data appear to be linear?

(b) Use the *regression* feature of a spreadsheet software program to find a linear model for the data.

(c) Use the model to estimate the value (in 1982–1984 dollars) of 1 dollar paid by consumers in 2007 and in 2008. Discuss the reliability of your estimates based on your scatter plot and the graph of your linear model for the data.

55. Health Services The numbers of employees E (in thousands) in the health services industry for the years 2000 to 2005 are shown in the table. *(Source: U.S. Department of Health and Human Services)*

Year	2000	2001	2002
Employees, E	12,718	13,134	13,556

Year	2003	2004	2005
Employees, E	13,893	14,190	14,523

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 0$ represent 2000. Do the data appear to be linear?

(b) Use the *regression* feature of a graphing utility to find a linear model for the data.

(c) Use the model to estimate the numbers of employees in 2007 and 2009.

(d) Graph the linear model along with the scatter plot of the data. Comparing the data with the model, are the predictions in part (c) most likely to be high, low, or just about right? Explain your reasoning.

56. Health Care The total yearly health care expenditures E (in billions of dollars) in the United States for the years 1996 to 2005 are shown in the table. *(Source: U.S. Centers for Medicare and Medicaid Services)*

Year	1996	1997	1998	1999
Expenditures, E	1073	1125	1191	1265

Year	2000	2001	2002
Expenditures, E	1353	1470	1603

Year	2003	2004	2005
Expenditures, E	1733	1859	1988

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 6$ represent 1996. Do the data appear to be linear?

(b) Use the *regression* feature of a graphing utility to find a linear model for the data.

(c) Use the model to estimate the health care expenditures in 2006, 2007, and 2008.

(d) Graph the linear model along with the scatter plot. Use the trend in the scatter plot to explain why the predictions from the model differ from the following 2007 government projections for the same expenditures: \$2164 billion in 2006, \$2320 billion in 2007, and \$2498 billion in 2008.

57. Think About It Annual data from three years are used to create linear models for the population and the yearly snowfall of Reno, Nevada. Which model is more likely to give better predictions for future years? Discuss the appropriateness of using only three data points in each situation.

Section 2.4

Functions

- Determine if an equation or a set of ordered pairs represents a function.
- Use function notation and evaluate a function.
- Find the domain of a function.
- Write a function that relates quantities in an application problem.

Introduction to Functions

Many everyday phenomena involve two quantities that are related to each other by some rule of correspondence. Here are some examples.

1. The simple interest I earned on \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.
2. The distance d traveled on a bicycle in 2 hours is related to the speed s of the bicycle by the formula $d = 2s$.
3. The area A of a circle is related to its radius r by the formula $A = \pi r^2$.

Not all correspondences between two quantities have simple mathematical formulas. For instance, people commonly match up athletes with jersey numbers and hours of the day with temperatures. In each of these cases, however, there is some rule of correspondence that matches each item from one set with exactly one item from a different set. Such a rule of correspondence is called a **function.**

Definition of a Function

A **function** f from a set A to a set B is a rule of correspondence that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

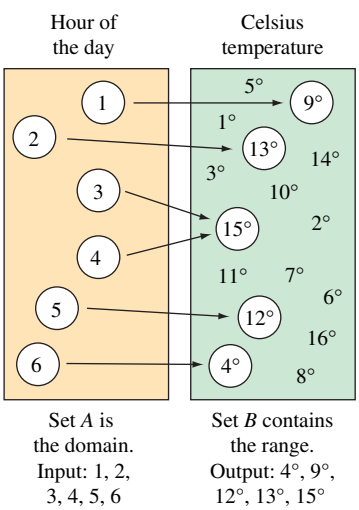


FIGURE 2.39 Function from Set A to Set B

To get a better idea of this definition, look at the function that relates the time of day to the temperature in Figure 2.39. This function can be represented by the following set of ordered pairs.

$$\{(1, 9°), (2, 13°), (3, 15°), (4, 15°), (5, 12°), (6, 4°)\}$$

In each ordered pair, the first coordinate (x-value) is the input and the second coordinate (y-value) is the output. In this example, note the following characteristics of a function.

1. Each element of A (the domain) must be matched with an element of B (the range).
2. Some elements of B may not be matched with any element of A.
3. Two or more elements of A may be matched with the same element of B.
4. An element of A cannot be matched with two different elements of B.

In the following two examples, you are asked to decide whether different correspondences are functions. To do this, you must decide whether each element of the domain A is matched with exactly one element of the range B. If any element of A is matched with two or more elements of B, the correspondence is not a function. For example, people are not a function of their birthday month because many people are born in any given month.

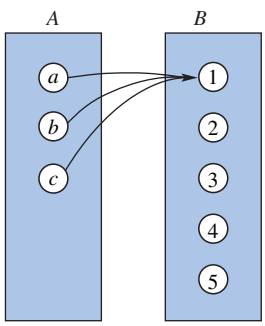
Example 1 Testing for Functions

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4, 5\}$. Which of the following sets of ordered pairs or figures represent functions from set A to set B?

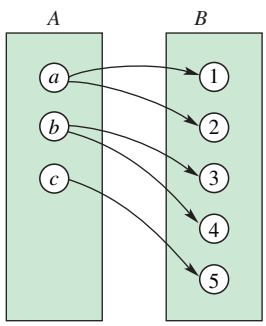
a. $\{(a, 2), (b, 3), (c, 4)\}$

b. $\{(a, 4), (b, 5)\}$

c.



d.



SOLUTION

a. This collection of ordered pairs *does* represent a function from A to B. Each element of A is matched with exactly one element of B.

b. This collection of ordered pairs *does not* represent a function from A to B. Not every element of A is matched with an element of B.

c. This figure *does* represent a function from A to B. It does not matter that each element of A is matched with the same element of B.

d. This figure *does not* represent a function from A to B. The element a of A is matched with *two* elements of B. This is also true of the element b.

✓CHECKPOINT 1

Let $A = \{a, b, c, d\}$ and $B = \{1, 3, 5, 7\}$. Does the set of ordered pairs $\{(a, 3), (b, 7), (c, 1), (d, 3)\}$ represent a function from set A to set B? ■

Representing functions by sets of ordered pairs is a common practice in *discrete mathematics*. In algebra, however, it is more common to represent functions by equations or formulas involving two variables. For instance, the equation

$$y = x^2 \qquad y \text{ is a function of } x.$$

represents the variable y as a function of the variable x. In this equation, x is the **independent variable** and y is the **dependent variable.** The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

Example 2 Testing for Functions Represented by Equations

Which of the equations represent(s) y as a function of x?

a. $x^2 + y = 1$

b. $-x + y^2 = 1$

SOLUTION To determine whether y is a function of x, try to solve for y in terms of x.

a. Solving for y yields

$$x^2 + y = 1 \qquad \text{Write original equation.}$$
$$y = 1 - x^2. \qquad \text{Solve for } y.$$

To each value of x there corresponds exactly one value of y. So, y *is* a function of x.

b. Solving for y yields

$$-x + y^2 = 1 \qquad \text{Write original equation.}$$
$$y^2 = 1 + x \qquad \text{Add } x \text{ to each side.}$$
$$y = \pm\sqrt{1 + x}. \qquad \text{Solve for } y.$$

The $\pm$ indicates that to a given value of x there correspond two values of y. So, y *is not* a function of x.

✓CHECKPOINT 2

Does the equation $y - 2 = x^2$ represent y as a function of x? ■

Function Notation

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 1 - x^2$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation.**

Input	*Output*	*Equation*
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as the **value of f at x** or simply **f of x.** The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *value* of the function at x. For instance, the function given by

$$f(x) = 3 - 2x$$

has *function values* denoted by $f(-1)$, $f(0)$, $f(2)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $\quad f(-1) = 3 - 2(-1) = 3 + 2 = 5.$

For $x = 0$, $\quad f(0) = 3 - 2(0) = 3 - 0 = 3.$

For $x = 2$, $\quad f(2) = 3 - 2(2) = 3 - 4 = -1.$

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7, \quad f(t) = t^2 - 4t + 7, \quad \text{and} \quad g(s) = s^2 - 4s + 7$$

all define the same function. In fact, the role of the independent variable in a function is simply that of a "placeholder." Consequently, the function above could be described by the form

$$f(\quad) = (\quad)^2 - 4(\quad) + 7.$$

Example 3 Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$. Find the following.

a. $g(2)$ **b.** $g(t)$ **c.** $g(x + 2)$

SOLUTION

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$g(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

b. Replacing x with t yields the following.

$$g(t) = -(t)^2 + 4(t) + 1 = -t^2 + 4t + 1$$

c. Replacing x with $x + 2$ yields the following.

$$\begin{aligned} g(x + 2) &= -(x + 2)^2 + 4(x + 2) + 1 \\ &= -(x^2 + 4x + 4) + 4x + 8 + 1 \\ &= -x^2 - 4x - 4 + 4x + 8 + 1 \\ &= -x^2 + 5 \end{aligned}$$

STUDY TIP

In Example 3(c), note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

✓CHECKPOINT 3

Let $h(x) = 2x^2 + x - 4$. Find $h(-1)$. ■

A function defined by two or more equations over a specified domain is called a **piecewise-defined function.**

Example 4 A Piecewise-Defined Function

Evaluate the function when $x = -1$, 0, and 1.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

SOLUTION Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$$f(-1) = (-1)^2 + 1 = 2.$$

For $x = 0$, use $f(x) = x - 1$ to obtain $f(0) = (0) - 1 = -1$. For $x = 1$, use $f(x) = x - 1$ to obtain $f(1) = (1) - 1 = 0$. The graph of the function is shown in Figure 2.40.

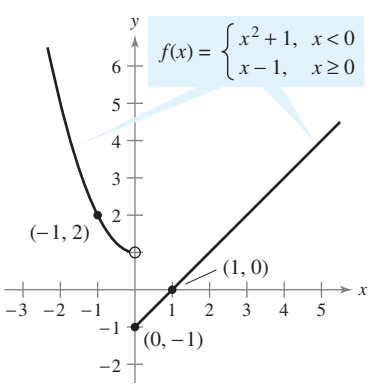


FIGURE 2.40

✓CHECKPOINT 4

Evaluate the function in Example 4 when $x = -3$ and 3. ■

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DISCOVERY

Use a graphing utility to graph $y = \sqrt{4 - x^2}$. What is the domain of this function? Then graph $y = \sqrt{x^2 - 4}$. What is the domain of this function? Do the domains of these two functions overlap? If so, for what values?

Finding the Domain of a Function

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function given by

$$f(x) = \frac{1}{x^2 - 4}$$

Domain excludes x-values that result in division by zero.

has an implied domain that consists of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain results from the restrictions needed to avoid even roots of negative numbers. For example, the function given by

$$f(x) = \sqrt{x}$$

Domain excludes x-values that result in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* result in the even root of a negative number.

Example 5 Finding the Domain of a Function

Find the domain of each function.

a. f: $\{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

b. $g(x) = \dfrac{1}{x + 5}$

c. Volume of a sphere: $V = \frac{4}{3}\pi r^3$

d. $h(x) = \sqrt{4 - x^2}$

e. $r(x) = \sqrt[3]{x + 3}$

SOLUTION

a. The domain of f consists of all first coordinates in the set of ordered pairs.

$$\text{Domain} = \{-3, -1, 0, 2, 4\}$$

b. Excluding x-values that yield zero in the denominator, the domain of g is the set of all real numbers x such that $x \neq -5$.

c. Because this function represents the volume of a sphere, the values of the radius r must be positive. So, the domain is the set of all real numbers r such that $r > 0$.

d. This function is defined only for x-values for which $4 - x^2 \geq 0$. Using the methods described in Section 1.7, you can conclude that $-2 \leq x \leq 2$. So, the domain of h is the interval $[-2, 2]$.

e. Because the cube root of any real number is defined, the domain of r is the set of all real numbers, or $(-\infty, \infty)$.

✓CHECKPOINT 5

Find the domain of the function $f(x) = 6 - x^3$. ■

In Example 5(c), note that the domain of a function may be implied by the physical context. For instance, from the equation $V = \frac{4}{3}\pi r^3$, you would have no reason to restrict r to positive values, but the physical context implies that a sphere cannot have a negative or zero radius.

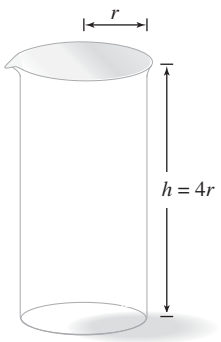


FIGURE 2.41

Applications

Example 6 The Dimensions of a Container

You are working with a cylindrical beaker in a chemistry lab experiment. The height of the beaker is 4 times the radius, as shown in Figure 2.41.

a. Write the volume of the beaker as a function of the radius r.

b. Write the volume of the beaker as a function of the height h.

SOLUTION

a. $V = \pi r^2 h = \pi r^2(4r) = 4\pi r^3$ V is a function of r.

b. $V = \pi\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{16}$ V is a function of h.

✓CHECKPOINT 6

In Example 6, suppose the radius is twice the height. Write the volume of the beaker as a function of the height h. ■

Example 7 The Path of a Baseball

A baseball is hit 3 feet above home plate at a velocity of 100 feet per second and an angle of 45°. The path of the baseball is given by the function

$$y = -0.0032x^2 + x + 3$$

where y and x are measured in feet. Will the baseball clear a 10-foot fence located 300 feet from home plate?

SOLUTION When $x = 300$, the height of the baseball is given by

$$y = -0.0032(300)^2 + 300 + 3 = 15 \text{ feet.}$$

The ball will clear the fence, as shown in Figure 2.42.

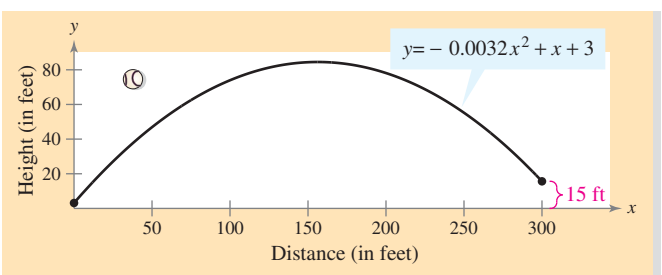


FIGURE 2.42

Notice that in Figure 2.42, the baseball is not at the point $(0, 0)$ before it is hit. This is because the original problem states that the baseball was hit 3 feet above the ground.

✓CHECKPOINT 7

In Example 7, will the baseball clear a 35-foot fence located 280 feet from home plate? ■

Example 8 **Patents**

The number P (in thousands) of patents issued increased in a linear pattern from 1998 to 2001. Then, in 2002, the pattern changed from a linear to a quadratic pattern (see Figure 2.43). These two patterns can be approximated by the function

$$P = \begin{cases} 6.96t + 106.9, & 8 \le t \le 11 \\ -6.550t^2 + 168.27t - 892.1, & 12 \le t \le 15 \end{cases}$$

with $t = 8$ corresponding to 1998. Use this function to approximate the total number of patents issued between 1998 and 2005. *(Source: U.S. Patent and Trademark Office)*

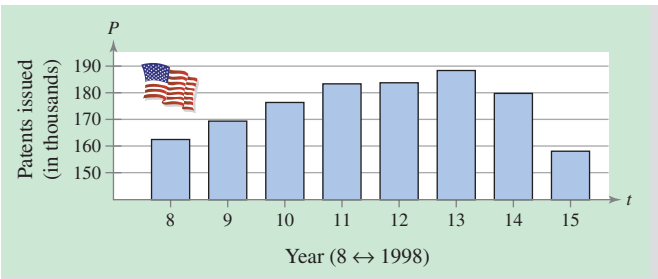


FIGURE 2.43

SOLUTION For 1998 to 2001, use the equation $P = 6.96t + 106.9$ to approximate the number of patents issued, as shown in the table. For 2002 to 2005, use the equation $P = -6.550t^2 + 168.27t - 892.1$ to approximate the number of patents issued, as shown in the table.

t	8	9	10	11	12	13	14	15
P	162.6	169.5	176.5	183.5	183.9	188.5	179.9	158.2

$P = 6.96t + 106.9$ $P = -6.550t^2 + 168.27t - 892.1$

To approximate the total number of patents issued from 1998 to 2005, add the amounts for each of the years, as follows.

$$162.6 + 169.5 + 176.5 + 183.5 + 183.9 + 188.5 + 179.9 + 158.2 = 1402.6$$

Because the number of patents issued is measured in thousands, you can conclude that the total number of patents issued between 1998 and 2005 was approximately 1,402,600.

✓CHECKPOINT 8

The number of cat cadavers purchased for dissection in a biology class from 2000 to 2008 can be modeled by the function

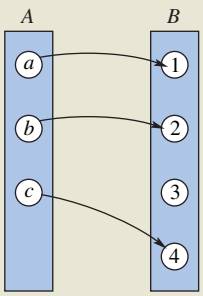
$$C = \begin{cases} 2t + 48, & 0 \le t \le 3 \\ 4t + 42, & 4 \le t \le 8 \end{cases}$$

with $t = 0$ corresponding to 2000. Use the function to approximate the total number of cat cadavers purchased from 2000 to 2008. ■

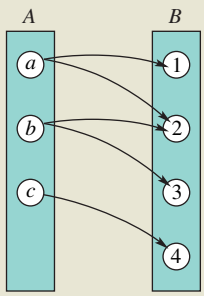
Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

For instance, let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$.



The set of ordered pairs $\{(a, 1), (b, 2), (c, 4)\}$ *is a function.*



The set of ordered pairs $\{(a, 1), (a, 2), (b, 2), (b, 3), (c, 4)\}$ *is not a function.*

Function Notation: $y = f(x)$

f is the **name** of the function.

y is the **dependent variable.**

x is the **independent variable.**

$f(x)$ is the **value of the function at x.**

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined. If x is in the domain of f, then f is said to be **defined** at x. If x is not in the domain of f, then f is said to be **undefined** at x.

Range: The **range** of a function is the set of all values (outputs) assumed by the dependent variable (that is, the set of all function values).

Implied Domain: If f is defined by an algebraic expression and the domain is not specified, the **implied domain** consists of all real numbers for which the expression is defined.

CONCEPT CHECK

1. Let $A = \{0, 2, 4, 6\}$ and $B = \{1, 3, 5, 7, 9\}$. Give an example of a set of ordered pairs that represent a function from set A to set B.
2. Is $f(2)$ equivalent to $2 \cdot f(x)$ for every function f? Explain.
3. Give an example of a function whose domain is the set of all real numbers x such that $x \neq 6$.
4. You want to write the area of a rectangle as a function of the width w. What information is needed? Explain.

Skills Review 2.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 1.1, 1.5, and 1.7.

In Exercises 1–4, simplify the expression.

1. $2(-3)^3 + 4(-3) - 7$

2. $4(-1)^2 - 5(-1) + 4$

3. $(x + 1)^2 + 3(x + 1) - 4 - (x^2 + 3x - 4)$

4. $(x - 2)^2 - 4(x - 2) - (x^2 - 4)$

In Exercises 5 and 6, solve for y in terms of x.

5. $2x + 5y - 7 = 0$

6. $y^2 = x^2$

In Exercises 7–10, solve the inequality.

7. $x^2 - 4 \geq 0$

8. $9 - x^2 \geq 0$

9. $x^2 + 2x + 1 \geq 0$

10. $x^2 - 3x + 2 \geq 0$

Exercises 2.4

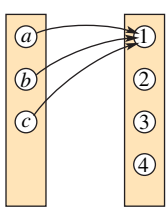
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, decide whether the set of figures represents a function from A to B.

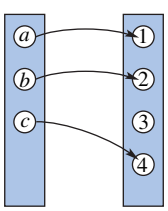
$A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$

Give reasons for your answers.

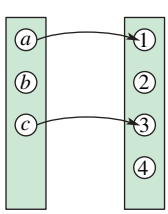
1.



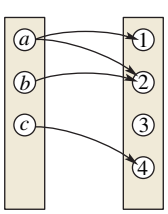
2.



3.



4.



In Exercises 5–8, decide whether the set of ordered pairs represents a function from A to B.

$A = \{0, 1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2\}$

Give reasons for your answers.

5. $\{(0, 1), (1, -2), (2, 0), (3, 2)\}$

6. $\{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$

7. $\{(0, 0), (1, 0), (2, 0), (3, 0)\}$

8. $\{(0, 2), (3, 0), (1, 1)\}$

In Exercises 9–12, decide whether the set of ordered pairs represents a function from A to B.

$A = \{a, b, c\}$ and $B = \{0, 1, 2, 3\}$

Give reasons for your answers.

9. $\{(a, 1), (c, 2), (c, 3), (b, 3)\}$

10. $\{(a, 1), (b, 2), (c, 3)\}$

11. $\{(1, a), (0, a), (2, c), (3, b)\}$

12. $\{(c, 0), (b, 0), (a, 3)\}$

In Exercises 13–16, the domain of f is the set

$A = \{-2, -1, 0, 1, 2\}$.

Write the function as a set of ordered pairs.

13. $f(x) = x^2$

14. $f(x) = \dfrac{2x}{x^2 + 1}$

15. $f(x) = \sqrt{x + 2}$

16. $f(x) = |x + 1|$

In Exercises 17–26, determine whether the equation represents y as a function of x.

17. $x^2 + y^2 = 4$

18. $x = y^2$

19. $x^2 + y = 4$

20. $x + y^2 = 4$

21. $2x + 3y = 4$

22. $x^2 + y^2 - 2x - 4y + 1 = 0$

23. $y^2 = x^2 - 1$

24. $y = \sqrt{x + 5}$

25. $x^2y - x^2 + 4y = 0$

26. $xy - y - x - 2 = 0$

In Exercises 27–30, fill in the blank and simplify.

27. $f(x) = 6 - 4x$

(a) $f(3) = 6 - 4(\quad)$

(b) $f(-7) = 6 - 4(\quad)$

(c) $f(t) = 6 - 4(\quad)$

(d) $f(c + 1) = 6 - 4(\quad)$

28. $f(s) = \dfrac{1}{s + 1}$

(a) $f(4) = \dfrac{1}{(\quad) + 1}$

(b) $f(0) = \dfrac{1}{(\quad) + 1}$

(c) $f(4x) = \dfrac{1}{(\quad) + 1}$

(d) $f(x + 1) = \dfrac{1}{(\quad) + 1}$

29. $g(x) = \dfrac{1}{x^2 - 2x}$

(a) $g(1) = \dfrac{1}{(\quad)^2 - 2(\quad)}$

(b) $g(-3) = \dfrac{1}{(\quad)^2 - 2(\quad)}$

(c) $g(t) = \dfrac{1}{(\quad)^2 - 2(\quad)}$

(d) $g(t + 1) = \dfrac{1}{(\quad)^2 - 2(\quad)}$

30. $f(t) = \sqrt{25 - t^2}$

(a) $f(3) = \sqrt{25 - (\quad)^2}$

(b) $f(5) = \sqrt{25 - (\quad)^2}$

(c) $f(x + 5) = \sqrt{25 - (\quad)^2}$

(d) $f(2x) = \sqrt{25 - (\quad)^2}$

In Exercises 31–44, evaluate the function at each specified value of the independent variable and simplify.

31. $f(x) = 2x - 3$

(a) $f(1)$ (b) $f(-3)$

(c) $f(x - 1)$ (d) $f\left(\frac{1}{4}\right)$

32. $g(y) = 7 - 3y$

(a) $g(0)$ (b) $g\left(\frac{7}{3}\right)$

(c) $g(s)$ (d) $g(s + 2)$

33. $h(t) = t^2 - 2t$

(a) $h(2)$ (b) $h(-1)$

(c) $h(x + 2)$ (d) $h(1.5)$

34. $k(b) = 2b^2 + 7b + 3$

(a) $k(0)$ (b) $k\left(-\frac{1}{2}\right)$

(c) $k(a)$ (d) $k(x + 2)$

35. $V(r) = \frac{4}{3}\pi r^3$

(a) $V(3)$ (b) $V(0)$

(c) $V\left(\frac{3}{2}\right)$ (d) $V(2r)$

36. $A(s) = \dfrac{\sqrt{3}s^2}{4}$

(a) $A(1)$ (b) $A(0)$

(c) $A(2x)$ (d) $A(3)$

37. $f(y) = 3 - \sqrt{y}$

(a) $f(4)$ (b) $f(100)$

(c) $f(4x^2)$ (d) $f(0.25)$

38. $f(x) = \sqrt{x + 3} - 2$

(a) $f(-3)$ (b) $f(1)$

(c) $f(x - 3)$ (d) $f(x + 4)$

39. $c(x) = \dfrac{1}{x^2 - 16}$

(a) $c(4)$ (b) $c(0)$

(c) $c(y + 2)$ (d) $c(y - 2)$

40. $q(t) = \dfrac{2t^2 + 3}{t^2}$

(a) $q(2)$ (b) $q(0)$

(c) $q(x)$ (d) $q(-x)$

41. $f(x) = \dfrac{|x|}{x}$

(a) $f(2)$ (b) $f(-2)$

(c) $f(x^2)$ (d) $f(x - 1)$

42. $f(x) = |x| + 4$

(a) $f(2)$ (b) $f(-2)$

(c) $f(x^2)$ (d) $f(x + 2)$

43. $f(x) = \begin{cases} 3x - 1, & x < 0 \\ 2x + 3, & x \geq 0 \end{cases}$

(a) $f(-1)$ (b) $f(0)$

(c) $f(-2)$ (d) $f(2)$

44. $f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ 2x - 3, & x > 1 \end{cases}$

(a) $f(-2)$ (b) $f(1)$

(c) $f\left(\frac{3}{2}\right)$ (d) $f(0)$

In Exercises 45–52, find all real values of x such that $f(x) = 0$.

45. $f(x) = 15 - 3x$

46. $f(x) = \dfrac{2x - 5}{3}$

47. $f(x) = x^2 - 9$

48. $f(x) = 2x^2 - 11x + 5$

49. $f(x) = x^3 - x$

50. $f(x) = x^3 - 3x^2 - 4x + 12$

51. $f(x) = \dfrac{3}{x - 1} + \dfrac{4}{x - 2}$

52. $f(x) = 3 + \dfrac{2}{x - 1}$

In Exercises 53–66, find the domain of the function.

53. $g(x) = 1 - 2x^2$

54. $f(x) = 5x^2 + 2x - 1$

55. $h(t) = \dfrac{4}{t}$

56. $s(y) = \dfrac{3y}{y + 5}$

57. $g(y) = \sqrt[3]{y - 10}$

58. $f(t) = \sqrt[3]{t + 4}$

59. $f(x) = \sqrt[4]{1 - x^2}$

60. $g(x) = \sqrt{x + 1}$

61. $g(x) = \dfrac{1}{x} - \dfrac{3}{x + 2}$

62. $h(x) = \dfrac{10}{x^2 - 2x}$

63. $f(x) = \dfrac{\sqrt{x + 1}}{x - 2}$

64. $f(s) = \dfrac{\sqrt{s - 1}}{s - 4}$

65. $f(x) = \dfrac{x - 4}{\sqrt{x}}$

66. $f(x) = \dfrac{x - 5}{\sqrt{x^2 - 9}}$

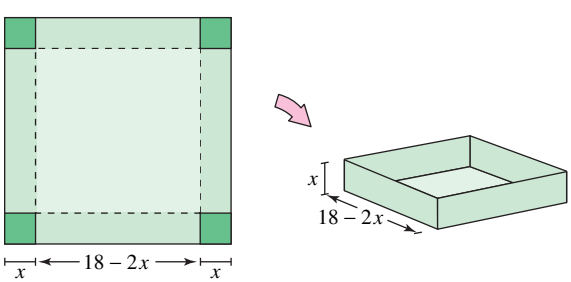
67. Consider $f(x) = \sqrt{x - 2}$ and $g(x) = \sqrt[3]{x - 2}$. Why are the domains of f and g different?

68. A student says that the domain of

$$f(x) = \frac{\sqrt{x + 1}}{x - 3}$$

is all real numbers except $x = 3$. Is the student correct? Explain.

69. Volume of a Box An open box is to be made from a square piece of material 18 inches on a side by cutting equal squares from the corners and turning up the sides (see figure).

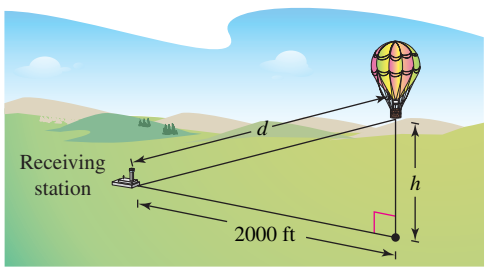


(a) Write the volume V of the box as a function of its height x.

(b) What is the domain of the function?

(c) Determine the volume of a box with a height of 4 inches.

70. Height of a Balloon A balloon carrying a transmitter ascends vertically from a point 2000 feet from the receiving station (see figure). Let d be the distance between the balloon and the receiving station. Write the height h of the balloon as a function of d. What is the domain of this function?



71. Cost, Revenue, and Profit A company produces a product for which the variable cost is \$11.75 per unit and the fixed costs are \$112,000. The product sells for \$21.95 per unit. Let x be the number of units produced and sold.

(a) Add the variable cost and the fixed costs to write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Use the formula

$$P = R - C$$

to write the profit P as a function of the number of units sold.

72. Cost, Revenue, and Profit A company produces a product for which the variable cost is \$9.85 per unit and the fixed costs are \$85,000. The product sells for \$19.95 per unit. Let x be the number of units produced and sold.

(a) Add the variable cost and the fixed costs to write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Use the formula

$$P = R - C$$

to write the profit P as a function of the number of units sold.

73. Path of a Ball The height y (in feet) of a baseball thrown by a child is given by

$$y = -\frac{1}{10}x^2 + 3x + 6$$

where x is the horizontal distance (in feet) from where the ball was thrown. Will the ball fly over the head of another child 30 feet away trying to catch the ball? (Assume that the child who is trying to catch the ball holds a baseball glove at a height of 5 feet.)

74. Path of a Salmon Part of the life cycle of a salmon is migration for reproduction. Salmon are anadromous fish. This means that they swim from the ocean to fresh water streams to lay their eggs. During migration, salmon must jump waterfalls to reach their destination. The path of a jumping salmon is given by

$$h = -0.42x^2 + 2.52x$$

where h is the height (in feet) and x is the horizontal distance (in feet) from where the salmon left the water. Will the salmon clear a waterfall that is 3 feet high if it leaves the water 4 feet from the waterfall?

75. National Defense The national defense budget expenses for veterans V (in billions of dollars) in the United States from 1990 to 2005 can be approximated by the model

$$V = \begin{cases} -0.326t^2 + 3.40t + 28.7, & 0 \le t \le 6 \\ 0.441t^2 - 6.23t + 62.6, & 7 \le t \le 15 \end{cases}$$

where t represents the year, with $t = 0$ corresponding to 1990. Use the model to find total veteran expenses in 1995 and 2005. *(Source: U.S. Office of Management and Budget)*

76. Mobile Homes The number N (in thousands) of mobile homes manufactured for residential use in the United States from 1991 to 2005 can be approximated by the model

$$N = \begin{cases} 29.08t + 157.0, & 1 \le t \le 7 \\ 4.902t^2 - 151.70t + 1289.2, & 8 \le t \le 15 \end{cases}$$

where t represents the year, with $t = 1$ corresponding to 1991. Use the model to find the total number of mobile homes manufactured between 1991 and 2005. *(Source: U.S. Census Bureau)*

77. Total Sales The total sales S (in millions of dollars) for the Cheesecake Factory for the years 1999 to 2005 are shown in the table. *(Source: Cheesecake Factory)*

Year	1999	2000	2001	2002
Sales, S	347.5	438.3	539.1	652.0

Year	2003	2004	2005
Sales, S	773.8	969.2	1177.6

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 9$ corresponding to 1999.

(b) Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data.

(c) Use each model to approximate the total sales for each year from 1999 to 2005. Compare the values generated by each model with the actual values shown in the table. Which model is a better fit? Justify your answer.

78. Book Value per Share The book values per share B (in dollars) for Analog Devices for the years 1996 to 2005 are shown in the table. *(Source: Analog Devices)*

Year	BV/share, B
1996	2.72
1997	3.36
1998	3.52
1999	4.62
2000	6.44

Year	BV/share, B
2001	7.83
2002	7.99
2003	8.88
2004	10.11
2005	10.06

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data.

(c) Use each model to approximate the book value per share for each year from 1996 to 2005. Compare the values generated by each model with the actual values shown in the table. Which model is a better fit? Justify your answer.

79. Average Cost The inventor of a new game determines that the variable cost of producing the game is \$2.95 per unit and the fixed costs are \$8000. The inventor sells each game for \$8.79. Let x be the number of games sold.

(a) Write the total cost C as a function of the number of games sold.

(b) Write the average cost per unit $\overline{C} = C/x$ as a function of x.

(c) Complete the table.

x	100	1000	10,000	100,000
$\overline{C}$				

(d) Write a paragraph analyzing the data in the table. What do you observe about the average cost per unit as x gets larger?

80. Average Cost A manufacturer determines that the variable cost for a new product is \$2.05 per unit and the fixed costs are \$57,000. The product is to be sold for \$5.89 per unit. Let x be the number of units sold.

(a) Write the total cost C as a function of the number of units sold.

(b) Write the average cost per unit $\overline{C} = C/x$ as a function of x.

(c) Complete the table.

x	100	1000	10,000	100,000
$\overline{C}$				

(d) Write a paragraph analyzing the data in the table. What do you observe about the average cost per unit as x gets larger?

81. Charter Bus Fares For groups of 80 or more people, a charter bus company determines the rate per person (in dollars) according to the formula

$$\text{Rate} = 8 - 0.05(n - 80) \quad n \geq 80$$

where n is the number of people in the group.

(a) Write the total revenue R for the bus company as a function of n.

(b) Complete the table.

n	90	100	110	120	130	140	150
R							

(c) Write a paragraph analyzing the data in the table.

82. Ripples in a Pond A stone is thrown into the middle of a calm pond, causing ripples to form in concentric circles. The radius r of the outermost ripple increases at the rate of 0.75 foot per second.

(a) Write a function for the radius r of the circle formed by the outermost ripple in terms of time t.

(b) Write a function for the area A enclosed by the outermost ripple. Complete the table.

Time, t	1	2	3	4	5
Radius, r (in feet)					
Area, A (in square feet)					

(c) Compare the ratios $A(2)/A(1)$ and $A(4)/A(2)$. What do you observe? Based on your observation, predict the area when $t = 8$. Verify by checking $t = 8$ in the area function.

83. *MAKE A DECISION: DIVIDENDS* The dividends D (in dollars) per share declared by Coca-Cola for the years 1990 to 2005 are shown in the table. *(Source: Coca-Cola Company)*

Year	Dividend, D	Year	Dividend, D
1990	0.20	1998	0.60
1991	0.24	1999	0.64
1992	0.28	2000	0.68
1993	0.34	2001	0.72
1994	0.39	2002	0.80
1995	0.44	2003	0.88
1996	0.50	2004	1.00
1997	0.56	2005	1.12

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 1990.

(b) Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data.

(c) Use the graphing utility to graph each model from part (b) with the data.

(d) Which model do you think better fits the data? Explain your reasoning.

(e) Use the model you selected in part (d) to estimate the dividends per share in 2006 and 2007. Coca-Cola predicts the dividends per share in 2006 and 2007 will be \$1.24 and \$1.32, respectively. How well do your estimates match the ones given by Coca-Cola?

MAKE A DECISION In Exercises 84 and 85, determine whether the statements use the word *function* in ways that are *mathematically* correct. Explain your reasoning.

84. (a) The sales tax on a purchased item is a function of the selling price.

(b) Your score on the next algebra exam is a function of the number of hours you study for the exam.

85. (a) The amount in your savings account is a function of your salary.

(b) The speed at which a free-falling baseball strikes the ground is a function of its initial height.

86. Extended Application To work an extended application analyzing the sales per share of St. Jude Medical, Inc. for the years 1991 to 2005, visit this text's website at *college.hmco.com/info/larsonapplied*. *(Source: St. Jude Medical, Inc.)*

Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

1. $(-3, 2),\ (4, -5)$ **2.** $(1.3, -4.5),\ (-3.7, 0.7)$ **3.** $(4, -2), \left(-1, -\frac{5}{2}\right)$

4. A city had a population of 233,134 in 2004 and 244,288 in 2007. Predict the population in 2009. Explain your reasoning.

In Exercises 5–8, find an equation of the line that passes through the given point and has the indicated slope. Then sketch the line.

	Point	*Slope*		*Point*	*Slope*
5.	$(3, 5)$	$m = \frac{2}{3}$	**6.**	$(-2, 4)$	$m = 0$
7.	$(2, -3)$	m is undefined.	**8.**	$(-2, -5)$	$m = -2$

In Exercises 9–11, sketch the graph of the equation. Identify any intercepts and symmetry.

9. $y = 9 - x^2$ **10.** $y = x\sqrt{x + 4}$ **11.** $y = |x - 3|$

In Exercises 12 and 13, find the standard form of the equation of the circle.

12. Center: $(2, -3)$; radius: 4

13. Center: $\left(0, -\frac{1}{2}\right)$; point on circle: $\left(-1, \frac{3}{2}\right)$

14. Write the equation $x^2 + y^2 - 2x + 4y - 4 = 0$ in standard form. Then sketch the circle.

In Exercises 15 and 16, evaluate the function as indicated and simplify.

15. $f(x) = 3(x + 2) - 4$

(a) $f(0)$ (b) $f(-3)$

16. $g(t) = 2t^3 - t^2$

(a) $g(1)$ (b) $g(-2)$

In Exercises 17 and 18, find the domain of the function.

17. $h(x) = \sqrt{x - 4}$

18. $f(x) = \dfrac{x}{x + 2}$

In Exercises 19 and 20, use the U.S. Department of Agriculture's estimates for the federal costs C of food stamps (in millions of dollars) shown in the table. *(Source: U.S. Department of Agriculture)*

Year	Cost, C (in millions of dollars)
2000	14,983
2001	15,547
2002	18,256
2003	21,404
2004	24,622
2005	28, 567

Table for 19 and 20

19. Let $t = 0$ represent 2000. Use a graphing utility to create a scatter plot of the data and use the *regression* feature to find a linear model and a quadratic model for the data.

20. Use each model you found in Exercise 19 to predict the federal costs of food stamps in 2006 and 2007.

21. Write the area A of a circle as a function of its circumference C.

Section 2.5

Graphs of Functions

- Find the domain and range using the graph of a function.
- Identify the graph of a function using the Vertical Line Test.
- Describe the increasing and decreasing behavior of a function.
- Find the relative minima and relative maxima of the graph of a function.
- Classify a function as even or odd.
- Identify six common graphs and use them to sketch the graph of a function.

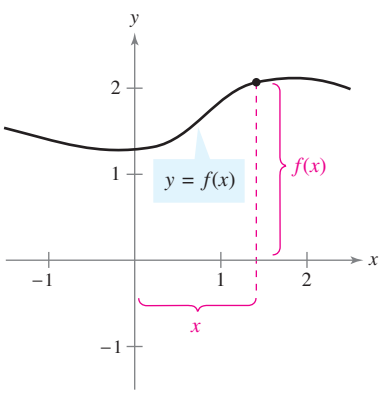


FIGURE 2.44

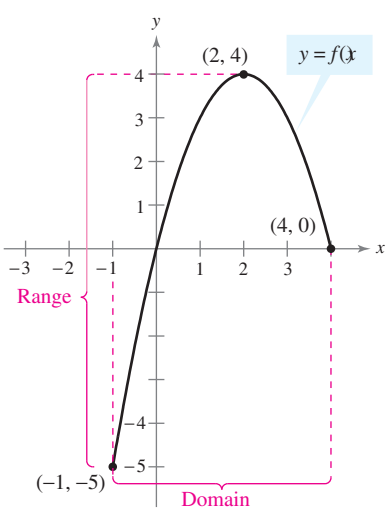


FIGURE 2.45

The Graph of a Function

In Section 2.4, you studied functions from an algebraic point of view. In this section, you will study functions from a graphical perspective.

The **graph of a function** f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f. As you study this section, remember that

$$x = \text{the directed distance from the } y\text{-axis}$$

$$f(x) = \text{the directed distance from the } x\text{-axis}$$

as shown in Figure 2.44. If the graph of a function has an x-intercept at $(a, 0)$, then a is a **zero** of the function. In other words, the zeros of a function are the values of x for which $f(x) = 0$. For instance, the function given by $f(x) = x^2 - 4$ has two zeros: -2 and 2.

The **range** of a function (the set of values assumed by the dependent variable) is often easier to determine graphically than algebraically. This technique is illustrated in Example 1.

Example 1 Finding the Domain and Range of a Function

Use the graph of the function f, shown in Figure 2.45, to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

SOLUTION

a. Because the graph does not extend beyond $x = -1$ (on the left) and $x = 4$ (on the right), the domain of f is all x in the interval $[-1, 4]$.

b. Because $(-1, -5)$ is a point on the graph of f, it follows that

$$f(-1) = -5.$$

Similarly, because $(2, 4)$ is a point on the graph of f, it follows that

$$f(2) = 4.$$

c. Because the graph does not extend below $f(-1) = -5$ or above $f(2) = 4$, the range of f is the interval $[-5, 4]$.

✓CHECKPOINT 1

Use the graph of $f(x) = x^2 - 3$ to find the domain and range of f. ■

By the definition of a function, at most one y-value corresponds to a given x-value. This means that the graph of a function cannot have two or more different points with the same x-coordinate, and no two points on the graph of a function can be vertically above or below each other. It follows, then, that a vertical line can intersect the graph of a function at most once. This observation provides a convenient visual test called the **Vertical Line Test** for functions.

Vertical Line Test for Functions

A set of points in a coordinate plane is the graph of y as a function of x if and only if no vertical line intersects the graph at more than one point.

Example 2 Vertical Line Test for Functions

Use the Vertical Line Test to decide whether the graphs in Figure 2.46 represent y as a function of x.

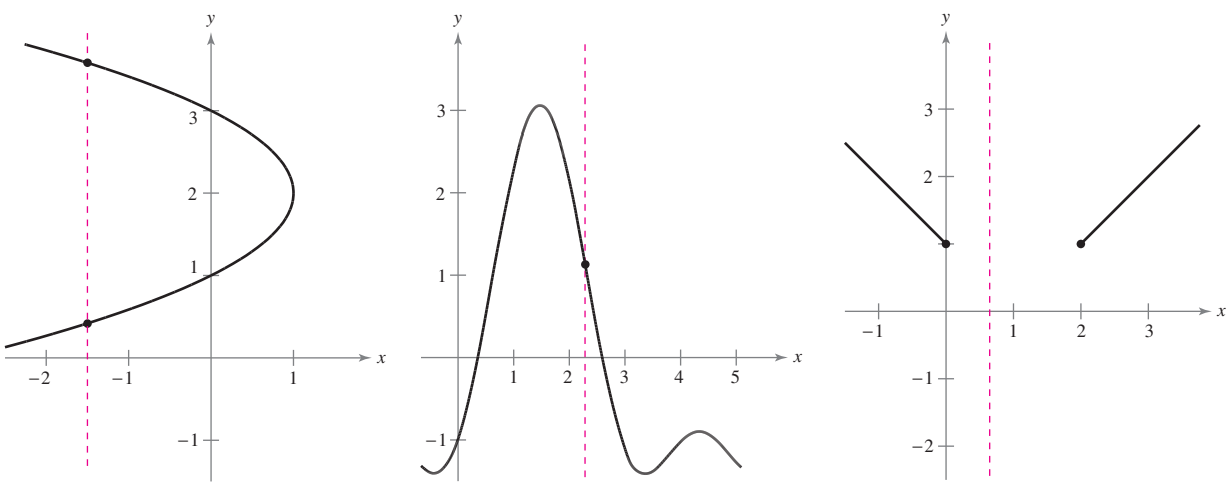


FIGURE 2.46

SOLUTION

a. This *is not* a graph of y as a function of x because you can find a vertical line that intersects the graph twice. That is, for a particular input x, there is more than one output y.

b. This *is* a graph of y as a function of x because every vertical line intersects the graph at most once. That is, for a particular input x, there is at most one output y.

c. This *is* a graph of y as a function of x. That is, for a particular input x, there is at most one output y. Note that if a vertical line does not intersect the graph, it simply means that the function is undefined for that particular value of x.

✓CHECKPOINT 2

Use the Vertical Line Test to decide whether the graph of $x^2 + y = 2$ represents y as a function of x. ■

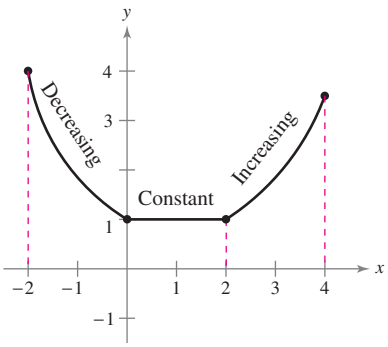


FIGURE 2.47

Increasing and Decreasing Functions

The more you know about the graph of a function, the more you know about the function itself. Consider the graph that is shown in Figure 2.47, for example. As you move from *left to right,* this graph decreases, then is constant, and then increases.

Increasing, Decreasing, and Constant Functions

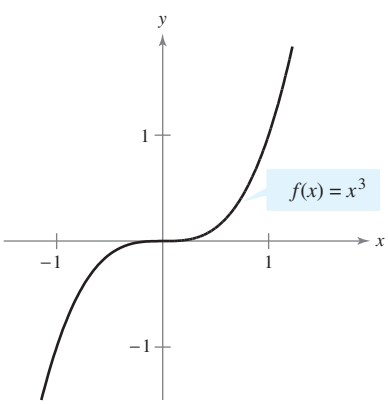
A function f is **increasing** on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

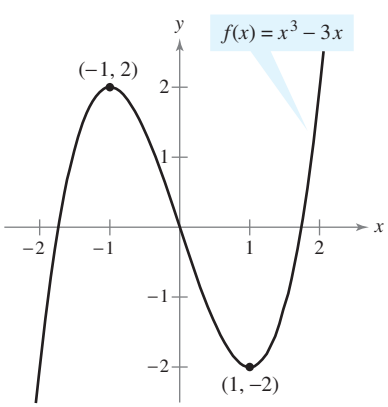
A function f is **constant** on an interval if, for any x_1 and x_2 in the interval, $f(x_1) = f(x_2)$.

Example 3 Increasing and Decreasing Functions

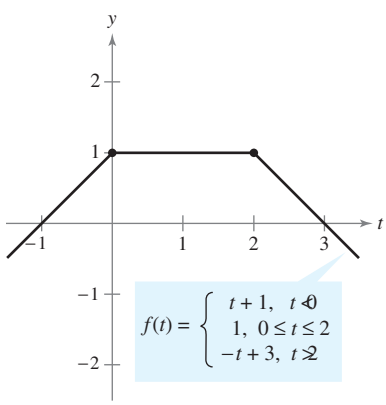
Describe the increasing or decreasing behavior of each function shown in Figure 2.48.



(a)



(b)



(c)

FIGURE 2.48

SOLUTION

a. This function is increasing over the entire real line.

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $(0, 2)$, and decreasing on the interval $(2, \infty)$.

✓CHECKPOINT 3

Describe the increasing or decreasing behavior of the function $f(x) = x^2 + 3x$. ■

The points at which a function changes its increasing, decreasing, or constant behavior are helpful in determining the **relative minimum** or **relative maximum** values of the function.

Definition of Relative Minimum and Relative Maximum

A function value $f(a)$ is called a **relative minimum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \leq f(x).$$

A function value $f(a)$ is called a **relative maximum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \geq f(x).$$

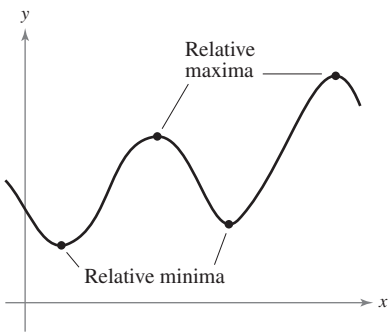


FIGURE 2.49

Figure 2.49 shows several examples of relative minima and relative maxima. In Section 3.1, you will study a technique for finding the *exact point* at which a second-degree polynomial function has a relative minimum or relative maximum. For the time being, however, you can use a graphing utility to find reasonable approximations of these points.

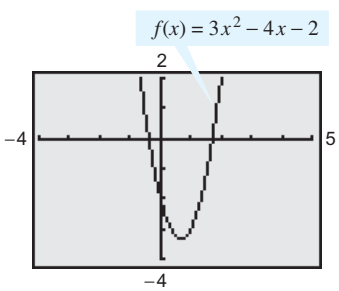


FIGURE 2.50

Example 4 Approximating a Relative Minimum

Use a graphing utility to approximate the relative minimum of the function given by $f(x) = 3x^2 - 4x - 2$.

SOLUTION The graph of f is shown in Figure 2.50. By using the *zoom* and *trace* features of a graphing utility, you can estimate that the function has a relative minimum at the point

$$(0.67, -3.33). \qquad \text{Relative minimum}$$

Later, in Section 3.1, you will be able to determine that the exact point at which the relative minimum occurs is $\left(\frac{2}{3}, -\frac{10}{3}\right)$.

✓CHECKPOINT 4

Use a graphing utility to approximate the relative maximum of the function given by $f(x) = -x^2 + 4x - 2$. ■

TECHNOLOGY

For instructions on how to use the *table* feature and the *minimum* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

You can also use the *table* feature of a graphing utility to approximate numerically the relative minimum of the function in Example 4. Using a table that begins at 0.6 and increments the value of x by 0.01, you can approximate the minimum of $f(x) = 3x^2 - 4x - 2$ to be -3.33, which occurs at $(0.67, -3.33)$. A third way to find the relative minimum is to use the *minimum* feature of a graphing utility.

TECHNOLOGY

If you use a graphing utility to estimate the x- and y-values of a relative minimum or relative maximum, the *zoom* feature will often produce graphs that are nearly flat. To overcome this problem, you can manually change the vertical setting of the viewing window. The graph will stretch vertically if the values of Ymin and Ymax are closer together.

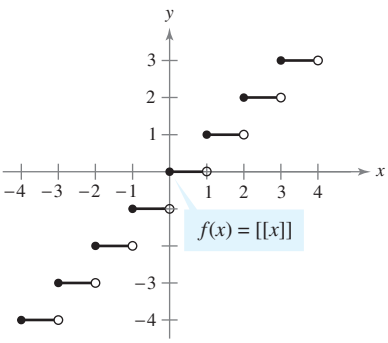


FIGURE 2.51 Greatest Integer Function

Step Functions

The **greatest integer function** is denoted by $[\![x]\!]$ and is defined as

$$f(x) = [\![x]\!] = \text{the greatest integer less than or equal to } x.$$

The graph of this function is shown in Figure 2.51. Note that the graph of the greatest integer function jumps vertically one unit at each integer and is constant (a horizontal line segment) between each pair of consecutive integers. Because of the jumps in its graph, the greatest integer function is an example of a type of function called a **step function.** Some values of the greatest integer function are as follows.

$$[\![-1]\!] = -1 \qquad [\![-0.5]\!] = -1$$
$$[\![0]\!] = 0 \qquad [\![0.5]\!] = 0$$
$$[\![1]\!] = 1 \qquad [\![1.5]\!] = 1$$

The range of the greatest integer function is the set of all integers.

If you use a graphing utility to graph a step function, you should set the utility to *dot* mode rather than *connected* mode.

Example 5 The Price of a Telephone Call

The cost of a long-distance telephone call is \$0.10 for up to, but not including, the first minute and \$0.05 for each additional minute (or portion of a minute). The greatest integer function

$$C = 0.10 + 0.05[\![t]\!], \quad t > 0$$

can be used to model the cost of this call, where C is the total cost of the call (in dollars) and t is the length of the call (in minutes).

a. Sketch the graph of this function.

b. How long can you talk without spending more than \$1?

SOLUTION

a. For calls up to, but not including, 1 minute, the cost is \$0.10. For calls between 1 and 2 minutes, the cost is \$0.15, and so on.

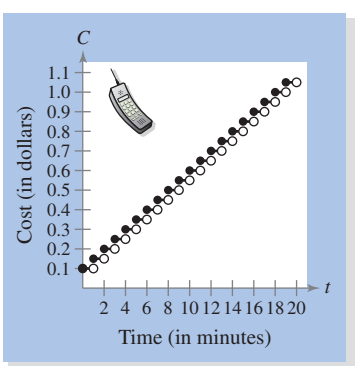


FIGURE 2.52

Length of call, t	Cost of call, C
$0 < t < 1$	\$0.10
$1 \le t < 2$	\$0.15
$2 \le t < 3$	\$0.20
⋮	⋮
$19 \le t < 20$	\$1.05

Using these and other values, you can sketch the graph shown in Figure 2.52.

b. From the graph, you can see that your phone call must be less than 19 minutes to avoid spending more than \$1.

✓CHECKPOINT 5

In Example 5, suppose the cost of each additional minute (or portion of a minute) is \$0.07. Sketch the graph of this function. How long can you talk without spending more than \$1? ■

Even and Odd Functions

DISCOVERY

Graph each function with a graphing utility. Determine whether the function is *odd, even,* or *neither.*

$f(x) = x^2 - x^4$

$g(x) = 2x^3 + 1$

$h(x) = x^5 - 2x^3 + x$

$k(x) = x^5 - 2x^4 + x - 2$

$j(x) = 2 - x^6 - x^8$

$p(x) = x^9 + 3x^5 - x^3 + x$

What do you notice about the equations of functions that are odd? What do you notice about the equations of functions that are even? Can you describe a way to identify a function as odd or even by inspecting its equation? Can you describe a way to identify a function as neither odd nor even by inspecting its equation?

In Section 2.1, you studied different types of symmetry of a graph. A function is said to be **even** if its graph is symmetric with respect to the y-axis and **odd** if its graph is symmetric with respect to the origin. The symmetry tests in Section 2.1 yield the following tests for even and odd functions. Even though symmetry with respect to the x-axis is introduced in Section 2.1, it will not be discussed here because a graph that is symmetric about the x-axis is not a function.

Tests for Even and Odd Functions

A function given by $y = f(x)$ is even if, for each x in the domain of f,

$$f(-x) = f(x).$$

A function given by $y = f(x)$ is odd if, for each x in the domain of f,

$$f(-x) = -f(x).$$

Example 6 Even and Odd Functions

Decide whether each function is even, odd, or neither.

a. $g(x) = x^3 - x$ **b.** $h(x) = x^2 + 1$

SOLUTION

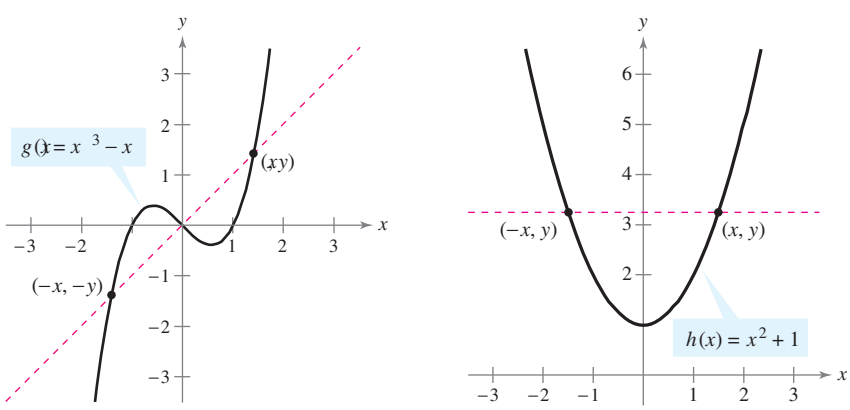
a. The function given by $g(x) = x^3 - x$ is odd because

$$g(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -g(x).$$

b. The function given by $h(x) = x^2 + 1$ is even because

$$h(-x) = (-x)^2 + 1 = x^2 + 1 = h(x).$$

The graphs of the two functions are shown in Figure 2.53.



(a) Odd function (symmetric about origin) (b) Even function (symmetric about y-axis)

FIGURE 2.53

✓CHECKPOINT 6

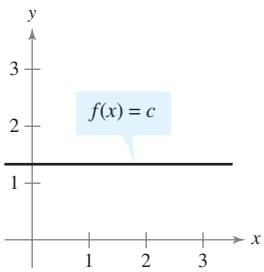
Decide whether the function $f(x) = -2x^2 + x - 1$ is even, odd, or neither. ■

Common Graphs

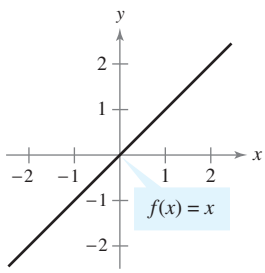
Figure 2.54 shows the graphs of six common functions. You need to be familiar with these graphs. They can be used as an aid when sketching other graphs. For instance, the graph of the absolute value function given by

$$f(x) = |x - 2|$$

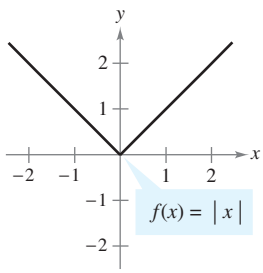
is $\vee$-shaped.



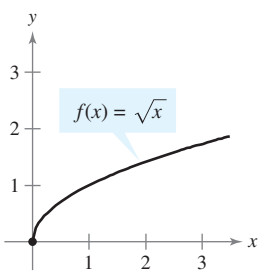
(a) Constant function



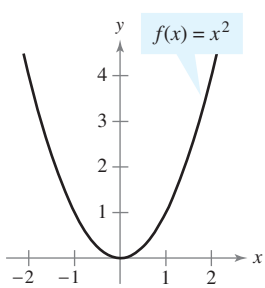
(b) Identity function



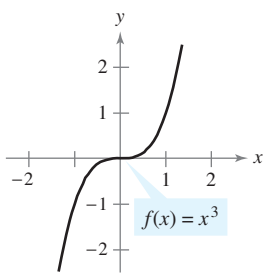
(c) Absolute value function



(d) Square root function



(e) Squaring function



(f) Cubing function

FIGURE 2.54

CONCEPT CHECK

In Exercises 1 and 2, determine whether the statement is true or false. Justify your answer.

1. If $a < 0$, then $f(0)$ is the relative maximum of the function $f(x) = ax^2$.
2. The graph of the greatest integer function is increasing over its entire domain.
3. Is the function represented by the following set of ordered pairs *even, odd,* or *neither?*

 $\{(1, 4), (-1, 4)\}$
4. The line $x = 1$ does not intersect the graph of f. Can you conclude that f is a function? Explain.

Skills Review 2.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.4, 1.5, and 2.4.

1. Find $f(2)$ for $f(x) = -x^3 + 5x$.

2. Find $f(6)$ for $f(x) = x^2 - 6x$.

3. Find $f(-x)$ for $f(x) = 3/x$.

4. Find $f(-x)$ for $f(x) = x^2 + 3$.

In Exercises 5 and 6, solve the equation.

5. $x^3 - 16x = 0$

6. $2x^2 - 3x + 1 = 0$

In Exercises 7–10, find the domain of the function.

7. $g(x) = 4(x - 4)^{-1}$

8. $f(x) = 2x/(x^2 - 9x + 20)$

9. $h(t) = \sqrt[4]{5 - 3t}$

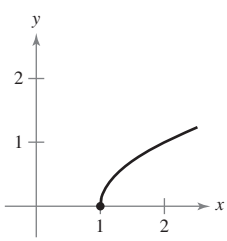
10. $f(t) = t^3 + 3t - 5$

Exercises 2.5

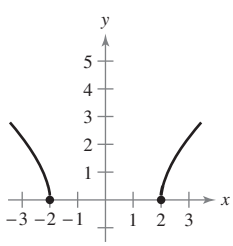
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find the domain and range of the function. Then evaluate f at the given x-value.

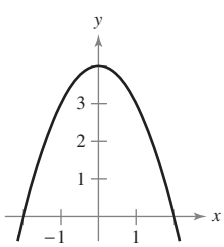
1. $f(x) = \sqrt{x - 1}$, $x = 1$



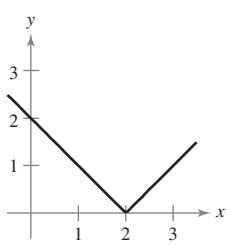
2. $f(x) = \sqrt{x^2 - 4}$, $x = -2$



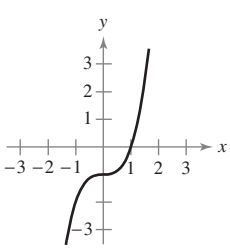
3. $f(x) = 4 - x^2$, $x = 0$



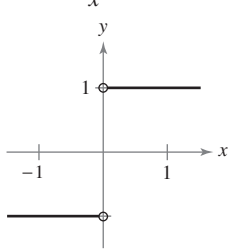
4. $f(x) = |x - 2|$, $x = 2$



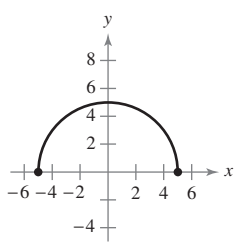
5. $f(x) = x^3 - 1$, $x = 0$



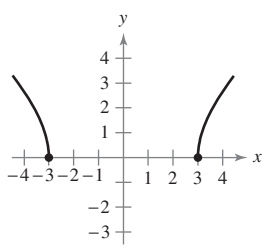
6. $f(x) = \dfrac{|x|}{x}$, $x = 5$



7. $f(x) = \sqrt{25 - x^2}$, $x = 0$

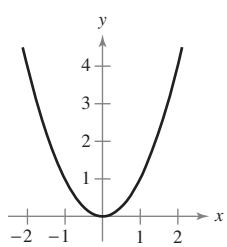


8. $f(x) = \sqrt{x^2 - 9}$, $x = 3$

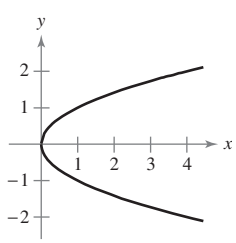


In Exercises 9–12, use the Vertical Line Test to decide whether y is a function of x.

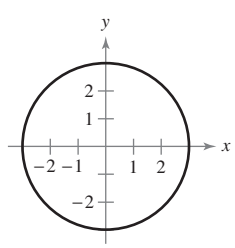
9. $y = x^2$



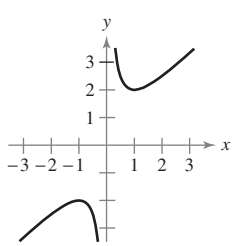
10. $x - y^2 = 0$



11. $x^2 + y^2 = 9$

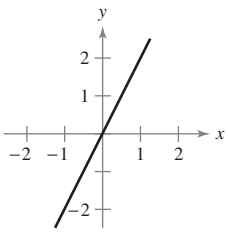


12. $x^2 = xy - 1$

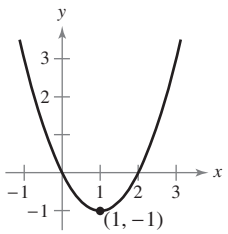


In Exercises 13–20, describe the increasing and decreasing behavior of the function. Find the point or points where the behavior of the function changes.

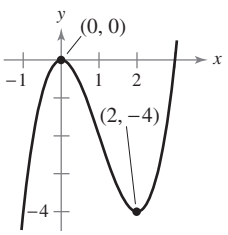
13. $f(x) = 2x$



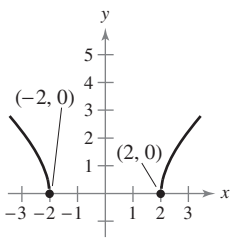
14. $f(x) = x^2 - 2x$



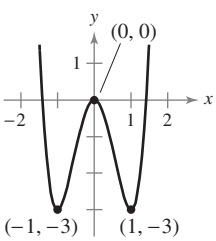
15. $f(x) = x^3 - 3x^2$



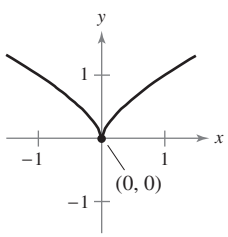
16. $f(x) = \sqrt{x^2 - 4}$



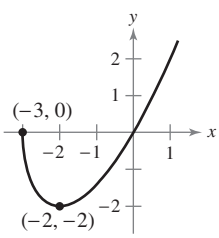
17. $f(x) = 3x^4 - 6x^2$



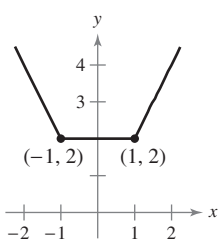
18. $f(x) = x^{2/3}$



19. $y = x\sqrt{x + 3}$



20. $y = |x + 1| + |x - 1|$



Ⓣ In Exercises 21–26, use a graphing utility to graph the function, approximate the relative minimum or maximum of the function, and estimate the open intervals on which the function is increasing or decreasing.

21. $f(x) = x^2 - 4x + 1$

22. $f(x) = -x^2 + 6x + 3$

23. $f(x) = x^3 - 3x^2$

24. $f(x) = -x^3 + 3x + 1$

25. $f(x) = \frac{1}{4}(-4x^4 - 5x^3 + 10x^2 + 8x + 6)$

26. $f(x) = \frac{1}{4}(x^4 + x^3 - 10x^2 + 2x - 15)$

In Exercises 27–32, decide whether the function is even, odd, or neither.

27. $f(x) = x^6 - 2x^2 + 3$

28. $f(t) = t^2 + 3t - 10$

29. $g(x) = x^3 - 5x$

30. $h(x) = x^3 + 3$

31. $f(x) = x\sqrt{4 - x^2}$

32. $g(s) = 4s^{2/3}$

In Exercises 33–36, evaluate the function at each specified value of the independent variable.

33. $f(x) = [\![x]\!]$

(a) $f(2)$ (b) $f(2.5)$

(c) $f(-2.5)$ (d) $f(-4)$

34. $f(x) = [\![-x]\!]$

(a) $f(3)$ (b) $f(6.1)$

(c) $f(-5.9)$ (d) $f(-9)$

35. $f(x) = [\![x - 1.8]\!]$

(a) $f(4)$ (b) $f(3.7)$

(c) $f(-5.8)$ (d) $f(-6.3)$

36. $f(x) = [\![x + 0.3]\!]$

(a) $f(2.9)$ (b) $f(4.6)$

(c) $f(-2.3)$ (d) $f(-4.2)$

In Exercises 37–50, sketch the graph of the function and determine whether the function is even, odd, or neither.

37. $f(x) = 3$

38. $g(x) = x$

39. $f(x) = 5 - 3x$

40. $h(x) = x^2 - 4$

41. $g(s) = \frac{s^3}{4}$

42. $f(t) = -t^4$

43. $f(x) = \sqrt{1 - x}$

44. $g(t) = \sqrt[3]{t - 1}$

45. $f(x) = x^{3/2}$

46. $f(x) = |x + 2|$

47. $f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ 3x - 1, & x > 1 \end{cases}$

48. $f(x) = \begin{cases} 2x - 1, & x \leq -1 \\ x^2 - 1, & x > -1 \end{cases}$

49. $f(x) = \begin{cases} x + 1, & x \le 0 \\ 4, & 0 < x \le 2 \\ 3x - 1, & x > 2 \end{cases}$

50. $f(x) = \begin{cases} 2x - 1, & x \le 1 \\ 3, & 1 < x \le 3 \\ 2x + 1, & x > 3 \end{cases}$

In Exercises 51–64, sketch the graph of the function.

51. $f(x) = 4 - x$

52. $f(x) = 4x + 2$

53. $f(x) = x^2 - 9$

54. $f(x) = x^2 - 4x$

55. $f(x) = 1 - x^4$

56. $f(x) = x^4 - 4x^2$

57. $f(x) = \frac{1}{3}(3 + |x|)$

58. $f(x) = -1(1 + |x|)$

59. $f(x) = \sqrt{x + 3}$

60. $f(x) = \sqrt{x - 1}$

61. $f(x) = -[\![x]\!]$

62. $f(x) = 2[\![x]\!]$

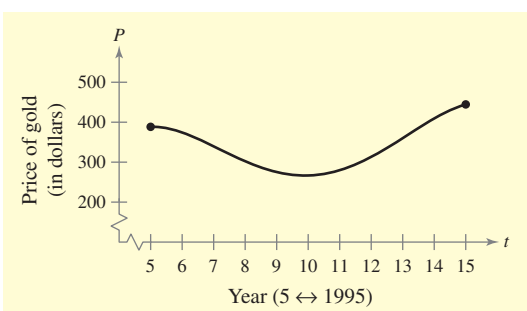
63. $f(x) = [\![x - 1]\!]$

64. $f(x) = [\![x + 1]\!]$

65. ***MAKE A DECISION: PRICE OF GOLD*** The price P (in dollars) of an ounce of gold from 1995 to 2005 can be approximated by the model

$$P = -0.203513t^4 + 8.27786t^3 - 115.1479t^2 + 635.832t - 819.60, \quad 5 \le t \le 15$$

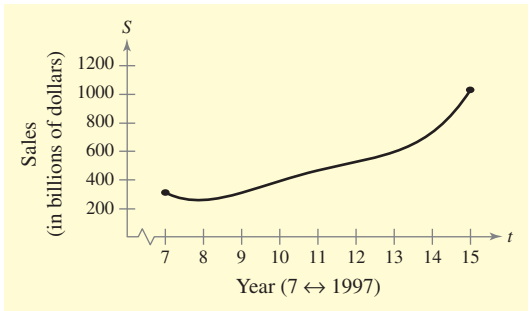
where t represents the year, with $t = 5$ corresponding to 1995. Use the graph of P to find the maximum price of gold between 1995 and 2005. During which years was the price decreasing? During which years was the price increasing? Is it realistic to assume that the price of gold will continue to follow this model? *(Source: World Gold Council)*



66. ***MAKE A DECISION: SALES*** The sales S (in billions of dollars) of petroleum and coal products from 1997 to 2005 can be approximated by the model

$$S = 1.34668t^4 - 57.7219t^3 + 918.390t^2 - 6355.84t + 16{,}367.4, \quad 7 \le t \le 15$$

where t represents the year, with $t = 7$ corresponding to 1997. Use the graph of S to find the maximum sales of these products between 1997 and 2005. During which years were sales decreasing? During which years were sales increasing? Is it realistic to assume that sales will continue to follow this model? *(Source: U.S. Census Bureau)*



67. **Lung Volume** The change in volume V (in milliliters) of the lungs as they expand and contract during a breath can be approximated by the model

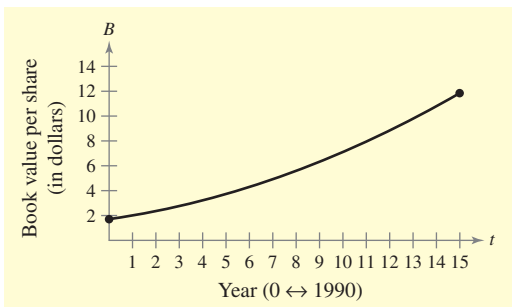
$$V = (-6.549s^2 + 26.20s - 3.8)^2, \quad 0 \le s \le 4$$

where s represents the number of seconds. Graph the volume function with a graphing utility and use the *trace* feature to estimate the number of seconds in which the volume is increasing and in which the volume is decreasing. Find the maximum change in volume between 0 and 4 seconds.

68. **Book Value** For the years 1990 to 2005, the book value B (in dollars) of a share of Wells Fargo stock can be approximated by the model

$$B = 0.0272t^2 + 0.268t + 1.71, \quad 0 \le t \le 15$$

where t represents the year, with $t = 0$ corresponding to 1990 (see figure). *(Source: Wells Fargo)*



(a) Estimate the maximum book value per share from 1990 to 2005.

(b) Estimate the minimum book value per share from 1990 to 2005.

(c) Verify your estimates from parts (a) and (b) with a graphing utility.

69. **Reasoning** When finding a maximum or minimum value in Exercises 6568, why should you also check the endpoints of the function?

70. **Reasoning** Assume that the book value B in Exercise 68 continues to follow the model through 2007. In which year is B at a maximum?

71. Maximum Profit The marketing department of a company estimates that the demand for a product is given by $p = 100 - 0.0001x$, where p is the price per unit and x is the number of units. The cost C of producing x units is given by $C = 350{,}000 + 30x$, and the profit P for producing and selling x units is given by

$$P = R - C = xp - C.$$

Sketch the graph of the profit function and estimate the number of units that would produce a maximum profit. Verify your estimate using a graphing utility.

72. Maximum Profit The marketing department of a company estimates that the demand for a product is given by $p = 125 - 0.0002x$, where p is the price per unit and x is the number of units. The cost C of producing x units is given by $C = 225{,}000 + 80x$, and the profit P for producing and selling x units is given by

$$P = R - C = xp - C.$$

Sketch the graph of the profit function and estimate the number of units that would produce a maximum profit. Verify your estimate using a graphing utility.

73. Cost of Overnight Delivery The cost of sending an overnight package from New York to Atlanta is \$9.80 for up to, but not including, the first pound and \$2.50 for each additional pound (or portion of a pound). A model for the total cost C of sending the package is $C = 9.8 + 2.5[\![x]\!]$, $x > 0$, where x is the weight of the package (in pounds). Sketch the graph of this function.

74. Cost of Overnight Delivery The cost of sending an overnight package from Los Angeles to Miami is \$10.75 for up to, but not including, the first pound and \$3.95 for each additional pound (or portion of a pound). A model for the total cost C of sending the package is $C = 10.75 + 3.95[\![x]\!]$, $x > 0$, where x is the weight of the package (in pounds). Sketch the graph of this function.

75. Strategic Reserve The total volume V (in millions of barrels) of the Strategic Oil Reserve R in the United States from 1995 to 2005 can be approximated by the model

$$V = \begin{cases} -2.722t^3 + 61.18t^2 - 451.5t + 1660, & 5 \le t \le 10 \\ 34.7t + 179, & 11 \le t \le 15 \end{cases}$$

where t represents the year, with $t = 5$ corresponding to 1995. Sketch the graph of this function. *(Source: U.S. Energy Information Administration)*

76. Grade Level Salaries The 2007 salary S (in dollars) for federal employees at the Step 1 level can be approximated by the model

$$S = \begin{cases} 2904.3x + 12{,}155, & x = 1, 2, \ldots, 10 \\ 11{,}499.2x - 81{,}008, & x = 11, \ldots, 15 \end{cases}$$

where x represents the "GS" grade. Sketch a *bar graph* that represents this function. *(Source: U.S. Office of Personnel Management)*

77. Air Travel The total numbers (in thousands) of U.S. airline delays, cancellations, and diversions for the years 1995 to 2005 are given by the following ordered pairs. *(Source: U.S. Bureau of Transportation Statistics)*

(1995, 5327.4) (1996, 5352.0) (1997, 5411.8)
(1998, 5384.7) (1999, 5527.9) (2000, 5683.0)
(2001, 5967.8) (2002, 5271.4) (2003, 6488.5)
(2004, 7129.3) (2005, 7140.6)

(a) Use the *regression* feature of a graphing utility to find a quadratic model for the data from 1995 to 2001. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data from 2002 to 2005. Let t represent the year, with $t = 12$ corresponding to 2002.

(c) Use your results from parts (a) and (b) to construct a piecewise model for all of the data.

78. Revenues The revenues of Symantec Corporation (in millions of dollars) from 1996 to 2005 are given by the following ordered pairs. *(Source: Symantec Corporation)*

(1996, 472.2) (1997, 578.4) (1998, 633.8)
(1999, 745.7) (2000, 853.6) (2001, 1071.4)
(2002, 1406.9) (2003, 1870.1) (2004, 2582.8)
(2005, 4143.4)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data from 1996 to 2000. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data from 2001 to 2005. Let t represent the year, with $t = 11$ corresponding to 2001.

(c) Use your results from parts (a) and (b) to construct a piecewise model for all of the data.

79. If f is an even function, determine whether g is even, odd, or neither. Explain.

(a) $g(x) = -f(x)$

(b) $g(x) = f(-x)$

(c) $g(x) = f(x) - 2$

(d) $g(x) = f(x - 2)$

Think About It In Exercises 80–83, find the coordinates of a second point on the graph of a function f if the given point is on the graph and the function is (a) even and (b) odd.

80. $\left(-\frac{3}{2}, 4\right)$

81. $\left(-\frac{5}{3}, -7\right)$

82. $(4, 9)$

83. $(5, -1)$

Section 2.6
Transformations of Functions

- Use vertical and horizontal shifts to sketch graphs of functions.
- Use reflections to sketch graphs of functions.
- Use nonrigid transformations to sketch graphs of functions.

Vertical and Horizontal Shifts

Many functions have graphs that are simple transformations of the common graphs that are summarized on page 214. For example, you can obtain the graph of $h(x) = x^2 + 2$ by shifting the graph of $f(x) = x^2$ *upward* two units, as shown in Figure 2.55. In function notation, h and f are related as follows.

$$h(x) = x^2 + 2 = f(x) + 2 \qquad \text{Upward shift of two units}$$

Similarly, you can obtain the graph of $g(x) = (x - 2)^2$ by shifting the graph of $f(x) = x^2$ to the *right* two units, as shown in Figure 2.56. In this case, the functions g and f have the following relationship.

$$g(x) = (x - 2)^2 = f(x - 2) \qquad \text{Right shift of two units}$$

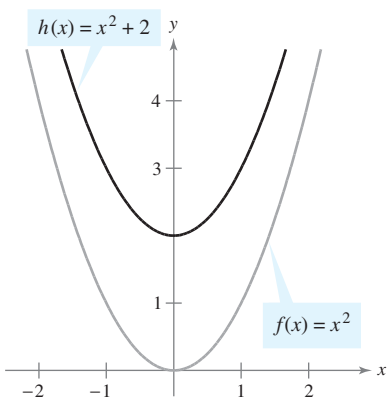


FIGURE 2.55 Vertical Shift Upward

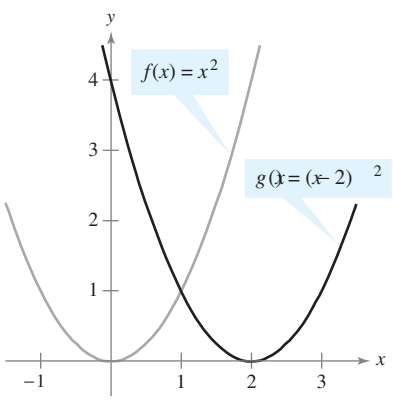


FIGURE 2.56 Horizontal Shift to the Right

The following list summarizes this discussion about horizontal and vertical shifts.

STUDY TIP

In items 3 and 4, be sure you see that $h(x) = f(x - c)$ corresponds to a *right* shift and $h(x) = f(x + c)$ corresponds to a *left* shift.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical** and **horizontal shifts** of the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units **upward:** $h(x) = f(x) + c$
2. Vertical shift c units **downward:** $h(x) = f(x) - c$
3. Horizontal shift c units to the **right:** $h(x) = f(x - c)$
4. Horizontal shift c units to the **left:** $h(x) = f(x + c)$

Some graphs can be obtained from a combination of vertical and horizontal shifts, as demonstrated in Example 1(b). Vertical and horizontal shifts generate a *family of functions,* each with the same shape but at different locations in the plane.

Example 1 Shifts in the Graph of a Function

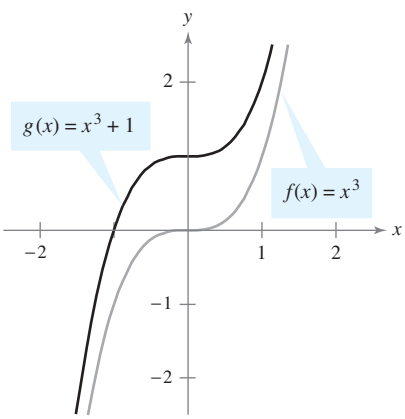
Use the graph of $f(x) = x^3$ to sketch the graph of each function.

a. $g(x) = x^3 + 1$ **b.** $h(x) = (x + 2)^3 + 1$

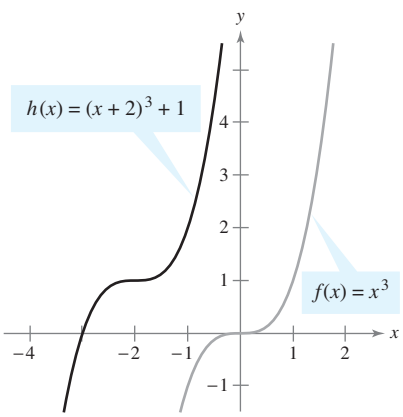
SOLUTION

a. Relative to the graph of $f(x) = x^3$, the graph of $g(x) = x^3 + 1$ is an upward shift of one unit, as shown in Figure 2.57(a).

b. Relative to the graph of $f(x) = x^3$, the graph of $h(x) = (x + 2)^3 + 1$ involves a left shift of two units *and* an upward shift of one unit, as shown in Figure 2.57(b).



(a) Vertical shift: one unit upward



(b) Horizontal shift: two units left; Vertical shift: one unit upward

FIGURE 2.57

Note that the functions f, g, and h belong to the family of cubic functions.

DISCOVERY

The point $(2, 4)$ is on the graph of $f(x) = x^2$. Predict the location of this point if the following transformations are performed.

a. $f(x - 4)$

b. $f(x) + 1$

c. $f(x + 1) - 2$

Use a graphing utility to verify your predictions. Can you find a general description that represents an ordered pair that has been shifted horizontally? vertically?

✓CHECKPOINT 1

Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of $g(x) = \sqrt{x - 1} - 1$. ■

TECHNOLOGY

Graphing utilities are ideal tools for exploring transformations of functions. Try to predict how the graphs of g and h relate to the graph of f. Graph f, g, and h in the same viewing window to confirm your prediction.

a. $f(x) = x^2$, $g(x) = (x - 4)^2$, $h(x) = (x - 4)^2 + 3$

b. $f(x) = x^2$, $g(x) = (x + 1)^2$, $h(x) = (x + 1)^2 - 2$

c. $f(x) = x^2$, $g(x) = (x + 4)^2$, $h(x) = (x + 4)^2 + 2$

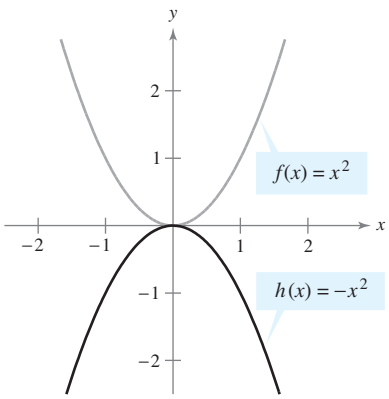


FIGURE 2.58 Reflection

Reflections

The second common type of transformation is a **reflection.** For instance, if you consider the x-axis to be a mirror, the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$, as shown in Figure 2.58.

Reflections in the Coordinate Axes

Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

1. **Reflection in the x-axis:** $g(x) = -f(x)$
2. **Reflection in the y-axis:** $h(x) = f(-x)$

Example 2 Reflections of the Graph of a Function

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = -\sqrt{x}$ **b.** $h(x) = \sqrt{-x}$

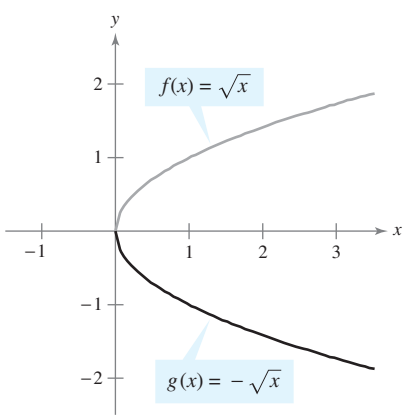
SOLUTION

a. The graph of g is a reflection of the graph of f in the x-axis because

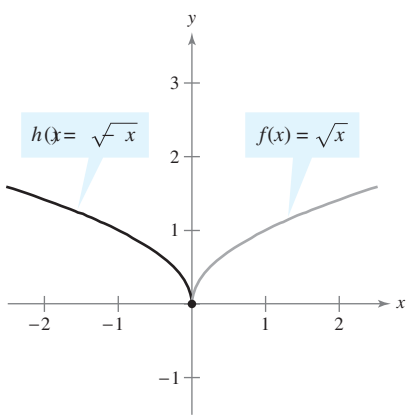
$$g(x) = -\sqrt{x} = -f(x).$$ See Figure 2.59(a).

b. The graph of h is a reflection of the graph of f in the y-axis because

$$h(x) = \sqrt{-x} = f(-x).$$ See Figure 2.59(b).



(a) Reflection in x-axis



(b) Reflection in y-axis

FIGURE 2.59

When sketching the graph of a function involving square roots, remember that the domain must be restricted to exclude numbers that make the radicand negative. For instance, here are the domains of the functions in Example 2.

Domain of $g(x) = -\sqrt{x}$: $x \geq 0$

Domain of $h(x) = \sqrt{-x}$: $x \leq 0$

TECHNOLOGY

You will find programs for several models of graphing utilities that will give you practice working with reflections, horizontal shifts, and vertical shifts at the website for this text at *college.hmco.com/info/larsonapplied.* These programs will graph the function

$$y = R(x + H)^2 + V$$

where $R = \pm 1$, H is an integer between -6 and 6, and V is an integer between -3 and 3. Each time you run the program, different values of R, H, and V are possible. From the graph, you should be able to determine the values of R, H, and V.

✓CHECKPOINT 2

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $g(x) = -|x|$

b. $h(x) = |-x|$ ■

Example 3 Reflections and Shifts

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

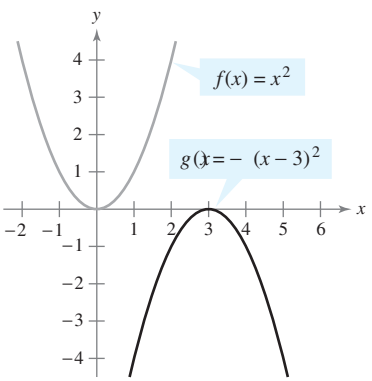
a. $g(x) = -(x - 3)^2$ **b.** $h(x) = -x^2 + 2$

SOLUTION

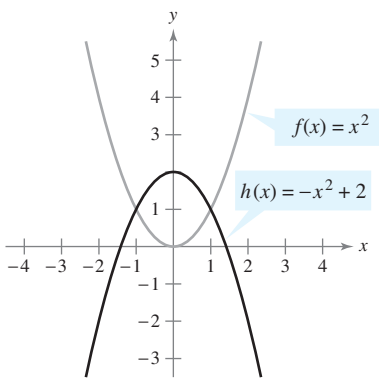
a. To sketch the graph of $g(x) = -(x - 3)^2$, first shift the graph of $f(x) = x^2$ to the right three units. Then reflect the result in the x-axis.

b. To sketch the graph of $h(x) = -x^2 + 2$, first reflect the graph of $f(x) = x^2$ in the x-axis. Then shift the result upward two units.

The graphs of both functions are shown in Figure 2.60.



(a) Shift and then reflect in x-axis



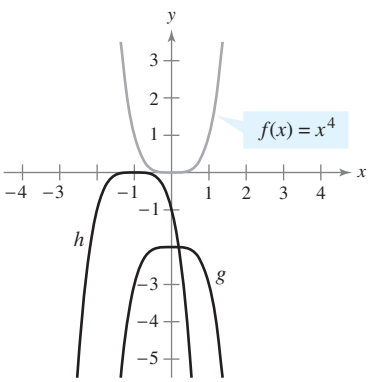
(b) Reflect in x-axis and then shift

FIGURE 2.60

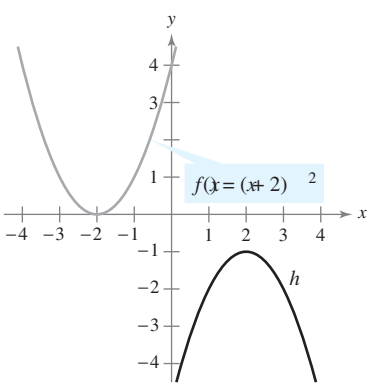
✓CHECKPOINT 3

Use the graph of $f(x) = x^3$ to sketch the graph of each function.

a. $g(x) = -(x + 2)^3$ **b.** $h(x) = -x^3 - 3$ ■



(a)



(b)

FIGURE 2.61

✓CHECKPOINT 4

The graph labeled h in Figure 2.61(b) is a transformation of the graph of $f(x) = (x + 2)^2$. Find an equation for the function h. ■

Example 4 Finding Equations from Graphs

The graphs labeled g and h in Figure 2.61(a) are transformations of the graph of $f(x) = x^4$. Find an equation for each function.

SOLUTION The graph of g is a reflection in the x-axis *followed* by a downward shift of two units of the graph of $f(x) = x^4$. So, the equation for g is $g(x) = -x^4 - 2$. The graph of h is a horizontal shift of one unit to the left *followed* by a reflection in the x-axis of the graph of $f(x) = x^4$. So, the equation for h is $h(x) = -(x + 1)^4$.

Can you think of another way to find an equation for g in Example 4? If you were to shift the graph of f upward two units and then reflect the graph in the x-axis, you would obtain the equation $g(x) = -(x^4 + 2)$. The Distributive Property yields $g(x) = -x^4 - 2$, which is the same equation obtained in Example 4.

DISCOVERY

Use a graphing utility to graph $f(x) = 2x^2$. Compare this graph with the graph of $h(x) = x^2$. Describe the effect of multiplying x^2 by a number greater than 1. Then graph $g(x) = \frac{1}{2}x^2$. Compare this with the graph of $h(x) = x^2$. Describe the effect of multiplying x^2 by a number greater than 0 but less than 1. Can you think of an easy way to remember this generalization? Use the *table* feature of a graphing utility to compare the values of $f(x)$, $g(x)$, and $h(x)$. What do you notice? How does this relate to the vertical stretch or vertical shrink of the graph of a function?

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are **rigid** transformations because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the xy-plane. A **nonrigid** transformation is one that causes a *distortion* change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $g(x) = cf(x)$, where the transformation is a **vertical stretch** if $|c| > 1$ and a **vertical shrink** if $0 < |c| < 1$.

Example 5 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$ **b.** $g(x) = \frac{1}{3}|x|$

SOLUTION

a. Relative to the graph of $f(x) = |x|$, the graph of

$$h(x) = 3|x| = 3f(x)$$

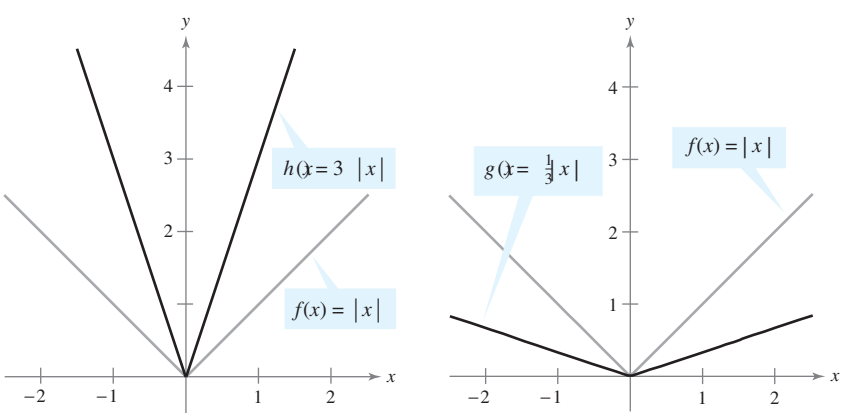
is a vertical stretch (each y-value is multiplied by 3) of the graph of f.

b. Similarly, the graph of

$$g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$$

is a vertical shrink $\left(\text{each } y\text{-value is multiplied by } \frac{1}{3}\right)$ of the graph of f.

The graphs of both functions are shown in Figure 2.62.



(a) (b)

FIGURE 2.62

✓CHECKPOINT 5

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = 4\sqrt{x}$ **b.** $h(x) = \frac{1}{4}\sqrt{x}$

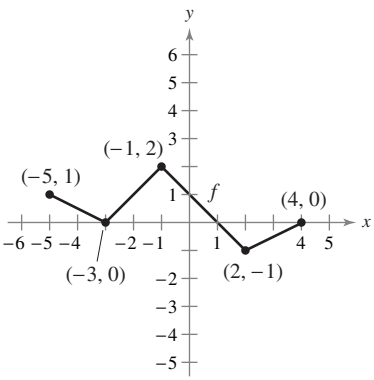


FIGURE 2.63

Example 6 Rigid and Nonrigid Transformations

Use the graph of f shown in Figure 2.63 to sketch each graph.

a. $g(x) = f(x - 2) + 1$

b. $h(x) = \frac{1}{2}f(x)$

SOLUTION

a. The graph of g is a horizontal shift to the right two units and a vertical shift upward one unit of the graph of f. The graph of g is shown in Figure 2.64(a).

b. The graph of h is a vertical shrink of the graph of f. The graph of h is shown in Figure 2.64(b).

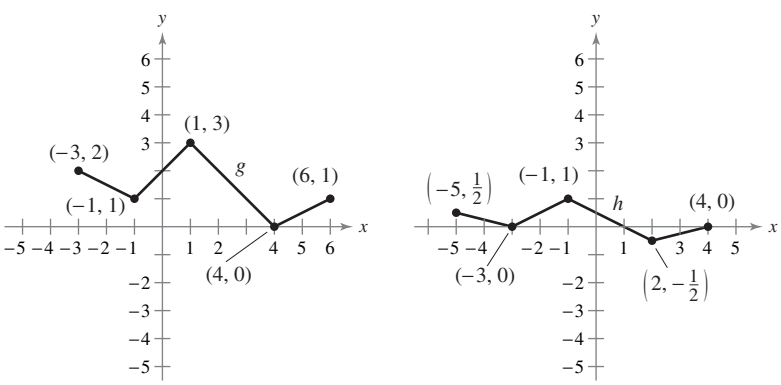
For $x = -5$, $h(-5) = \frac{1}{2}f(-5) = \frac{1}{2}(1) = \frac{1}{2}$.

For $x = -3$, $h(-3) = \frac{1}{2}f(-3) = \frac{1}{2}(0) = 0$.

For $x = -1$, $h(-1) = \frac{1}{2}f(-1) = \frac{1}{2}(2) = 1$.

For $x = 2$, $h(2) = \frac{1}{2}f(2) = \frac{1}{2}(-1) = -\frac{1}{2}$.

For $x = 4$, $h(4) = \frac{1}{2}f(4) = \frac{1}{2}(0) = 0$.



(a) (b)

FIGURE 2.64

✓CHECKPOINT 6

Use the graph of g shown in Figure 2.64(a) to sketch the graph of $p(x) = 2g(x) - 1$. ■

CONCEPT CHECK

In Exercises 1–4, determine whether the statement is true or false. Explain your reasoning.

1. A rigid transformation preserves the basic shape of a graph.
2. The graph of $g(x) = x^2 + 5$ is a vertical shift downward five units of the graph of $f(x) = x^2$.
3. The graph of $g(x) = (x - 1)^2$ is a horizontal shift to the left one unit of the graph of $f(x) = x^2$.
4. The graph of $g(x) = 2x^2$ is an example of a nonrigid transformation of the graph of $f(x) = x^2$.

Skills Review 2.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.4, 1.5, 2.4, and 2.5.

In Exercises 1 and 2, evaluate the function at the indicated value.

1. Find $f(3)$ for $f(x) = x^2 - 4x + 15$.

2. Find $f(-x)$ for $f(x) = 2x/(x - 3)$.

In Exercises 3 and 4, solve the equation.

3. $-x^3 + 10x = 0$

4. $3x^2 + 2x - 8 = 0$

In Exercises 5–10, sketch the graph of the function.

5. $f(x) = -2$

6. $f(x) = -x$

7. $f(x) = x + 5$

8. $f(x) = 2 - x$

9. $f(x) = 3x - 4$

10. $f(x) = 9x + 10$

Exercises 2.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, describe the sequence of transformations from $f(x) = x^2$ to g. Then sketch the graph of g by hand. Verify with a graphing utility.

1. $g(x) = x^2 - 4$

2. $g(x) = x^2 + 1$

3. $g(x) = (x + 2)^2$

4. $g(x) = (x - 3)^2$

5. $g(x) = (x - 2)^2 + 2$

6. $g(x) = (x + 1)^2 - 3$

7. $g(x) = -x^2 + 1$

8. $g(x) = -(x - 2)^2$

In Exercises 9–16, describe the sequence of transformations from $f(x) = |x|$ to g. Then sketch the graph of g by hand. Verify with a graphing utility.

9. $g(x) = |x| + 2$

10. $g(x) = |x| - 3$

11. $g(x) = |x - 1|$

12. $g(x) = |x + 4|$

13. $g(x) = -|x| + 3$

14. $g(x) = 5 - |x - 1|$

15. $g(x) = |x + 1| - 3$

16. $g(x) = |x - 2| + 2$

In Exercises 17–24, describe the sequence of transformations from $f(x) = \sqrt{x}$ to g. Then sketch the graph of g by hand. Verify with a graphing utility.

17. $g(x) = \sqrt{x - 3}$

18. $g(x) = \sqrt{x + 4}$

19. $g(x) = \sqrt{x - 3} + 1$

20. $g(x) = \sqrt{x + 5} - 2$

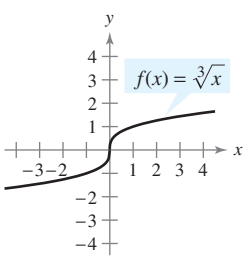
21. $g(x) = \sqrt{2x}$

22. $g(x) = \sqrt{2x} - 5$

23. $g(x) = 2 - \sqrt{x - 4}$

24. $g(x) = \sqrt{-x} + 1$

In Exercises 25–34, describe the sequence of transformations from $f(x) = \sqrt[3]{x}$ to y. Then sketch the graph of y by hand. Verify with a graphing utility.



25. $y = \sqrt[3]{-x}$

26. $y = -\sqrt[3]{x}$

27. $y = \sqrt[3]{x} - 1$

28. $y = \sqrt[3]{x + 1}$

29. $y = 2 - \sqrt[3]{x + 1}$

30. $y = -\sqrt[3]{x - 1} - 4$

31. $y = \sqrt[3]{x + 1} - 1$

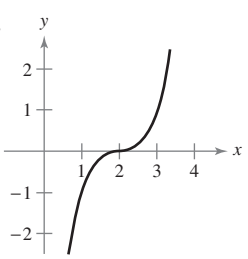
32. $y = 2\sqrt[3]{x - 2} + 1$

33. $y = \frac{1}{2}\sqrt[3]{x}$

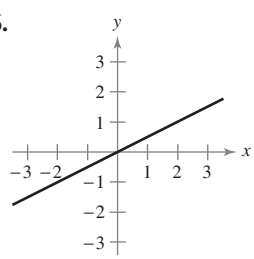
34. $y = \frac{1}{2}\sqrt[3]{x} - 3$

In Exercises 35–40, identify the transformation shown in the graph and the associated common function. Write the equation of the graphed function.

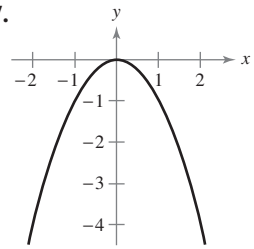
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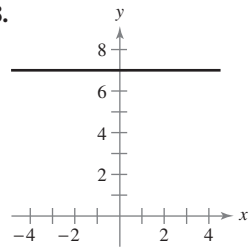
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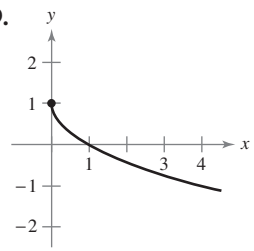
37.



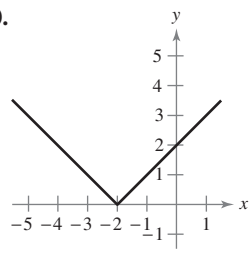
38.



39.



40.



41. Use a graphing utility to graph f for $c = -2, 0$, and 2 in the same viewing window.

(a) $f(x) = \frac{1}{2}x + c$

(b) $f(x) = \frac{1}{2}(x - c)$

(c) $f(x) = \frac{1}{2}(cx)$

In each case, compare the graph with the graph of $y = \frac{1}{2}x$.

42. Use a graphing utility to graph f for $c = -2, 0$, and 2 in the same viewing window.

(a) $f(x) = x^3 + c$

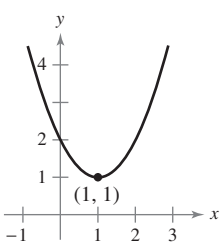
(b) $f(x) = (x - c)^3$

(c) $f(x) = (x - 2)^3 + c$

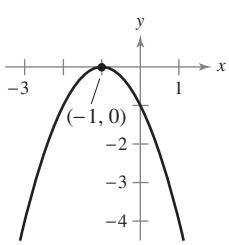
In each case, compare the graph with the graph of $y = x^3$.

43. Use the graph of $f(x) = x^2$ to write equations for the functions whose graphs are shown.

(a)

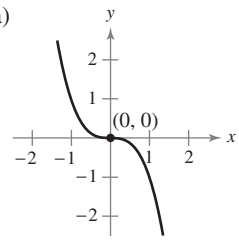


(b)

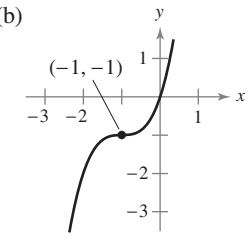


44. Use the graph of $f(x) = x^3$ to write equations for the functions whose graphs are shown.

(a)



(b)



45. Use the graph of f (see figure) to sketch each graph.

(a) $y = f(x) + 2$ (b) $y = -f(x)$

(c) $y = f(x - 2)$ (d) $y = f(x + 3)$

(e) $y = 2f(x)$ (f) $y = f(-x)$

46. Use the graph of f (see figure) to sketch each graph.

(a) $y = f(x) - 1$ (b) $y = f(x + 1)$

(c) $y = f(x - 1)$ (d) $y = -f(x - 2)$

(e) $y = f(-x)$ (f) $y = \frac{1}{2}f(x)$

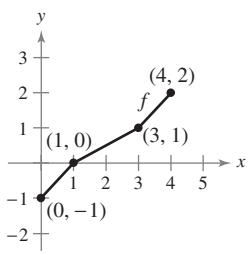


Figure for 45

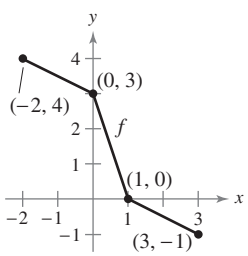


Figure for 46

47. Use the graph of f (see figure) to sketch each graph.

(a) $y = f(-x)$ (b) $y = f(x) + 4$

(c) $y = 2f(x)$ (d) $y = -f(x - 4)$

(e) $y = f(x) - 3$ (f) $y = -f(x) - 1$

48. Use the graph of f (see figure) to sketch each graph.

(a) $y = f(x - 5)$ (b) $y = -f(x) + 3$

(c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x + 1)$

(e) $y = f(-x)$ (f) $y = f(x) - 5$

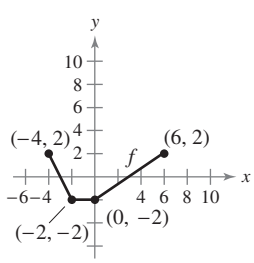


Figure for 47

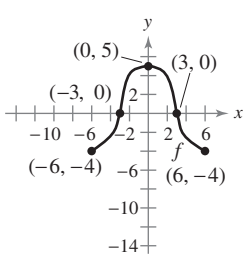


Figure for 48

In Exercises 49–52, consider the graph of $f(x) = x^3$. Use your knowledge of rigid and nonrigid transformations to write an equation for each of the following descriptions. Verify with a graphing utility.

49. The graph of f is shifted two units downward.

50. The graph of f is shifted three units to the left.

51. The graph of f is vertically stretched by a factor of 4.

52. The graph of f is vertically shrunk by a factor of $\frac{1}{3}$.

In Exercises 53–56, consider the graph of $f(x) = |x|$. Use your knowledge of rigid and nonrigid transformations to write an equation for each of the following descriptions. Verify with a graphing utility.

53. The graph of f is shifted three units to the right and two units upward.

54. The graph of f is reflected in the x-axis, shifted two units to the left, and shifted three units upward.

55. The graph of f is vertically stretched by a factor of 4 and reflected in the x-axis.

56. The graph of f is vertically shrunk by a factor of $\frac{1}{3}$ and shifted two units to the left.

In Exercises 57–60, consider the graph of $g(x) = \sqrt{x}$. Use your knowledge of rigid and nonrigid transformations to write an equation for each of the following descriptions. Verify with a graphing utility.

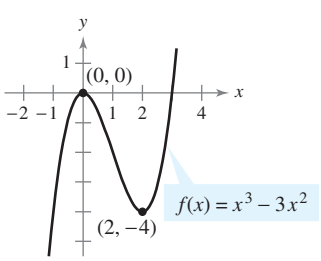
57. The graph of g is shifted four units to the right and three units downward.

58. The graph of g is reflected in the x-axis, shifted two units to the left, and shifted one unit upward.

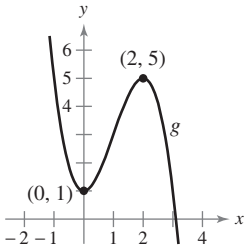
59. The graph of g is vertically shrunk by a factor of $\frac{1}{2}$ and shifted three units to the right.

60. The graph of g is vertically stretched by a factor of 2, reflected in the x-axis, and shifted three units upward.

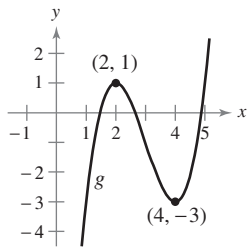
In Exercises 61 and 62, use the graph of $f(x) = x^3 - 3x^2$ to write an equation for the function g shown in the graph.



61.



62.



63. The point $(3, 9)$ on the graph of $f(x) = x^2$ has been shifted to the point $(4, 7)$ after a rigid transformation. Identify the shift and write the new function g in terms of f.

64. The point $(8, 2)$ on the graph of $f(x) = \sqrt[3]{x}$ has been shifted to the point $(5, 0)$ after a rigid transformation. Identify the shift and write the new function h in terms of f.

65. Profit A company's weekly profit P (in hundreds of dollars) from a product is given by the model

$$P(x) = 80 + 20x - 0.5x^2, \quad 0 \le x \le 20$$

where x is the amount (in hundreds of dollars) spent on advertising.

(a) Use a graphing utility to graph the profit function.

(b) The company estimates that taxes and operating costs will increase by an average of \$2500 per week during the next year. Rewrite the profit equation to reflect this expected decrease in profits. Identify the type of transformation applied to the graph of the equation.

(c) Rewrite the profit equation so that x measures advertising expenditures in dollars. [Find $P(x/100)$.] Identify the type of transformation applied to the graph of the profit function.

66. Automobile Aerodynamics The number of horsepower H required to overcome wind drag on an automobile is approximated by

$$H(x) = 0.002x^2 + 0.005x - 0.029, \quad 10 \le x \le 100$$

where x is the speed of the car (in miles per hour).

(a) Use a graphing utility to graph the function.

(b) Rewrite the horsepower function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.] Identify the type of transformation applied to the graph of the horsepower function.

67. Exploration Use a graphing utility to graph the six functions below in the same viewing window. Describe any similarities and differences you observe among the graphs.

(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$

(d) $y = x^4$ (e) $y = x^5$ (f) $y = x^6$

68. Reasoning Use the results of Exercise 67 to make a conjecture about the shapes of the graphs of $y = x^7$ and $y = x^8$. Use a graphing utility to verify your conjecture.

Section 2.7

The Algebra of Functions

- Find the sum, difference, product, and quotient of two functions.
- Form the composition of two functions and determine its domain.
- Identify a function as the composition of two functions.
- Use combinations and compositions of functions to solve application problems.

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two functions can be combined to create new functions. For example, the functions given by $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ can be combined as follows.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4 \quad \text{Sum}$$

$$f(x) - g(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2 \quad \text{Difference}$$

$$f(x)g(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3 \quad \text{Product}$$

$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1, \quad g(x) \neq 0 \quad \text{Quotient}$$

The domain of an arithmetic combination of the functions f and g consists of all real numbers that are common to the domains of f and g.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the **sum, difference, product,** and **quotient** of f and g are defined as follows.

1. *Sum:* $(f + g)(x) = f(x) + g(x)$
2. *Difference:* $(f - g)(x) = f(x) - g(x)$
3. *Product:* $(fg)(x) = f(x) \cdot g(x)$
4. *Quotient:* $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$.

SOLUTION

$$(f + g)(x) = f(x) + g(x) = (2x + 1) + (x^2 + 2x - 1) = x^2 + 4x$$

✓CHECKPOINT 1

Given $f(x) = x^2 - 4$ and $h(x) = x^2 + x + 3$, find $(f + h)(x)$. ■

Example 2 **Finding the Difference of Two Functions**

Given the functions

$$f(x) = 2x + 1 \text{ and } g(x) = x^2 + 2x - 1$$

find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

SOLUTION The difference of the functions f and g is given by

$$\begin{aligned}(f - g)(x) &= f(x) - g(x) && \text{Definition of difference of two functions}\\ &= (2x + 1) - (x^2 + 2x - 1) && \text{Substitute for } f(x) \text{ and } g(x).\\ &= -x^2 + 2. && \text{Simplify.}\end{aligned}$$

When $x = 2$, the value of this difference is

$$(f - g)(2) = -(2)^2 + 2 = -2.$$

STUDY TIP

Note that in Example 2, $(f - g)(2)$ can also be evaluated as follows.

$$\begin{aligned}(f - g)(2) &= f(2) - g(2)\\ &= [2(2) + 1]\\ &\quad - [2^2 + 2(2) - 1]\\ &= 5 - 7\\ &= -2\end{aligned}$$

✓CHECKPOINT 2

Given $f(x) = x^2 - 4$ and $h(x) = x^2 + x + 3$, find $(f - h)(x)$. Then evaluate the difference when $x = 3$. ■

In Examples 1 and 2, both f and g have domains that consist of all real numbers. So, the domains of $(f + g)$ and $(f - g)$ are also the set of all real numbers. Remember that any restrictions on the domains of f and g must be considered when forming the sum, difference, product, or quotient of f and g.

Example 3 **The Quotient of Two Functions**

Find the domains of $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{g}{f}\right)(x)$ for the functions

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{4 - x^2}.$$

SOLUTION The quotient of f and g is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}}$$

and the quotient of g and f is given by

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these two domains is $[0, 2]$, which implies that the domains of f/g and g/f are as follows. Notice that the domains differ slightly.

$$\text{Domain of } \frac{f}{g}\text{: } [0, 2) \qquad \text{Domain of } \frac{g}{f}\text{: } (0, 2]$$

✓CHECKPOINT 3

Find the domains of $\left(\frac{f}{h}\right)(x)$ and $\left(\frac{h}{f}\right)(x)$ for the functions $f(x) = x - 1$ and $h(x) = x - 3$. ■

Composition of Functions

Another way to combine two functions is to form the **composition** of one with the other. For instance, if $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is given by

$$f(g(x)) = f(x + 1) = (x + 1)^2.$$

This composition is denoted as $f \circ g$ and is read as "f composed with g."

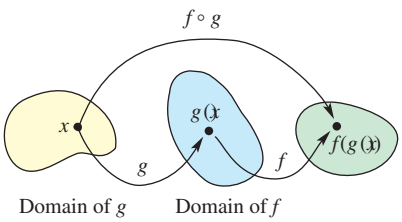


FIGURE 2.65

Definition of the Composition of Two Functions

The **composition** of the functions f and g is given by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 2.65.)

From the definition above, it follows that the domain of $f \circ g$ is always a subset of the domain of g, and the range of $f \circ g$ is always a subset of the range of f.

Example 4 Composition of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, find the following.

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

SOLUTION

a. The composition of f with g is as follows.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(4 - x^2) && \text{Definition of } g(x)\\ &= (4 - x^2) + 2 && \text{Definition of } f(x)\\ &= -x^2 + 6 && \text{Simplify.}\end{aligned}$$

b. The composition of g with f is as follows.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(x + 2) && \text{Definition of } f(x)\\ &= 4 - (x + 2)^2 && \text{Definition of } g(x)\\ &= 4 - (x^2 + 4x + 4) && \text{Expand.}\\ &= -x^2 - 4x && \text{Simplify.}\end{aligned}$$

Note that, in this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

✓CHECKPOINT 4

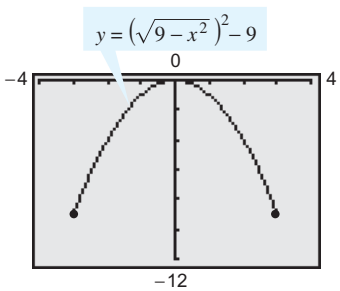
Given $f(x) = x^2 - 2$ and $g(x) = x + 1$, find $(f \circ g)(x)$. ■

TECHNOLOGY

In Example 5, the domain of the composite function is $[-3, 3]$. To convince yourself of this, use a graphing utility to graph

$$y = \left(\sqrt{9 - x^2}\right)^2 - 9$$

as shown in the figure below. Notice that the graphing utility does not extend the graph to the left of $x = -3$ or to the right of $x = 3$.



Example 5 Finding the Domain of a Composite Function

Find the composition $(f \circ g)(x)$ for the functions given by

$$f(x) = x^2 - 9 \quad \text{and} \quad g(x) = \sqrt{9 - x^2}.$$

Then find the domain of $f \circ g$.

SOLUTION The composition of the functions is as follows.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt{9 - x^2}\right) \\ &= \left(\sqrt{9 - x^2}\right)^2 - 9 \\ &= 9 - x^2 - 9 \\ &= -x^2\end{aligned}$$

From this result, it might appear that the domain of the composition is the set of all real numbers. However, because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $f \circ g$ is $[-3, 3]$.

✓CHECKPOINT 5

Find the composition $(f \circ g)(x)$ for the functions given by $f(x) = \sqrt{x}$ and $g(x) = 3 - x^2$. Then find the domain of $f \circ g$. ■

In Examples 4 and 5, you formed the composition of two functions. To "decompose" a composite function, look for an "inner" function and an "outer" function. For instance, the function h given by

$$h(x) = (3x - 5)^3$$

is the composition of f with g, where $f(x) = x^3$ and $g(x) = 3x - 5$. That is,

$$h(x) = (3x - 5)^3 = [g(x)]^3 = f(g(x)).$$

In the function h, $g(x) = 3x - 5$ is the *inner* function and $f(x) = x^3$ is the *outer* function.

Example 6 Identifying a Composite Function

Write the function given by $h(x) = \dfrac{1}{(x - 2)^2}$ as a composition of two functions.

SOLUTION One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2} = x^{-2}.$$

Then you can write

$$h(x) = \frac{1}{(x - 2)^2} = (x - 2)^{-2} = f(x - 2) = f(g(x)).$$

✓CHECKPOINT 6

Write the function given by $h(x) = (x - 1)^2 + 2$ as a composition of two functions. ■

Applications



Andy Williams/Getty Images

The Capitol building in Washington, D.C. is where each state's Congressional representatives convene. In recent years, no party has had a strong majority, which can make it difficult to pass legislation.

Example 7 Political Makeup of the U.S. Senate

Consider three functions R, D, and I that represent the numbers of Republicans, Democrats, and Independents, respectively, in the U.S. Senate from 1967 to 2005. Sketch the graphs of R, D, and I and the sum of R, D, and I in the same coordinate plane. The numbers of senators from each political party are shown below.

Year	R	D	I
1967	36	64	0
1969	42	58	0
1971	44	54	2
1973	42	56	2
1975	37	61	2
1977	38	61	1
1979	41	58	1
1981	53	46	1
1983	54	46	0
1985	53	47	0

Year	R	D	I
1987	45	55	0
1989	45	55	0
1991	44	56	0
1993	43	57	0
1995	52	48	0
1997	55	45	0
1999	55	45	0
2001	50	50	0
2003	51	48	1
2005	55	44	1

SOLUTION The graphs of R, D, and I are shown in Figure 2.66. Note that the sum of R, D, and I is the constant function $R + D + I = 100$. This follows from the fact that the number of senators in the United States is 100 (two from each state).

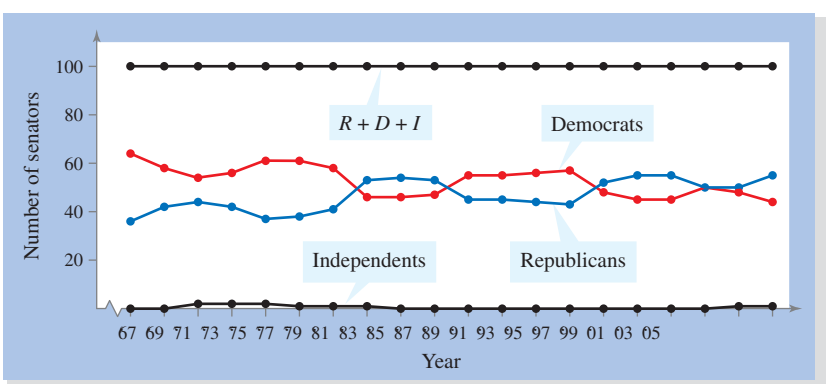


FIGURE 2.66 Numbers of U.S. Senators by Political Party

✓CHECKPOINT 7

In Example 7, consider the function f given by $f = 100 - (R + D)$. What does f represent in the context of the real-life situation? ■

Example 8 Bacteria Count

The number of bacteria in a certain food is given by

$$N(T) = 20T^2 - 80T + 500, \quad 2 \le T \le 14$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 4t + 2, \quad 0 \le t \le 3$$

where t is the time in hours. Find (a) the composition $N(T(t))$, (b) the number of bacteria in the food when $t = 2$ hours, and (c) how long the food can remain unrefrigerated before the bacteria count reaches 2000.

SOLUTION

a. $N(T(t)) = 20(4t + 2)^2 - 80(4t + 2) + 500$

$= 20(16t^2 + 16t + 4) - 320t - 160 + 500$

$= 320t^2 + 320t + 80 - 320t - 160 + 500$

$= 320t^2 + 420$

b. When $t = 2$, the number of bacteria is

$$N(T(2)) = 320(2)^2 + 420 = 1280 + 420 = 1700.$$

c. The bacteria count will reach $N = 2000$ when $320t^2 + 420 = 2000$. By solving this equation, you can determine that the bacteria count will reach 2000 when $t \approx 2.2$ hours. So, the food can remain unrefrigerated for about 2 hours and 12 minutes.

✓CHECKPOINT 8

In Example 8, how long can the food remain unrefrigerated before the bacteria count reaches 1000? ■

CONCEPT CHECK

1. Given $g(x) = x^2 + 3x$ and $f(x) = 2x + 3$, describe and correct the error in finding $(g - f)(x)$.

$$(g - f)(x) = x^2 + 3x - 2x + 3$$
$$= x^2 + x + 3$$

2. Given $f(x) = x^2$ and $g(x) = 2x - 1$, describe and correct the error in finding $(f \circ g)(x)$.

$$(f \circ g)(x) = f(g(x)) = 2(x^2) - 1 = 2x^2 - 1$$

3. Explain why the domain of the composition $f \circ g$ is a subset of the domain of g.

4. Are the domains of the functions given by $h(x) = \sqrt{x - 3}$ and $g(x) = \dfrac{1}{\sqrt{x - 3}}$ the same? Explain.

Skills Review 2.7

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 0.7.

In Exercises 1–10, perform the indicated operations and simplify the result.

1. $\frac{1}{x} + \frac{1}{1-x}$

2. $\frac{2}{x+3} - \frac{2}{x-3}$

3. $\frac{3}{x-2} - \frac{2}{x(x-2)}$

4. $\frac{x}{x-5} + \frac{1}{3}$

5. $(x-1)\left(\frac{1}{\sqrt{x^2-1}}\right)$

6. $\left(\frac{x}{x^2-4}\right)\left(\frac{x^2-x-2}{x^2}\right)$

7. $(x^2-4) \div \left(\frac{x+2}{5}\right)$

8. $\left(\frac{x}{x^2+3x-10}\right) \div \left(\frac{x^2+3x}{x^2+6x+5}\right)$

9. $\frac{(1/x)+5}{3-(1/x)}$

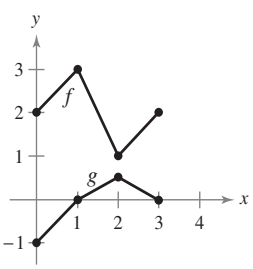
10. $\frac{(x/4)-(4/x)}{x-4}$

Exercises 2.7

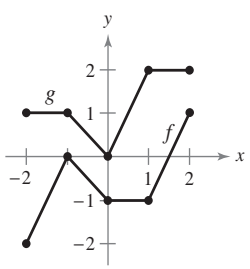
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, use the graphs of f and g to graph $h(x) = (f + g)(x)$.

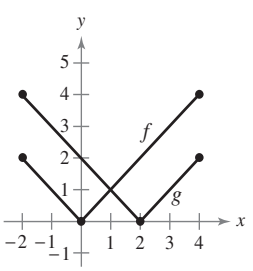
1.



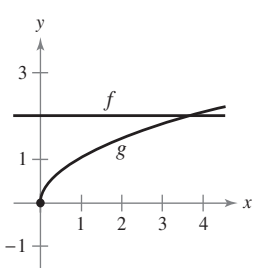
2.



3.



4.



In Exercises 5–12, find (a) $(f+g)(x)$, (b) $(f-g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

5. $f(x) = x + 1, \quad g(x) = x - 1$

6. $f(x) = 2x - 3, \quad g(x) = 1 - x$

7. $f(x) = x^2, \quad g(x) = 1 - x$

8. $f(x) = 2x + 3, \quad g(x) = x^2 - 1$

9. $f(x) = x^2 + 5, \quad g(x) = \sqrt{1-x}$

10. $f(x) = \sqrt{x^2-4}, \quad g(x) = \frac{x^2}{x^2+1}$

11. $f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x^2}$

12. $f(x) = \frac{x}{x+1}, \quad g(x) = x^3$

In Exercises 13–24, evaluate the function for $f(x) = 2x + 1$ and $g(x) = x^2 - 2$.

13. $(f+g)(3)$

14. $(f-g)(-2)$

15. $(f-g)(2t)$

16. $(f+g)(t-1)$

17. $(fg)(-2)$

18. $(fg)(-6)$

19. $\left(\frac{f}{g}\right)(5)$

20. $\left(\frac{f}{g}\right)(0)$

21. $(f-g)(0)$

22. $(f+g)(1)$

23. $\left(\frac{f}{g}\right)(-1) - g(3)$

24. $(2f)(5) + (3g)(-4)$

In Exercises 25–28, find (a) $f \circ g$, (b) $g \circ f$, and (c) $f \circ f$.

25. $f(x) = 3x, \quad g(x) = 2x + 5$

26. $f(x) = 2x - 1, \quad g(x) = 7 - x$

27. $f(x) = x^2, \quad g(x) = 3x + 1$

28. $f(x) = x^3, \quad g(x) = \frac{1}{x}$

In Exercises 29–36, find (a) $f \circ g$ and (b) $g \circ f$.

29. $f(x) = \sqrt{x+4}, \quad g(x) = x^2$

30. $f(x) = \sqrt[3]{x-1}, \quad g(x) = x^3 + 1$

31. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 1$

32. $f(x) = \frac{1}{2}x + 1, \quad g(x) = 2x + 3$

33. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x}$

34. $f(x) = 2x - 3, \quad g(x) = 2x - 3$

35. $f(x) = |x|, \quad g(x) = x + 6$

36. $f(x) = x^{2/3}, \quad g(x) = x^6$

In Exercises 37–40, determine the domain of (a) f, (b) g, and (c) $f \circ g$.

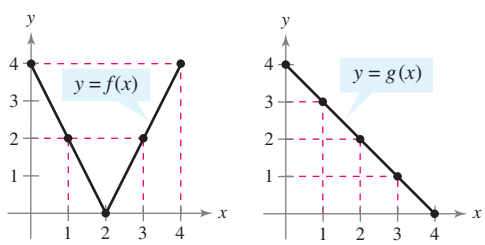
37. $f(x) = x^2 + 3, \quad g(x) = \sqrt{x}$

38. $f(x) = \sqrt[3]{x+1}, \quad g(x) = x^3$

39. $f(x) = \dfrac{1}{x^2}, \quad g(x) = x - 2$

40. $f(x) = \dfrac{5}{x^2 - 4}, \quad g(x) = x + 3$

In Exercises 41–44, use the graphs of f and g to evaluate the functions.



41. (a) $(f+g)(3)$ (b) $\left(\dfrac{f}{g}\right)(2)$

42. (a) $(f-g)(1)$ (b) $(fg)(4)$

43. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

44. (a) $(f \circ g)(0)$ (b) $(g \circ f)(3)$

In Exercises 45–52, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

45. $h(x) = (2x+1)^2$ **46.** $h(x) = (1-x)^3$

47. $h(x) = \sqrt[3]{x^2 - 4}$ **48.** $h(x) = \sqrt{9 - x}$

49. $h(x) = \dfrac{1}{x+2}$

50. $h(x) = \dfrac{4}{(5x+2)^2}$

51. $h(x) = (x+4)^2 + 2(x+4)$

52. $h(x) = (x+3)^{3/2}$

53. Stopping Distance While driving at x miles per hour, you are required to stop quickly to avoid an accident. The distance the car travels (in feet) during your reaction time is given by $R(x) = \frac{3}{4}x$. The distance the car travels (in feet) while you are braking is given by

$$B(x) = \frac{1}{15}x^2.$$

Find the function that represents the total stopping distance T. (*Hint:* $T = R + B$.) Graph the functions R, B, and T on the same set of coordinate axes for $0 \le x \le 60$.

54. Cost The weekly cost C of producing x units in a manufacturing process is given by the function

$$C(x) = 70x + 800.$$

The number of units x produced in t hours is given by

$$x(t) = 40t.$$

Find and interpret $(C \circ x)(t)$.

55. Cost The weekly cost C of producing x units in a manufacturing process is given by the function

$$C(x) = 50x + 495.$$

The number of units x produced in t hours is given by

$$x(t) = 30t.$$

Find and interpret $(C \circ x)(t)$.

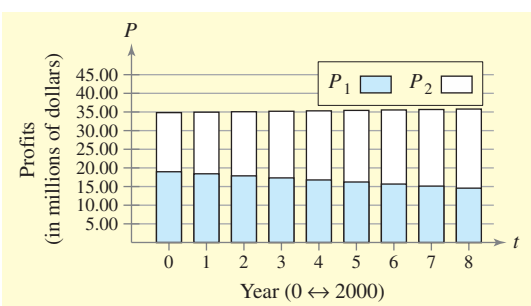
56. Comparing Profits A company has two manufacturing plants, one in New Jersey and the other in California. From 2000 to 2008, the profits for the manufacturing plant in New Jersey were decreasing according to the function

$$P_1 = 18.97 - 0.55t, \quad t = 0, 1, 2, 3, 4, 5, 6, 7, 8$$

where P_1 represents the profits (in millions of dollars) and t represents the year, with $t = 0$ corresponding to 2000. On the other hand, the profits for the manufacturing plant in California were increasing according to the function

$$P_2 = 15.85 + 0.67t, \quad t = 0, 1, 2, 3, 4, 5, 6, 7, 8.$$

Write a function that represents the overall company profits during the nine-year period. Use the *stacked bar graph* in the figure, which represents the total profits for the company during this nine-year period, to determine whether the overall company profits were increasing or decreasing.



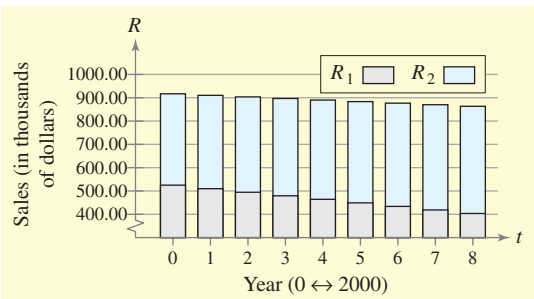
57. Comparing Sales You own two fast-food restaurants. During the years 2000 to 2008, the sales for the first restaurant were decreasing according to the function

$R_1 = 525 - 15.2t, \quad t = 0, 1, 2, 3, 4, 5, 6, 7, 8$

where R_1 represents the sales (in thousands of dollars) and t represents the year, with $t = 0$ corresponding to 2000. During the same nine-year period, the sales for the second restaurant were increasing according to the function

$R_2 = 392 + 8.5t, \quad t = 0, 1, 2, 3, 4, 5, 6, 7, 8.$

Write a function that represents the total sales for the two restaurants. Use the *stacked bar graph* in the figure, which represents the total sales during this nine-year period, to determine whether the total sales were increasing or decreasing.



58. Female Labor Force The table shows the marital status of women in the civilian labor force for the years 1995 to 2005. The numbers (in millions) of working women whose status is single, married, or other (widowed, divorced, or separated) are represented by the variables y_1, y_2, and y_3, respectively. *(Source: U.S. Bureau of Labor Statistics)*

Year	y_1	y_2	y_3
1995	15.5	33.4	12.1
1996	15.8	33.6	12.4
1997	16.5	33.8	12.7
1998	17.1	33.9	12.8
1999	17.6	34.4	12.9
2000	17.8	35.1	13.3
2001	18.0	35.2	13.6
2002	18.2	35.5	13.7
2003	18.4	36.0	13.8
2004	18.6	35.8	14.0
2005	19.2	35.9	14.2

(a) Create a stacked bar graph for the data.

(b) Use the *regression* feature of a graphing utility to find linear models for y_1, y_2, and y_3. Let t represent the year, with $t = 5$ corresponding to 1995.

(c) Use a graphing utility to graph the models for y_1, y_2, y_3, and

$y_4 = y_1 + y_2 + y_3$

in the same viewing window. Use y_4 to predict the total number of women in the work force in 2007 and 2009.

59. Cost, Revenue, and Profit The table shows the revenues y_1 (in thousands of dollars) and total costs y_2 (in thousands of dollars) for a sports memorabilia store for the years 1998 to 2008.

Year	y_1	y_2
1998	40.9	29.8
1999	46.3	32.9
2000	51.3	36.5
2001	55.9	39.9
2002	60.8	43.8
2003	65.9	46.9

Year	y_1	y_2
2004	71.0	51.1
2005	75.7	53.7
2006	80.8	57.6
2007	85.6	62.1
2008	90.7	68.7

(a) Use the *regression* feature of a graphing utility to find linear models for y_1 and y_2. Let t represent the year, with $t = 8$ corresponding to 1998.

(b) Use a graphing utility to graph the models for y_1, y_2, and

$y_3 = y_1 - y_2$

in the same viewing window. What does y_3 represent in the context of the problem? Determine the value of y_3 in 2010.

(c) Create a stacked bar graph for y_2 and y_3. What do the heights of the bars represent?

60. Bacteria Count The number of bacteria in a certain food product is given by

$N(T) = 10T^2 - 20T + 600, \quad 1 \le T \le 20$

where T is the temperature of the food. When the food is removed from the refrigerator, the temperature of the food is given by

$T(t) = 3t + 1$

where t is the time in hours. Find (a) the composite function $N(T(t))$ and (b) the time when the bacteria count reaches 1500.

61. Bacteria Count The number of bacteria in a certain food product is given by

$$N(T) = 25T^2 - 50T + 300, \quad 2 \le T \le 20$$

where T is the temperature of the food. When the food is removed from the refrigerator, the temperature of the food is given by

$$T(t) = 2t + 1$$

where t is the time in hours. Find (a) the composite function $N(T(t))$ and (b) the time when the bacteria count reaches 750.

62. Troubled Waters A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outermost ripple is given by

$$r(t) = 0.6t$$

where t is time in seconds after the pebble strikes the water. The area of the outermost circle is given by the function

$$A(r) = \pi r^2.$$

Find and interpret $(A \circ r)(t)$.

63. Consumer Awareness The suggested retail price of a new hybrid car is p dollars. The dealership advertises a factory rebate of $2000 and a 10% discount.

(a) Write a function R in terms of p giving the cost of the hybrid car after receiving the rebate from the factory.

(b) Write a function S in terms of p giving the cost of the hybrid car after receiving the dealership discount.

(c) Form the composite functions $(R \circ S)(p)$ and $(S \circ R)(p)$ and interpret each.

(d) Find $(R \circ S)(20{,}500)$ and $(S \circ R)(20{,}500)$. Which yields the lower cost for the hybrid car? Explain.

Price-Earnings Ratio In Exercises 64 and 65, the average annual price-earnings ratio for a corporation's stock is defined as the average price of the stock divided by the earnings per share. The average price of a corporation's stock is given as the function P and the earnings per share is given as the function E. Find the price-earnings ratios, P/E, for the years 2001 to 2005.

64. *Cheesecake Factory*

Year	2001	2002	2003	2004	2005
P	$18.34	$23.17	$23.63	$29.04	$33.90
E	$0.53	$0.64	$0.75	$0.88	$1.09

(Source: Cheesecake Factory)

65. *Jack in the Box*

Year	2001	2002	2003	2004	2005
P	$27.22	$28.19	$19.38	$25.20	$36.21
E	$2.11	$2.33	$2.04	$2.27	$2.48

(Source: Jack in the Box)

66. Find the domains of $(f/g)(x)$ and $(g/f)(x)$ for the functions

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{9 - x^2}.$$

Why do the two domains differ?

True or False? In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.

67. If $f(x) = x + 1$ and $g(x) = 6x$, then $(f \circ g)(x) = (g \circ f)(x)$.

68. If you are given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

Business Capsule



AP/Wide World Photos

SunPower Corporation develops and manufactures solar-electric power products. SunPower's new higher efficiency solar cells generate up to 50% more power than other solar technologies. SunPower's technology was developed by Dr. Richard Swanson and his students while he was Professor of Engineering at Stanford University. SunPower's 2006 revenues are projected to increase 300% from its 2005 revenues.

69. Research Project Use your campus library, the Internet, or some other reference source to find information about an alternative energy business experiencing strong growth similar to the example above. Write a brief report about the company or small business.

Section 2.8

Inverse Functions

- Determine if a function has an inverse function.
- Find the inverse function of a function.
- Graph a function and its inverse function.

Inverse Functions

Recall from Section 2.4 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$$f(x) = x + 4\text{: } \{(1, 5), (2, 6), (3, 7), (4, 8)\}$$

By interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A and can be written as follows.

$$f^{-1}(x) = x - 4\text{: } \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of f is equal to the range of f^{-1} and vice versa, as shown in Figure 2.67. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function, as follows.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$
$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$

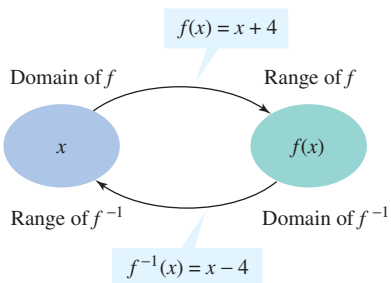


FIGURE 2.67

Example 1 Finding Inverse Functions Informally

Find the inverse function of $f(x) = 4x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

SOLUTION The given function *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is

$$f^{-1}(x) = \frac{x}{4}.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x$$

$$f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

✓CHECKPOINT 1

Find the inverse function of $f(x) = \frac{x}{6}$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function. ■

Example 2 **Finding Inverse Functions Informally**

Find the inverse function of

$$f(x) = x - 6.$$

Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

SOLUTION The given function *subtracts* 6 from each input. To 'undo' this function, you need to *add* 6 to each input. So, the inverse function of $f(x) = x - 6$ is

$$f^{-1}(x) = x + 6.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$\begin{aligned} f(f^{-1}(x)) &= f(x + 6) && \text{Substitute } x + 6 \text{ for } f^{-1}(x). \\ &= (x + 6) - 6 && \text{Substitute } x + 6 \text{ into } f(x). \\ &= x && \text{Identity function} \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(x - 6) && \text{Substitute } x - 6 \text{ for } f(x). \\ &= (x - 6) + 6 && \text{Substitute } x - 6 \text{ into } f^{-1}(x). \\ &= x && \text{Identity function} \end{aligned}$$

✓CHECKPOINT 2

Find the inverse function of $f(x) = x + 10$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function. ■

The formal definition of inverse function is as follows.

Definition of Inverse Function

Let f and g be two functions such that

$f(g(x)) = x$ for every x in the domain of g

and

$g(f(x)) = x$ for every x in the domain of f.

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f inverse"). So,

$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

Don't be confused by the use of -1 to denote the inverse function f^{-1}. In this text, f^{-1} *always* refers to the inverse function of the function f and *not* to the reciprocal of $f(x)$. That is,

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

If the function g is the inverse function of the function f, it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other.*

Example 3 Verifying Inverse Functions

Show that the following functions are inverse functions.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

SOLUTION

$$f(g(x)) = f\left(\sqrt[3]{\frac{x+1}{2}}\right) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$
$$= 2\left(\frac{x+1}{2}\right) - 1$$
$$= x + 1 - 1$$
$$= x$$

$$g(f(x)) = g(2x^3 - 1) = \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}}$$
$$= \sqrt[3]{\frac{2x^3}{2}}$$
$$= \sqrt[3]{x^3}$$
$$= x$$

DISCOVERY

Graph the equations from Example 3 and the equation $y = x$ on a graphing utility using a square viewing window.

$$y_1 = 2x^3 - 1$$
$$y_2 = \sqrt[3]{\frac{x+1}{2}}$$
$$y_3 = x$$

What do you observe about the graphs of y_1 and y_2?

✓CHECKPOINT 3

Show that the following functions are inverse functions.

$$f(x) = x^3 + 6 \quad \text{and} \quad g(x) = \sqrt[3]{x - 6}$$ ■

Example 4 Verifying Inverse Functions

Which of the functions given by

$$g(x) = \frac{x-2}{5} \quad \text{and} \quad h(x) = \frac{5}{x} + 2$$

is the inverse function of $f(x) = \dfrac{5}{x-2}$?

SOLUTION By forming the composition of f with g, you can see that

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{[(x-2)/5] - 2} = \frac{25}{x - 12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g is *not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{[(5/x) + 2] - 2} = \frac{5}{5/x} = x.$$

So, it appears that h is the inverse function of f. You can confirm this result by showing that the composition of h with f is also equal to the identity function. (Try doing this.)

✓CHECKPOINT 4

Which of the functions given by $g(x) = \dfrac{x+4}{3}$ and $h(x) = \dfrac{x}{3} + 4$ is the inverse function of $f(x) = 3x - 4$? ■

Finding Inverse Functions

For simple functions (such as the ones in Examples 1 and 2), you can find inverse functions by inspection. For more complicated functions it is best to use the following guidelines. The key step in these guidelines is switching the roles of x and y. This step corresponds to the fact that inverse functions have ordered pairs with the coordinates reversed.

STUDY TIP

Note in Step 3 of the guidelines for finding inverse functions that it is possible for a function to have no inverse function. For instance, the function given by $f(x) = x^2$ has no inverse function.

Finding Inverse Functions

1. In the equation for $f(x)$, replace $f(x)$ by y.
2. Interchange the roles of x and y.
3. Solve the new equation for y. If the new equation does not represent y as a function of x, the function f does not have an inverse function. If the new equation does represent y as a function of x, continue to Step 4.
4. Replace y by $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x = f^{-1}(f(x))$.

Example 5 Finding Inverse Functions

Find the inverse function of $f(x) = \dfrac{5 - 3x}{2}$.

SOLUTION

$$f(x) = \frac{5 - 3x}{2}$$ Write original function.

$$y = \frac{5 - 3x}{2}$$ Replace $f(x)$ by y.

$$x = \frac{5 - 3y}{2}$$ Interchange x and y.

$$2x = 5 - 3y$$ Multiply each side by 2.

$$3y = 5 - 2x$$ Isolate the y-term.

$$y = \frac{5 - 2x}{3}$$ Solve for y.

$$f^{-1}(x) = \frac{5 - 2x}{3}$$ Replace y by $f^{-1}(x)$.

Note that both f and f^{-1} have domains and ranges that consist of the entire set of real numbers. Check that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

✓CHECKPOINT 5

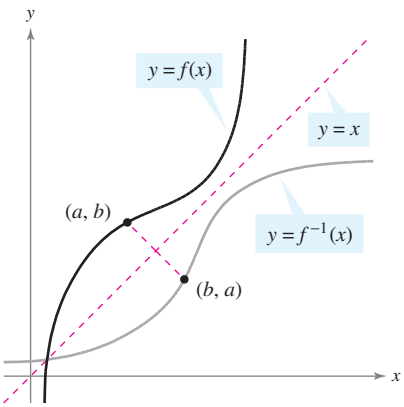
Find the inverse function of $f(x) = 4x + 5$. ■

TECHNOLOGY

Access the website for this text at *college.hmco.com/info/larsonapplied* for a graphing utility program that will graph a function f and its reflection in the line $y = x$. Programs for several models of graphing utilities are available.

The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a *reflection* of the graph of f in the line $y = x$, as shown in Figure 2.68.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

FIGURE 2.68

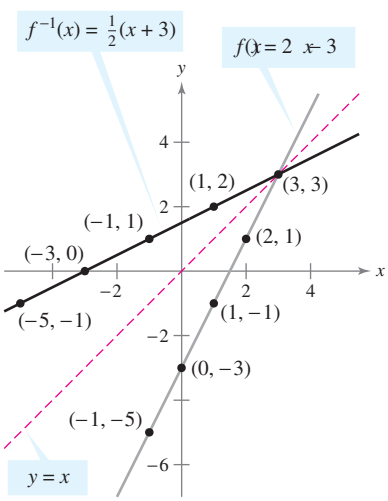


FIGURE 2.69

Example 6 The Graphs of f and f^{-1}

Sketch the graphs of the inverse functions given by

$$f(x) = 2x - 3 \quad \text{and} \quad f^{-1}(x) = \tfrac{1}{2}(x + 3)$$

in the same coordinate plane and show that the graphs are reflections of each other in the line $y = x$.

SOLUTION The graphs of f and f^{-1} are shown in Figure 2.69. Visually, it appears that the graphs are reflections of each other in the line $y = x$. You can further verify this reflective property by testing a few points on each graph. Note in the following list that if the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}.

Graph of $f(x) = 2x - 3$	*Graph of* $f^{-1}(x) = \frac{1}{2}(x + 3)$
$(0, -3)$	$(-3, 0)$
$(1, -1)$	$(-1, 1)$
$(2, 1)$	$(1, 2)$
$(3, 3)$	$(3, 3)$

✓CHECKPOINT 6

Sketch the graphs of the inverse functions given by $f(x) = \frac{2}{5}x + 2$ and $f^{-1}(x) = \frac{5}{2}x - 5$ in the same coordinate plane and show that the graphs are reflections of each other in the line $y = x$. ■

The Study Tip on page 241 mentioned that the function given by

$$f(x) = x^2$$

has no inverse function. What this means is that, *assuming the domain of f is the entire real line,* the function given by $f(x) = x^2$ has no inverse function. If the domain of f is restricted to the nonnegative real numbers, however, then f does have an inverse function, as demonstrated in Example 7.

Example 7 The Graphs of f and f^{-1}

Sketch the graphs of the inverse functions given by

$$f(x) = x^2, \quad x \geq 0, \quad \text{and} \quad f^{-1}(x) = \sqrt{x}$$

in the same coordinate plane and show that the graphs are reflections of each other in the line $y = x$.

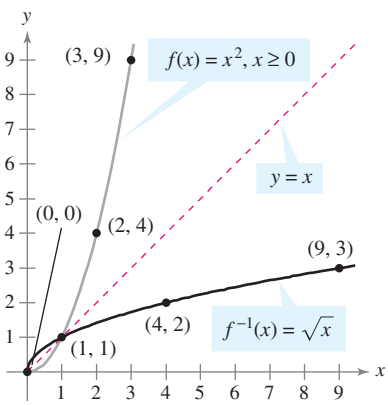


FIGURE 2.70

SOLUTION The graphs of f and f^{-1} are shown in Figure 2.70. Visually, it appears that the graphs are reflections of each other in the line $y = x$. You can further verify this reflective property by testing a few points on each graph. Note in the following list that if the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}.

Graph of $f(x) = x^2, \quad x \geq 0$	*Graph of* $f^{-1}(x) = \sqrt{x}$
$(0, 0)$	$(0, 0)$
$(1, 1)$	$(1, 1)$
$(2, 4)$	$(4, 2)$
$(3, 9)$	$(9, 3)$

You can verify algebraically that the functions are inverse functions of each other by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ as follows.

$$f(f^{-1}(x)) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x, \quad \text{if } x \geq 0$$

$$f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = x, \quad \text{if } x \geq 0$$

✓CHECKPOINT 7

Sketch the graphs of the inverse functions given by $f(x) = x^2 + 3$, $x \geq 0$, and $f^{-1}(x) = \sqrt{x - 3}$ in the same coordinate plane and show that the graphs are reflections of each other in the line $y = x$. ■

The guidelines for finding the inverse function of a function include an *algebraic* test for determining whether a function has an inverse function. The reflective property of the graphs of inverse functions gives you a *geometric* test for determining whether a function has an inverse function. This test is called the **Horizontal Line Test** for inverse functions.

Horizontal Line Test for Inverse Functions

A function f has an inverse function if and only if no *horizontal* line intersects the graph of f at more than one point.

Example 8 Applying the Horizontal Line Test

Use the graph of f to determine whether the function has an inverse function.

a. $f(x) = x^3 - 1$ **b.** $f(x) = x^2 - 1$

SOLUTION

a. The graph of the function given by

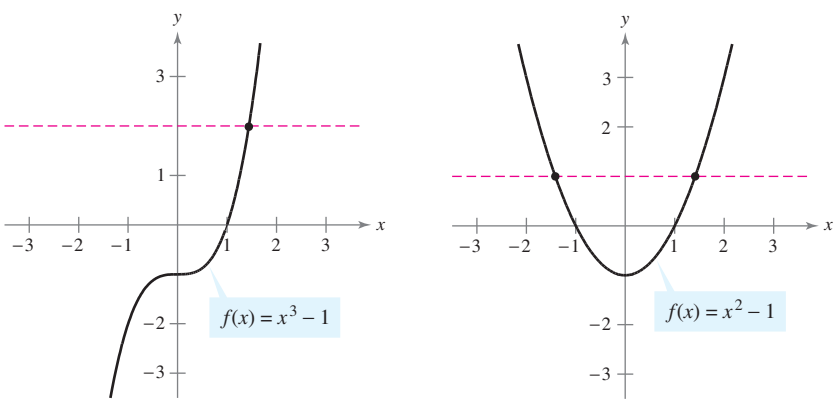
$$f(x) = x^3 - 1$$

is shown in Figure 2.71(a). Because no horizontal line intersects the graph of f at more than one point, you can conclude that f does have an inverse function.

b. The graph of the function given by

$$f(x) = x^2 - 1$$

is shown in Figure 2.71(b). Because it is possible to find a horizontal line that intersects the graph of f at more than one point, you can conclude that f does not have an inverse function.



(a) (b)

FIGURE 2.71

✓CHECKPOINT 8

Use the graph of f to determine whether the function has an inverse function.

a. $f(x) = |x|$

b. $f(x) = \sqrt{x}$ ■

CONCEPT CHECK

1. What can you say about the functions m and n given that $m(n(x)) = x$ for every x in the domain of n and $n(m(x)) = x$ for every x in the domain of m?
2. Given that the functions g and h are inverses of each other and (a, b) is a point on the graph of g, name a point on the graph of h.
3. Explain how to find an inverse function algebraically.
4. The line $y = 2$ intersects the graph of $f(x)$ at two points. Does f have an inverse? Explain.

Skills Review 2.8

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.4, 1.1, 1.5, and 2.4.

In Exercises 1–4, find the domain of the function.

1. $f(x) = \sqrt[3]{x+1}$

2. $f(x) = \sqrt{x+1}$

3. $g(x) = \dfrac{2}{x^2 - 2x}$

4. $h(x) = \dfrac{x}{3x+5}$

In Exercises 5–8, simplify the expression.

5. $2\left(\dfrac{x+5}{2}\right) - 5$

6. $7 - 10\left(\dfrac{7-x}{10}\right)$

7. $\sqrt[3]{2\left(\dfrac{x^3}{2} - 2\right) + 4}$

8. $\sqrt[5]{(x+2)^5} - 2$

In Exercises 9 and 10, solve for x in terms of y.

9. $y = \dfrac{2x-6}{3}$

10. $y = \sqrt[3]{2x-4}$

Exercises 2.8

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the inverse function of the function f given by the set of ordered pairs.

1. $\{(1, 4), (2, 5), (3, 6), (4, 7)\}$

2. $\{(6, 2), (5, 3), (4, 4), (3, 5)\}$

3. $\{(-1, 1), (-2, 2), (-3, 3), (-4, 4)\}$

4. $\{(6, -2), (5, -3), (4, -4), (3, -5)\}$

In Exercises 5–8, find the inverse function informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

5. $f(x) = 2x$

6. $f(x) = -\dfrac{x}{4}$

7. $f(x) = x - 5$

8. $f(x) = x + 7$

In Exercises 9–16, show that f and g are inverse functions by (a) using the definition of inverse functions and (b) graphing the functions. Make sure you test a few points, as shown in Examples 6 and 7.

9. $f(x) = 5x + 1, \quad g(x) = \dfrac{x-1}{5}$

10. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3-x}{4}$

11. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$

12. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

13. $f(x) = \sqrt{x-4}, \quad g(x) = x^2 + 4, \quad x \geq 0$

14. $f(x) = 9 - x^2, \quad x \geq 0$

$g(x) = \sqrt{9-x}, \quad x \leq 9$

15. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1-x}$

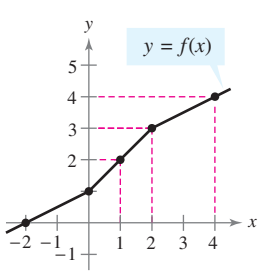
16. $f(x) = \dfrac{1}{1+x}, \quad x \geq 0$

$g(x) = \dfrac{1-x}{x}, \quad 0 < x \leq 1$

In Exercises 17–20, use the graph of f to complete the table and to sketch the graph of f^{-1}.

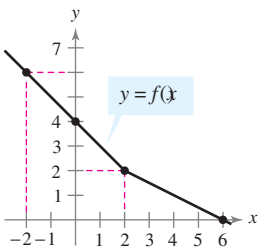
17.

x	0	1	2	3	4
$f^{-1}(x)$					



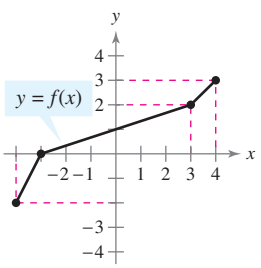
18.

x	0	2	4	6
$f^{-1}(x)$				



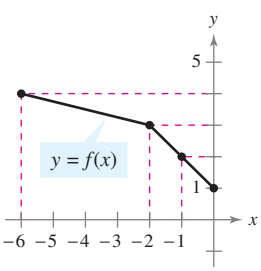
19.

x	-2	0	2	3
$f^{-1}(x)$				



20.

x	1	2	3	4
$f^{-1}(x)$				



In Exercises 21–36, determine whether the function has an inverse function. If it does, find its inverse function.

21. $f(x) = x^4$

22. $f(x) = \dfrac{1}{x^2}$

23. $g(x) = \dfrac{x}{8}$

24. $f(x) = 3x + 5$

25. $p(x) = -4$

26. $f(x) = \dfrac{3x + 4}{5}$

27. $f(x) = (x + 3)^2,\ x \geq -3$

28. $q(x) = (x - 5)^2$

29. $h(x) = \dfrac{1}{x}$

30. $f(x) = |x - 2|,\ x \leq 2$

31. $f(x) = \sqrt{2x + 3}$

32. $f(x) = \sqrt{x - 2}$

33. $g(x) = x^2 - x^4$

34. $f(x) = \dfrac{x^2}{x^2 + 1}$

35. $f(x) = 25 - x^2,\ x \leq 0$

36. $f(x) = 36 + x^2,\ x \leq 0$

Error Analysis In Exercises 37 and 38, a student has handed in the answer to a problem on a quiz. Find the error(s) in each solution and discuss how to explain each error to the student.

37. Find the inverse function f^{-1} of $f(x) = \sqrt{2x - 5}$.

$f(x) = \sqrt{2x - 5}$, so

$f^{-1}(x) = \dfrac{1}{\sqrt{2x - 5}}$

38. Find the inverse function f^{-1} of $f(x) = \frac{3}{5}x + \frac{1}{3}$.

$f(x) = \frac{3}{5}x + \frac{1}{3}$, so

$f^{-1}(x) = \frac{5}{3}x - 3$

(T) In Exercises 39–48, find the inverse function f^{-1} of the function f. Then, using a graphing utility, graph both f and f^{-1} in the same viewing window.

39. $f(x) = 2x - 3$

40. $f(x) = 5x + 2$

41. $f(x) = x^5$

42. $f(x) = x^3 + 1$

43. $f(x) = \sqrt{x}$

44. $f(x) = x^2,\ x \geq 0$

45. $f(x) = \sqrt{16 - x^2},\quad 0 \leq x \leq 4$

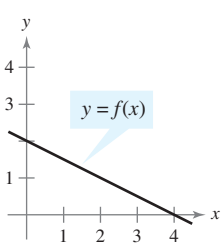
46. $f(x) = \dfrac{3}{x + 1}$

47. $f(x) = \sqrt[3]{x + 2}$

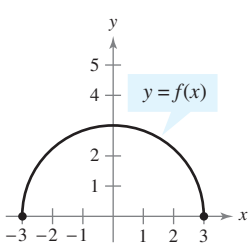
48. $f(x) = x^{3/5} - 2$

In Exercises 49–52, does the function have an inverse function? Explain your reasoning.

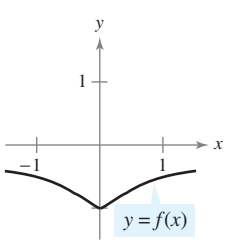
49.



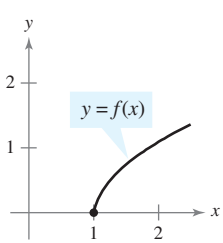
50.



51.



52.



In Exercises 53–58, graph the function and use the Horizontal Line Test to determine whether the function has an inverse function.

53. $g(x) = \dfrac{5 - 2x}{3}$ **54.** $f(x) = 10$

55. $h(x) = |x - 5|$ **56.** $g(x) = (x - 3)^2$

57. $f(x) = -\sqrt{9 - x^2}$ **58.** $f(x) = (x - 1)^3$

In Exercises 59–62, use the functions given by

$$f(x) = \frac{1}{8}x - 3 \quad \text{and} \quad g(x) = x^3$$

to find the value.

59. $(f^{-1} \circ g^{-1})(1)$ **60.** $(g^{-1} \circ f^{-1})(-3)$

61. $(f^{-1} \circ f^{-1})(6)$ **62.** $(g^{-1} \circ g^{-1})(-4)$

In Exercises 63–66, use the functions given by

$$f(x) = x + 4 \quad \text{and} \quad g(x) = 2x - 5$$

to find the composition of functions.

63. $g^{-1} \circ f^{-1}$ **64.** $f^{-1} \circ g^{-1}$

65. $(f \circ g)^{-1}$ **66.** $(g \circ f)^{-1}$

67. Cost With fixed daily costs of \$1500, the cost C for a T-shirt business to make x personalized T-shirts is given by $C(x) = 7.50x + 1500$. Find the inverse function $C^{-1}(x)$ and explain what it represents. Describe the domains of $C(x)$ and $C^{-1}(x)$.

68. Profit A company's profit P for producing x units is given by $P(x) = 47x - 5736$. Find the inverse function $P^{-1}(x)$ and explain what it represents. Describe the domains of $P(x)$ and $P^{-1}(x)$.

69. Movie Theaters The average prices of admission y (in dollars) to a movie theater for the years 1998 to 2005 are shown in the table. *(Source: Motion Picture Association of America, Inc.)*

Year	1998	1999	2000	2001
Admission price, y	4.69	5.08	5.39	5.66

Year	2002	2003	2004	2005
Admission price, y	5.81	6.03	6.21	6.41

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 8$ corresponding to 1998.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data.

(c) Algebraically find the inverse function of the model in part (b). Explain what this inverse function represents in a real-life context.

(d) Use the inverse function you found in part (c) to estimate the year in which the average admission price to a movie theater will reach \$8.00.

70. Lead Exposure A project is conducted to study the amount of lead accumulated in the bones of humans. The concentration L (in micrograms per gram of bone mineral) of lead found in the tibia of a man is measured every five years. The results are shown in the table.

Age	15	20	25	30	35	40
Lead, L	3.2	5.4	9.2	12.2	13.8	16.0

(a) Use a graphing utility to create a scatter plot of the data. Let x represent the age (in years) of the man.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data.

(c) Algebraically find the inverse function of the model in part (b). Explain what this inverse function represents in a real-life context.

(d) Use the inverse function you found in part (c) to estimate the age of the man when the concentration of lead in his tibia reaches 25 micrograms per gram of bone mineral.

71. Reasoning You are helping a friend to find the inverse function of a one-to-one function. He states that interchanging the roles of x and y is "cheating." Explain how you would use the graphs of $f(x) = x^2 + 1$, $x \geq 0$, and $f^{-1}(x) = \sqrt{x - 1}$ to justify that particular step in the process of finding an inverse function.

72. Diesel Mechanics The function given by

$$y = 0.03x^2 + 245.5, \quad 0 < x < 100$$

approximates the exhaust temperature y for a diesel engine in degrees Fahrenheit, where x is the percent load for the diesel engine. Solve the equation for x in terms of y and use the result to find the percent load for a diesel engine when the exhaust temperature is 410°F.

73. Earnings-Dividend Ratio From 1995 to 2005, the earnings per share for Wal-Mart Stores were approximately related to the dividends per share by the function given by

$$f(x) = \sqrt{0.0161x^3 + 0.008}, \quad 0.6 \leq x \leq 2.63$$

where f represents the dividends per share (in dollars) and x represents the earnings per share (in dollars). In 2004, Wal-Mart paid dividends of \$0.48 per share. Find the inverse function of f and use the inverse function to approximate the earnings per share in 2004. *(Source: Wal-Mart Stores, Inc.)*

Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 250. Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 2.1	Review Exercises
■ Plot points in the Cartesian plane, find the distance between two points, and find the midpoint of a line segment joining two points. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	*1–6*
■ Determine whether a point is a solution of an equation.	*7, 8*
■ Sketch the graph of an equation using a table of values.	*9, 10*
■ Find the x- and y-intercepts, and determine the symmetry, of the graph of an equation.	*11–16*
■ Write the equation of a circle in standard form. $(x - h)^2 + (y - k)^2 = r^2$	*17–20*
Section 2.2	
■ Find the slope of a line passing through two points. $m = \frac{y_2 - y_1}{x_2 - x_1}$	*21–24*
■ Use the point-slope form to find the equation of a line. $y - y_1 = m(x - x_1)$	*25–28*
■ Use the slope-intercept form to sketch a line. $y = mx + b$	*29–32*
■ Use slope to determine if lines are parallel or perpendicular, and write the equation of a line parallel or perpendicular to a given line. Parallel lines: $m_1 = m_2$ Perpendicular lines: $m_1 = -\frac{1}{m_2}$	*33–40*
Section 2.3	
■ Construct and use a linear model to relate quantities that vary directly. Direct variation: $y = mx$	*41–50*
■ Construct and use a linear model with slope as the rate of change.	*51–53*
■ Use a scatter plot to find a linear model that fits a set of data.	*54*

* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

Section 2.4	Review Exercises
■ Determine if an equation or a set of ordered pairs represents a function.	*55–60*
■ Use function notation, evaluate a function, and find the domain of a function.	*61–69*
■ Write a function that relates quantities in an application problem.	*70–72*
Section 2.5	
■ Find the domain and range using the graph of a function.	*73–76*
■ Identify the graph of a function using the Vertical Line Test.	*77–82*
■ Describe the increasing and decreasing behavior of a function.	*73–76, 92*
■ Find the relative maxima and relative minima of the graph of a function.	*73–76*
■ Classify a function as even or odd. In an even function, $f(-x) = f(x)$ In an odd function, $f(-x) = -f(x)$	*73–76*
■ Identify six common graphs and use them to sketch the graph of a function.	*83–91*
Section 2.6	
■ Use vertical and horizontal shifts, reflections, and nonrigid transformations to sketch graphs of functions.	*93–100*
Section 2.7	
■ Find the sum, difference, product, and quotient of two functions.	*101–106*
■ Form the composition of two functions and determine its domain.	*107–110*
■ Identify a function as the composition of two functions.	*111–114*
■ Use combinations and compositions of functions to solve application problems.	*115–118*
Section 2.8	
■ Verify that two functions are inverse functions of each other. $f(f^{-1}(x)) = x$ $f^{-1}(f(x)) = x$	*119, 120, 125–128*
■ Determine if a function has an inverse function.	*119–129*
■ Find the inverse function of a function.	*121–128*
■ Graph a function and its inverse function.	*121–128*
■ Find and use an inverse function in an application problem.	*129*

Study Strategies

■ **To Memorize or Not to Memorize?** When studying mathematics, you often need to memorize formulas, rules, and properties. The formulas that you use most often can become committed to memory through practice. Some formulas, however, are used infrequently or may be easily forgotten. When you are unsure of a formula, you may be able to *derive* it using other information that you know. For instance, if you forget the standard form of the equation of a circle, you can use the Distance Formula and properties of a circle to derive it, as shown on pages 164 and 165. If you also forget the Distance Formula, you can depict the distance between two generic points graphically and use the Pythagorean Theorem to derive the formula, as shown on page 158.

■ **Choose Convenient Values for Yearly Data** When you work with data involving years, you may want to reassign simpler values to represent the years. For instance, you might represent the years 1992 to 2009 by the x-values 2 to 19. If you sketch a graph of these data, be sure to account for this in the x-axis title: Year ($2 \leftrightarrow 1992$).

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

1. $(3, 2), (-3, -5)$

2. $(-9, 3), (5, 7)$

3. $(3.45, 6.55), (-1.06, -3.87)$

4. $(-6.7, -3.9), (5.1, 8.2)$

In Exercises 5 and 6, find x such that the distance between the points is 25.

5. $(10, 10), (x, -5)$

6. $(x, -5), (-15, 10)$

In Exercises 7 and 8, determine whether the point is a solution of the equation.

7. $y = 2x^2 - 7x - 15$

(a) $(5, 0)$ (b) $(-2, 7)$

8. $y = \sqrt{16 - x^2}$

(a) $(1, 5)$ (b) $(4, 0)$

In Exercises 9 and 10, complete the table. Use the resulting solution points to sketch the graph of the equation.

9. $y = -\frac{1}{2}x + 2$

x	-2	0	2	3	4
y					

10. $y = x^2 - 3x$

x	-1	0	1	2	3
y					

In Exercises 11–16, sketch the graph of the equation. Identify any intercepts and test for symmetry.

11. $y = x^2 + 3$

12. $y^2 = x$

13. $y = 3x - 4$

14. $y = \sqrt{9 - x}$

15. $y = x^3 + 1$

16. $y = |x - 3|$

In Exercises 17 and 18, find the standard form of the equation of the specified circle.

17. Center: $(-1, 2)$; radius: 6

18. Endpoints of the diameter: $(-2, -3), (4, 5)$

In Exercises 19 and 20, write the equation of the circle in standard form and sketch its graph.

19. $x^2 + y^2 - 4x + 6y - 12 = 0$

20. $4x^2 + 4y^2 - 4x - 8y - 11 = 0$

In Exercises 21–24, plot the points and find the slope of the line passing through the points.

21. $(3, 7), (2, -1)$

22. $(3, -2), (-1, -2)$

23. $(3, 4), (3, -2)$

24. $(-1, 5), (2, -3)$

In Exercises 25–28, find an equation of the line that passes through the point and has the indicated slope. Sketch the line.

	Point	*Slope*
25.	$(0, -5)$	$m = \frac{3}{2}$
26.	$(3, 0)$	$m = -\frac{2}{3}$
27.	$(-2, 6)$	$m = 0$
28.	$(5, 4)$	m is undefined.

In Exercises 29–32, find the slope and y-intercept (if possible) of the line specified by the equation. Then sketch the line.

29. $5x - 4y + 11 = 0$

30. $3y - 2 = 0$

31. $17 - 5x = 10$

32. $16x + 12y - 24 = 0$

In Exercises 33–36, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.

33. L_1: $(0, 3), (-2, 1)$; L_2: $(-8, -3), (4, 9)$

34. L_1: $(-3, -1), (2, 5)$; L_2: $(2, 1), (8, 6)$

35. L_1: $(3, 6), (-1, -5)$; L_2: $(-2, 3), (4, 7)$

36. L_1: $(-1, 2), (-1, 4)$; L_2: $(7, 3), (4, 7)$

In Exercises 37–40, write an equation of the line through the point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Line*
37.	$(3, -2)$	$5x - 4y = 8$
38.	$(-8, 3)$	$2x + 3y = 5$
39.	$(-1, -2)$	$y = 2$
40.	$(0, 5)$	$x = -3$

Direct Variation In Exercises 41–44, y is proportional to x. Use the x- and y-values to find a linear model that relates x and y.

41. $x = 3, y = 7$

42. $x = 5, y = 3.8$

43. $x = 10, y = 3480$

44. $x = 14, y = 1.95$

Direct Variation In Exercises 45–48, write a linear model that relates the variables.

45. A varies directly as r; $A = 30$ when $r = 6$.

46. y varies directly as z; $y = 7$ when $z = 14$.

47. a is proportional to b; $a = 15$ when $b = 20$.

48. m varies directly as n; $m = 12$ when $n = 21$.

49. Property Tax The property tax in a city is based on the assessed value of the property. A house that has an assessed value of \$80,000 has a property tax of \$2920. Find a mathematical model that gives the amount of property tax y in terms of the assessed value of the property x. Use the model to find the property tax on a house that has an assessed value of \$102,000.

50. Feet and Meters You are driving and you notice a billboard that indicates it is 1000 feet or 305 meters to the next restaurant of a national fast-food chain. Use this information to find a linear model that relates feet to meters. Use the model to complete the table.

Feet	20	50	100	120
Meters				

51. Fourth-Quarter Sales During the second and third quarters of the year, a business had sales of \$275,000 and \$305,500, respectively. Assume the growth of the sales follows a linear pattern. What will sales be during the fourth quarter?

52. Dollar Value The dollar value of a product in 2008 is \$75 and the item is expected to increase in value at a rate of \$5.95 per year. Write a linear equation that gives the dollar value of the product in terms of the year. Use this model to predict the dollar value of the product in 2010. (Let $t = 8$ represent 2008.)

53. Straight-Line Depreciation A small business purchases a piece of equipment for \$135,000. After 10 years, the equipment will have to be replaced. Its salvage value at that time is expected to be \$5500. Write a linear equation giving the value V of the equipment during the 10 years it will be used.

54. Sales The sales S (in millions of dollars) for Intuit Corporation for the years 2000 to 2005 are shown in the table. *(Source: Intuit Corporation)*

Year	Sales S (in millions of dollars)
2000	1093.8
2001	1261.5
2002	1358.3
2003	1650.7
2004	1867.7
2005	2079.9

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000. Do the data appear to be linear?

(b) Use the *regression* feature of a graphing utility to find a linear model for the data.

(c) Use the linear model from part (b) to predict sales in 2006 and 2007.

(d) Intuit Corporation predicts sales of \$2325 million for 2006 and \$2500 million for 2007. Do your estimates from part (c) agree with those of Intuit Corporation? Which set of estimates do you think is more reasonable? Explain.

In Exercises 55–58, decide whether the equation represents y as a function of x.

55. $3x - 4y = 12$

56. $y^2 = x^2 - 9$

57. $y = \sqrt{x + 3}$

58. $x^2 + y^2 - 6x + 8y = 0$

In Exercises 59 and 60, decide whether the set of ordered pairs represents a function from A to B.

$A = \{1, 2, 3\}$ $B = \{-3, -4, -7\}$

Give reasons for your answer.

59. $\{(1, -3), (2, -7), (3, -3)\}$

60. $\{(1, -4), (2, -3), (3, -9)\}$

In Exercises 61 and 62, evaluate the function at each specified value of the independent variable and simplify.

61. $f(x) = \sqrt{x + 4} - 5$

(a) $f(5)$ (b) $f(0)$ (c) $f(-4)$ (d) $f(x + 3)$

62. $f(x) = \begin{cases} 2x - 1, & x \leq 1 \\ x^2 + 2, & x > 1 \end{cases}$

(a) $f(0)$ (b) $f(1)$ (c) $f(3)$ (d) $f(-4)$

In Exercises 63–68, find the domain of the function.

63. $f(x) = 2x^2 + 7x + 3$

64. $g(t) = \dfrac{3}{t^2 - 4}$

65. $h(x) = \sqrt{x + 5}$

66. $f(t) = \sqrt[3]{t - 3}$

67. $g(t) = \dfrac{\sqrt{t - 1}}{t - 4}$

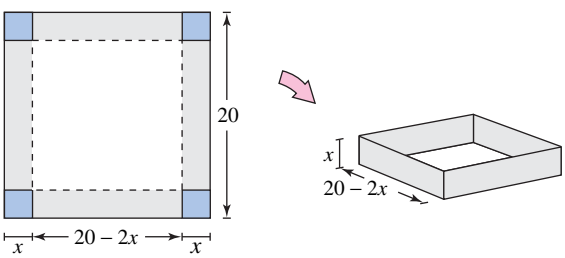
68. $h(x) = \sqrt[4]{16 - x^2}$

69. Reasoning A student has difficulty understanding why the domains of

$$h(x) = \frac{x^2 - 4}{x} \quad \text{and} \quad k(x) = \frac{x}{x^2 - 4}$$

are different. How would you explain their respective domains algebraically? How could you use a graphing utility to explain their domains?

70. Volume of a Box An open box is to be made from a square piece of material 20 inches on a side by cutting equal squares from the corners and turning the sides up (see figure).



(a) Write the volume V of the box as a function of its height x.

(b) What is the domain of this function?

(c) Use a graphing utility to graph the function.

71. Balance in an Account A person deposits $6500 in an account that pays 6.85% interest compounded quarterly.

(a) Write the balance of the account in terms of the time t that the principal is left in the account.

(b) What is the domain of this function?

72. Vertical Motion The velocity v (in feet per second) of a ball thrown vertically upward from ground level is given by

$$v(t) = -32t + 80$$

where t is the time (in seconds).

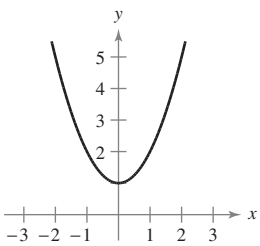
(a) Find the velocity when $t = 1$.

(b) Find the time when the ball reaches its maximum height. [*Hint:* Find the time when $v(t) = 0$.]

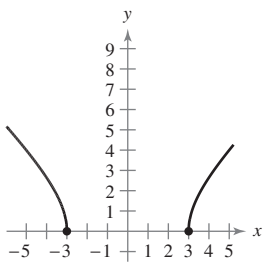
(c) Find the velocity when $t = 3$.

In Exercises 73–76, (a) determine the domain and range of the function, (b) determine the intervals over which the function is increasing, decreasing, or constant, (c) determine if the function is even, odd, or neither, and (d) approximate any relative minimum or relative maximum values of the function.

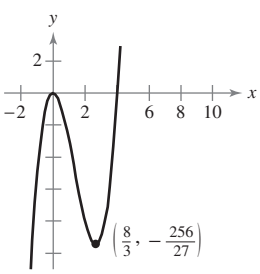
73. $f(x) = x^2 + 1$



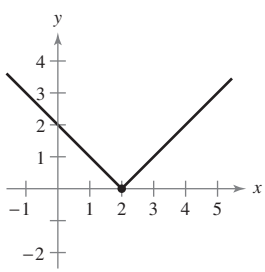
74. $f(x) = \sqrt{x^2 - 9}$



75. $f(x) = x^3 - 4x^2$

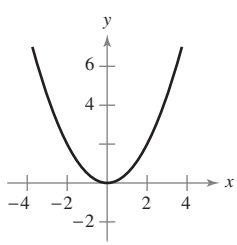


76. $f(x) = |x - 2|$

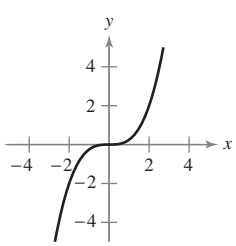


In Exercises 77–82, use the Vertical Line Test to decide whether y is a function of x.

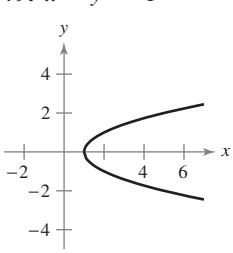
77. $y = \frac{1}{2}x^2$



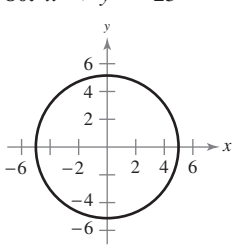
78. $y = \frac{1}{4}x^3$



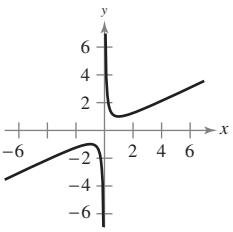
79. $x - y^2 = 1$



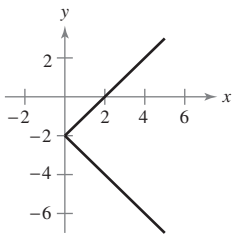
80. $x^2 + y^2 = 25$



81. $x^2 = 2xy - 1$



82. $x = |y + 2|$



In Exercises 83–90, sketch the graph of the function.

83. $f(x) = |x + 3|$

84. $g(x) = \sqrt{x^2 - 16}$

85. $h(x) = 2[\![x]\!] + 1$

86. $f(x) = 3$

87. $g(x) = \begin{cases} x + 2, & x < 0 \\ 2, & x = 0 \\ x^2 + 2, & x > 0 \end{cases}$

88. $g(x) = \begin{cases} 3x + 1, & x < -1 \\ x^2 - 3, & x \geq -1 \end{cases}$

89. $h(x) = x^2 - 3x$

90. $f(x) = \sqrt{9 - x^2}$

91. Cost of Overnight Delivery The cost of sending an overnight package from Los Angeles to Dallas is \$10.25 for up to, but not including, the first pound and \$2.75 for each additional pound (or portion of a pound). A model for the total cost C of sending the package is

$$C = 10.25 + 2.75[\![x]\!], \quad x > 0$$

where x is the weight of the package (in pounds). Sketch the graph of this function.

92. Revenue A company determines that the total revenue R (in hundreds of thousands of dollars) for the years 1997 to 2010 can be approximated by the function

$$R = -0.025t^3 + 0.8t^2 - 2.5t + 8.75, \quad 7 \leq t \leq 20$$

where t represents the year, with $t = 7$ corresponding to 1997. Graph the revenue function using a graphing utility and use the *trace* feature to estimate the years during which the revenue was increasing and the years during which the revenue was decreasing.

In Exercises 93 and 94, describe the sequence of transformations from $f(x) = x^2$ to g. Then sketch the graph of g.

93. $g(x) = -(x - 1)^2 - 2$

94. $g(x) = -x^2 + 3$

In Exercises 95 and 96, describe the sequence of transformations from $f(x) = \sqrt{x}$ to g. Then sketch the graph of g.

95. $g(x) = -\sqrt{x - 2}$

96. $g(x) = \sqrt{x} + 2$

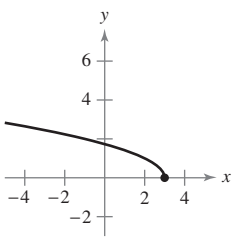
In Exercises 97 and 98, describe the sequence of transformations from $f(x) = \sqrt[3]{x}$ to g. Then sketch the graph of g.

97. $g(x) = \sqrt[3]{x + 2}$

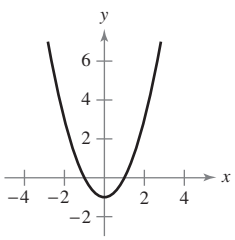
98. $g(x) = 2\sqrt[3]{x}$

In Exercises 99 and 100, identify the transformation shown in the graph and the associated common function. Write the equation of the graphed function.

99.



100.



In Exercises 101 and 102, find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $(f/g)(x)$. What is the domain of f/g?

101. $f(x) = 3x - 1, \quad g(x) = x^2 + 2x$

102. $f(x) = 3x, \quad g(x) = \sqrt{x^2 + 1}$

In Exercises 103–106, evaluate the function for $f(x) = x^2 + 3x$ and $g(x) = 2x - 5$.

103. $(f + g)(2)$

104. $(f - g)(-1)$

105. $(fg)(3)$

106. $\left(\frac{f}{g}\right)(0)$

In Exercises 107–110, find and determine the domains of (a) $f \circ g$ and (b) $g \circ f$.

107. $f(x) = x^2, \quad g(x) = x + 3$

108. $f(x) = 2x - 5, \quad g(x) = x^2 + 2$

109. $f(x) = \frac{1}{x}, \quad g(x) = 3x + x^2$

110. $f(x) = \frac{1}{x^2}, \quad g(x) = x^3$

In Exercises 111–114, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

111. $h(x) = (6x - 5)^2$

112. $h(x) = \sqrt[3]{x + 2}$

113. $h(x) = \frac{1}{(x - 1)^2}$

114. $h(x) = (x - 3)^3 + 2(x - 3)$

115. ***MAKE A DECISION: COMPARING SALES*** You own two dry cleaning establishments. From 2000 to 2008, the sales for one of the establishments were increasing according to the function

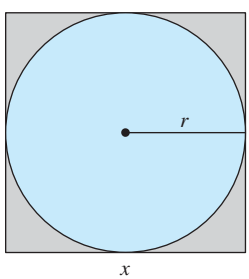
$$R_1 = 499.7 - 0.3t + 0.2t^2, \quad t = 0, 1, 2, 3, 4, 5, 6, 7, 8$$

where R_1 represents the sales (in thousands of dollars) and t represents the year, with $t = 0$ corresponding to 2000. During the same nine-year period, the sales for the second establishment were decreasing according to the function

$$R_2 = 300.8 - 0.62t, \quad t = 0, 1, 2, 3, 4, 5, 6, 7, 8.$$

Write a function that represents the total sales for the two establishments. Make a stacked bar graph to represent the total sales during this nine-year period. Were total sales increasing or decreasing?

116. **Area** A square concrete foundation is prepared as a base for a large cylindrical aquatic tank that is to be used in ecology experiments (see figure).



(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.

117. ***MAKE A DECISION*** You are a sales representative for an automobile manufacturer. You are paid an annual salary plus a bonus of 3% of your sales over $500,000. Consider the two functions given by

$$f(x) = x - 500{,}000$$

and

$$g(x) = 0.03x.$$

If x is greater than $500,000, does $f(g(x))$ or $g(f(x))$ represent your bonus? Explain.

118. **Bacteria** The number N of bacteria is given by $N(T) = 8T^2 - 14T + 200$, where T is the temperature (in degrees Fahrenheit). The temperature is $T(t) = 2t + 2$, where t is the time in hours. Find and interpret $(N \circ T)(t)$.

In Exercises 119 and 120, show that f and g are inverse functions of each other.

119. $f(x) = 3x + 5, \quad g(x) = \dfrac{x - 5}{3}$

120. $f(x) = \sqrt[3]{x - 3}, \quad g(x) = x^3 + 3$

In Exercises 121–124, determine whether the function has an inverse function. If it does, find the inverse function and graph f and f^{-1} in the same coordinate plane.

121. $f(x) = 3x^2$

122. $f(x) = \sqrt[3]{x + 1}$

123. $f(x) = \dfrac{1}{x}$

124. $f(x) = \dfrac{x^2}{x^2 - 9}$

In Exercises 125–128, (a) find f^{-1}, (b) sketch the graphs of f and f^{-1} on the same coordinate plane, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

125. $f(x) = \frac{1}{2}x - 3$

126. $f(x) = \sqrt{x + 1}$

127. $f(x) = x^2, \quad x \geq 0$

128. $f(x) = \sqrt[3]{x - 1}$

129. **Federal Student Aid** The average awards A (in dollars) of federal financial aid (including grants and loans) for the years 2000 to 2005 are shown in the table. *(Source: U.S. Department of Education)*

Year	Average award, A (in dollars)
2000	2925
2001	2982
2002	3089
2003	3208
2004	3316
2005	3425

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data.

(c) If the data can be modeled by a one-to-one function, find the inverse function of the model and use it to predict in what year the average award will be $3600.

Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, find the distance between the points and the midpoint of the line segment connecting the points.

1. $(-3, 2), (5, -2)$

2. $(3.25, 7.05), (-2.37, 1.62)$

3. Find the intercepts of the graph of $y = (x + 5)(x - 3)$.

4. Describe the symmetry of the graph of $y = \dfrac{x}{x^2 - 4}$.

5. Find an equation of the line through $(-3, 5)$ with a slope of $\frac{2}{3}$.

6. Write the equation of the circle in standard form and sketch its graph.

$x^2 + y^2 - 6x + 4y - 3 = 0$

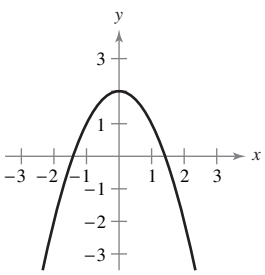


Figure for 9

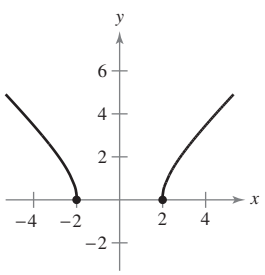


Figure for 10

In Exercises 7 and 8, decide whether the statement is true or false. Explain.

7. The equation $2x - 3y = 5$ identifies y as a function of x.

8. If $A = \{3, 4, 5\}$ and $B = \{-1, -2, -3\}$, the set $\{(3, -9), (4, -2), (5, -3)\}$ represents a function from A to B.

In Exercises 9 and 10, (a) find the domain and range of the function, (b) determine the intervals over which the function is increasing, decreasing, or constant, (c) determine whether the function is even or odd, and (d) approximate any relative minimum or relative maximum values of the function.

9. $f(x) = 2 - x^2$ (See figure.)

10. $g(x) = \sqrt{x^2 - 4}$ (See figure.)

In Exercises 11 and 12, sketch the graph of the function.

11. $g(x) = \begin{cases} x + 1, & x < 0 \\ 1, & x = 0 \\ x^2 + 1, & x > 0 \end{cases}$

12. $h(x) = (x - 3)^2 + 4$

In Exercises 13–16, use $f(x) = x^2 + 2$ and $g(x) = 2x - 1$ to find the function.

13. $(f - g)(x)$

14. $(fg)(x)$

15. $(f \circ g)(x)$

16. $g^{-1}(x)$

17. A business purchases a piece of equipment for \$30,000. After 5 years, the equipment will be worth only \$4000. Write a linear equation that gives the value V of the equipment during the 5 years.

Ⓣ **18. Population** The projected populations P (in millions) of children under the age of 5 in the United States for selected years from 2010 to 2050 are shown in the table. Use a graphing utility to create a scatter plot of the data and find a linear model for the data. Let t represent the year, with $t = 10$ corresponding to 2010. *(Source: U.S. Census Bureau)*

Year	Population, P
2010	21.4
2015	22.4
2020	22.9
2025	23.5
2030	24.3
2035	25.3
2040	26.3
2045	27.2
2050	28.1

Table for 18

3 Polynomial and Rational Functions



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Many professional athletes sign contracts with sportswear companies to promote clothing lines and footwear. Quadratic functions are often used to model real-life phenomena, such as the profit from selling a line of sportswear. You can use a quadratic model to determine how much money a company can spend on advertising to obtain a certain profit. (See Section 3.4, Example 9.)

3.1 Quadratic Functions and Models
3.2 Polynomial Functions of Higher Degree
3.3 Polynomial Division
3.4 Real Zeros of Polynomial Functions
3.5 Complex Numbers
3.6 The Fundamental Theorem of Algebra
3.7 Rational Functions

Applications

Polynomial and rational functions are used to model and solve many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Liver Transplants, Exercise 60, page 268
- Cost of Dental Care, Exercise 59, page 301
- Health Care Spending, Exercise 78, page 333

Section 3.1

Quadratic Functions and Models

- Sketch the graph of a quadratic function and identify its vertex and intercepts.
- Find a quadratic function given its vertex and a point on its graph.
- Construct and use a quadratic model to solve an application problem.

The Graph of a Quadratic Function

In this and the next section, you will study the graphs of polynomial functions.

Definition of a Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x with degree n.**

Polynomial functions are classified by degree. Recall that the degree of a polynomial is the highest degree of its terms. For instance, the polynomial function given by

$$f(x) = a, \quad a \neq 0$$ Constant function

has degree 0 and is called a **constant function.** In Chapter 2, you learned that the graph of this type of function is a horizontal line. The polynomial function given by

$$f(x) = ax + b, \quad a \neq 0$$ Linear function

has degree 1 and is called a **linear function.** In Chapter 2, you learned that the graph of the linear function given by $f(x) = ax + b$ is a line whose slope is a and whose y-intercept is $(0, b)$. In this section, you will study second-degree polynomial functions, which are called **quadratic functions.**

For instance, each of the following functions is a quadratic function.

$$f(x) = x^2 + 6x + 2 \quad g(x) = 2(x + 1)^2 - 3 \quad h(x) = (x - 2)(x + 1)$$

Definition of a Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function of x given by

$$f(x) = ax^2 + bx + c$$ Quadratic function

is called a quadratic function.

The graph of a quadratic function is called a **parabola.** It is "$\cup$"-shaped and can open upward or downward.

All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point at which the axis intersects the parabola is the **vertex** of the parabola, as shown in Figure 3.1. If the leading coefficient is positive, the graph of $f(x) = ax^2 + bx + c$ is a parabola that opens upward, and if the leading coefficient is negative, the graph of $f(x) = ax^2 + bx + c$ is a parabola that opens downward.

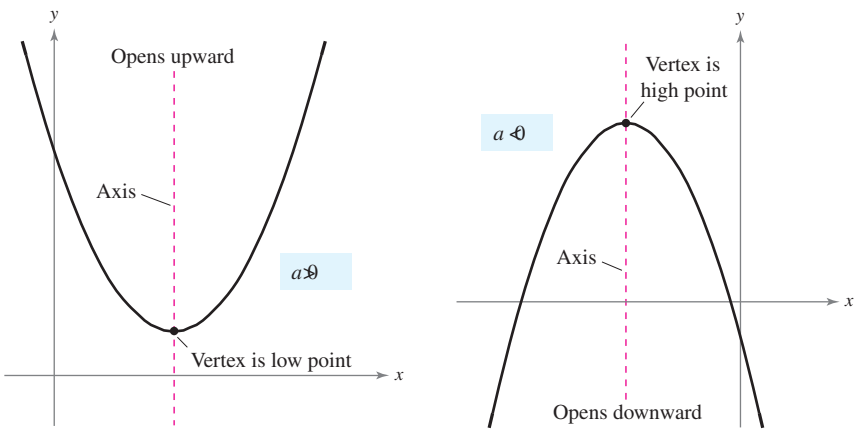
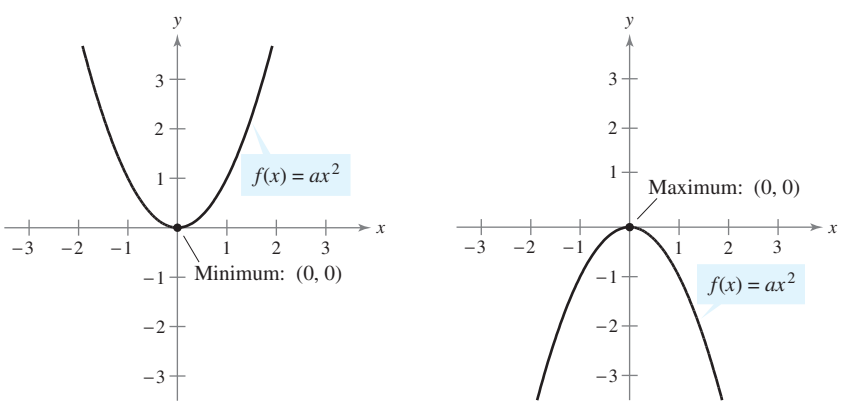


FIGURE 3.1

The simplest type of quadratic function is

$f(x) = ax^2$.

Its graph is a parabola whose vertex is $(0, 0)$. When $a > 0$, the vertex is the point with the *minimum* y-value on the graph, and when $a < 0$, the vertex is the point with the *maximum* y-value on the graph, as shown in Figure 3.2.



$a > 0$: Parabola opens upward $a < 0$: Parabola opens downward

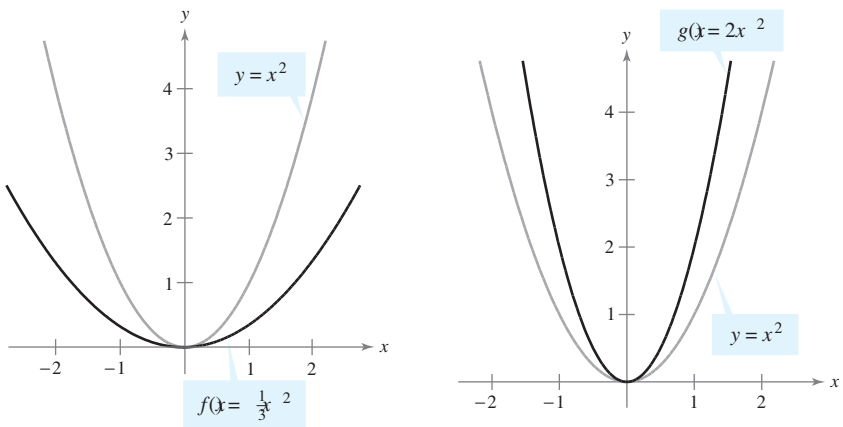
FIGURE 3.2

When sketching the graph of $f(x) = ax^2$, it is helpful to use the graph of $y = x^2$ as a reference, as discussed in Section 2.6. There you saw that when $a > 1$, the graph of $y = af(x)$ is a vertical stretch of the graph of $y = f(x)$. When $0 < a < 1$, the graph of $y = af(x)$ is a vertical shrink of the graph of $y = f(x)$. This is demonstrated again in Example 1.

Example 1 Sketching the Graph of a Quadratic Function

a. Compared with the graph of $y = x^2$, each output of $f(x) = \frac{1}{3}x^2$ vertically "shrinks" the graph by a factor of $\frac{1}{3}$, creating the wider parabola shown in Figure 3.3(a).

b. Compared with the graph of $y = x^2$, each output of $g(x) = 2x^2$ vertically "stretches" the graph by a factor of 2, creating the narrower parabola shown in Figure 3.3(b).



(a) (b)

FIGURE 3.3

✓CHECKPOINT 1

Sketch the graph of $f(x) = 4x^2$. Then compare the graph with the graph of $y = x^2$. ■

In Example 1, note that the coefficient a determines how widely the parabola given by $f(x) = ax^2$ opens. If $|a|$ is small, the parabola opens more widely than if $|a|$ is large.

Recall from Section 2.6 that the graphs of $y = f(x \pm c)$, $y = f(x) \pm c$, $y = -f(x)$, and $y = f(-x)$ are rigid transformations of the graph of $y = f(x)$. For instance, in Figure 3.4, notice how the graph of $y = x^2$ can be transformed to produce the graphs of $f(x) = -x^2 + 1$ and $g(x) = (x + 2)^2 - 3$.

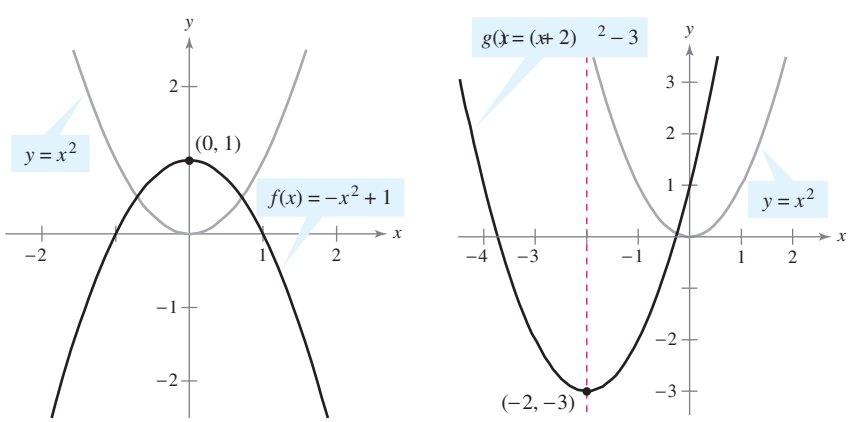


FIGURE 3.4

The Standard Form of a Quadratic Function

The **standard form** of a quadratic function is

$$f(x) = a(x - h)^2 + k.$$

This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola.

Standard Form of a Quadratic Function

The quadratic function given by

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

is said to be in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). If $a > 0$, the parabola opens upward, and if $a < 0$, the parabola opens downward.

To write a quadratic function in standard form, you can use the process of *completing the square,* as illustrated in Example 2.

Example 2 Graphing a Parabola in Standard Form

Sketch the graph of $f(x) = 2x^2 + 8x + 7$ and identify the vertex.

SOLUTION Begin by writing the quadratic function in standard form. The first step in completing the square is to factor out any coefficient of x^2 that is not 1.

$$\begin{aligned} f(x) &= 2x^2 + 8x + 7 && \text{Write original function.} \\ &= 2(x^2 + 4x) + 7 && \text{Factor 2 out of } x \text{ terms.} \\ &= 2(x^2 + 4x + 4 - 4) + 7 && \text{Add and subtract 4 within parentheses to complete the square.} \end{aligned}$$

$(4/2)^2$

After adding and subtracting 4 within the parentheses, you must now regroup the terms to form a perfect square trinomial. The -4 can be removed from inside the parentheses. But, because of the 2 outside the parentheses, you must multiply -4 by 2 as shown below.

$$\begin{aligned} f(x) &= 2(x^2 + 4x + 4) - 2(4) + 7 && \text{Regroup terms.} \\ &= 2(x^2 + 4x + 4) - 8 + 7 && \text{Simplify.} \\ &= 2(x + 2)^2 - 1 && \text{Standard form} \end{aligned}$$

From this form, you can see that the graph of f is a parabola that opens upward with vertex $(-2, -1)$. This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of $y = 2x^2$, as shown in Figure 3.5.

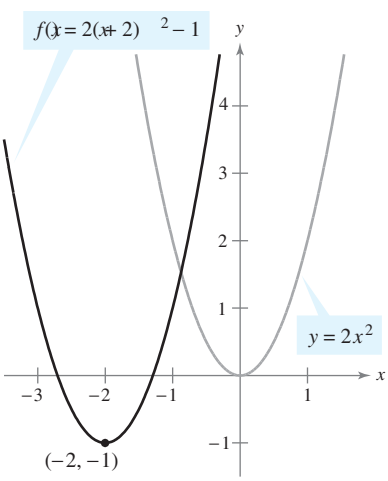


FIGURE 3.5

✓CHECKPOINT 2

Sketch the graph of $f(x) = 2x^2 - 12x + 20$ and identify the vertex. ■

Example 3 Graphing a Parabola in Standard Form

Sketch the graph of $f(x) = -x^2 + 6x - 8$ and identify the vertex.

SOLUTION As in Example 2, begin by writing the quadratic function in standard form.

$$\begin{aligned} f(x) &= -x^2 + 6x - 8 && \text{Write original function.} \\ &= -(x^2 - 6x) - 8 && \text{Factor } -1 \text{ out of } x \text{ terms.} \\ &= -(x^2 - 6x + 9 - 9) - 8 && \text{Add and subtract 9 within parentheses to complete the square.} \\ &\qquad\quad\ \ (-6/2)^2 \\ &= -(x^2 - 6x + 9) - (-9) - 8 && \text{Regroup terms.} \\ &= -(x - 3)^2 + 1 && \text{Standard form} \end{aligned}$$

So, the graph of f is a parabola that opens downward with vertex at $(3, 1)$, as shown in Figure 3.6.

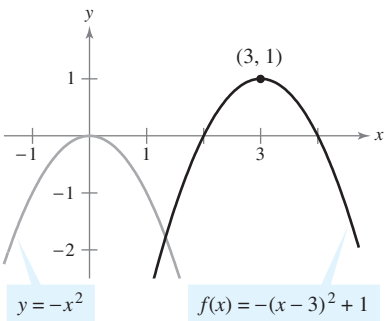


FIGURE 3.6

✓CHECKPOINT 3

Sketch the graph of $f(x) = -3x^2 + 12x + 1$ and identify the vertex. ■

Example 4 Finding an Equation of a Parabola

Find an equation of the parabola whose vertex is $(1, 2)$ and that passes through the point $(0, 0)$, as shown in Figure 3.7.

SOLUTION Because the parabola has a vertex at $(h, k) = (1, 2)$, the equation must have the form

$$f(x) = a(x - 1)^2 + 2. \qquad \text{Standard form}$$

Because the parabola passes through the point $(0, 0)$, it follows that when $x = 0$, $f(x)$ must equal 0. Substitute 0 for x and 0 for $f(x)$ to obtain the equation

$$0 = a(0 - 1)^2 + 2.$$

This equation can be solved easily for a, and you can see that

$$a = -2.$$

You can now write an equation of the parabola.

$$\begin{aligned} f(x) &= -2(x - 1)^2 + 2 && \text{Substitute for } a, h, \text{ and } k \text{ in standard form.} \\ &= -2x^2 + 4x && \text{Simplify.} \end{aligned}$$

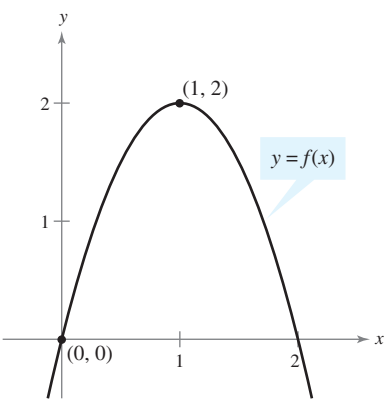


FIGURE 3.7

✓CHECKPOINT 4

Find an equation of the parabola whose vertex is $(3, 4)$ and that passes through the point $(2, 5)$. ■

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, you must solve the equation

$$ax^2 + bx + c = 0.$$

If the equation $ax^2 + bx + c$ does not factor, you can use the Quadratic Formula to determine the x-intercepts. Remember, however, that a parabola may have no x-intercepts.

TECHNOLOGY

Your graphing utility may have *minimum* and *maximum* features that determine the minimum and maximum points of the graph of a function. You can use these features to find the vertex of a parabola. For instructions on how to use the *minimum* and *maximum* features, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied*.

Applications

Many applications involve finding the maximum or minimum value of a quadratic function. By writing $f(x) = ax^2 + bx + c$ in standard form, you can determine that the vertex occurs at $x = -b/2a$.

Example 5 The Maximum Height of a Baseball

A baseball is hit 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The path of the baseball is given by

$$f(x) = -0.0032x^2 + x + 3$$

where $f(x)$ is the height of the baseball (in feet) and x is the distance from home plate (in feet). What is the maximum height reached by the baseball?

SOLUTION For this quadratic function, you have

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= -0.0032x^2 + x + 3. \end{aligned}$$

So, $a = -0.0032$ and $b = 1$. Because the function has a maximum when $x = -b/2a$, the baseball reaches its maximum height when it is

$$x = -\frac{b}{2a} = -\frac{1}{2(-0.0032)} = 156.25 \text{ feet}$$

from home plate. At this distance, the maximum height is

$$f(156.25) = -0.0032(156.25)^2 + 156.25 + 3 = 81.125 \text{ feet.}$$

The path of the baseball is shown in Figure 3.8.

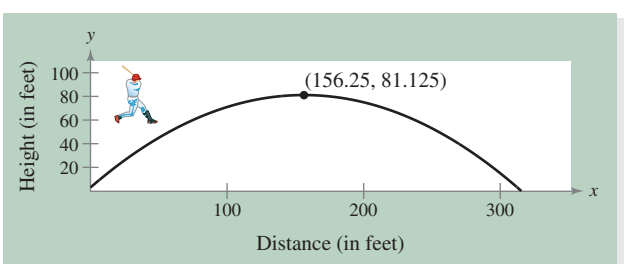


FIGURE 3.8

✓CHECKPOINT 5

In Example 5, suppose the baseball is hit at a velocity of 70 feet per second. The path of the baseball is given by $f(x) = -0.007x^2 + x + 4$, where $f(x)$ is the height of the baseball (in feet) and x is the distance from home plate (in feet). What is the maximum height reached by the baseball? ■

In Section 2.3 you plotted data points in the coordinate plane and estimated the best-fitting line. Fitting a quadratic model by this same process would be complicated. Most graphing utilities have a built-in statistical program that easily calculates the best-fitting quadratic model for a set of data points. Refer to the user's guide of your graphing utility for the required steps.

Example 6 Fitting a Quadratic Function to Data

Sparrow Population The table shows the numbers N of sparrows in a nature preserve for the years 1993 to 2008. Use a graphing utility to plot the data and find the quadratic model that best fits the data. Find the vertex of the graph of the quadratic model and interpret its meaning in the context of the problem. Let $x = 3$ represent the year 1993.

Year	1993	1994	1995	1996	1997	1998	1999	2000
x	3	4	5	6	7	8	9	10
Number, N	211	187	148	120	95	76	67	62

Year	2001	2002	2003	2004	2005	2006	2007	2008
x	11	12	13	14	15	16	17	18
Number, N	66	71	92	107	128	145	167	197

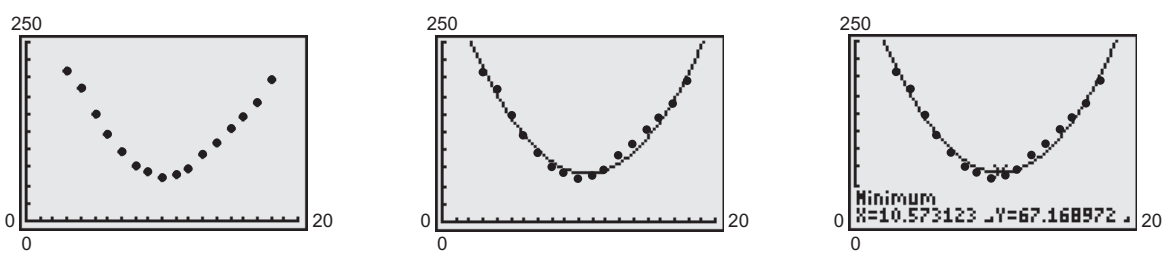
SOLUTION Begin by entering the data into your graphing utility and displaying the scatter plot. From the scatter plot that is shown in Figure 3.9(a) you can see that the points have a parabolic trend. Use the *quadratic regression* feature to find the quadratic function that best fits the data. The quadratic equation that best fits the data is

$$N = 2.53x^2 - 53.5x + 350, \quad 3 \le x \le 18.$$

Graph the data and the equation in the same viewing window, as shown in Figure 3.9(b). By using the *minimum* feature of your graphing utility, you can see that the vertex of the graph is approximately (10.6, 67.2), as shown in Figure 3.9(c). The vertex corresponds to the year in which the number of sparrows in the nature preserve was the least. So, in 2001, the number of sparrows in the nature preserve reached a minimum.

TECHNOLOGY

For instructions on how to use the *regression* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied*.



(a) (b) (c)

FIGURE 3.9

✓ CHECKPOINT 6

In Example 6, use the model to predict the number of sparrows in the nature preserve in 2011. ■

Example 7 Charitable Contributions

The percent of their income that a family gives to charities is related to their income level. For families with annual incomes between \$5000 and \$100,000, the percent P can be modeled by

$$P(x) = 0.0014x^2 - 0.1529x + 5.855, \quad 5 \le x \le 100$$

where x is the annual income (in thousands of dollars). Use the model to estimate the income that corresponds to the minimum percent of income given to charities.

SOLUTION One way to answer the question is to sketch the graph of the quadratic function, as shown in Figure 3.10. From this graph, it appears that the minimum percent corresponds to an income level of about \$55,000.

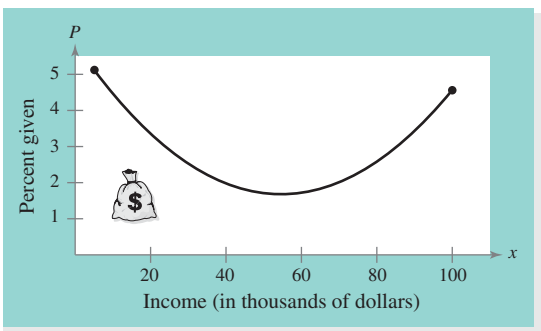


FIGURE 3.10

Another way to answer the question is to use the fact that the minimum point of the parabola occurs when $x = -b/2a$.

$$x = -\frac{b}{2a} = -\frac{-0.1529}{2(0.0014)} \approx 54.6$$

From this x-value, you can conclude that the minimum percent corresponds to an income level of about \$54,600.

✓CHECKPOINT 7

A manufacturer has daily production costs C (in dollars per unit) of $C = 0.15x^2 - 9x + 700$ where x is the number of units produced. How many units should be produced each day to yield a minimum cost per unit? ■

CONCEPT CHECK

1. Does the vertex of the graph of $f(x) = -3(x + 1)^2 - 1$ contain a minimum y-value or a maximum y-value? Explain.
2. Is the quadratic function given by $f(x) = 2(x - 1)^2 + 3$ written in standard form? Explain.
3. Write an equation of a parabola that is the graph of $y = x^2$ shifted right three units, downward one unit, and vertically stretched by a factor of 2.
4. The graph of the quadratic function given by $f(x) = a(x - 1)^2 + 3$ has two x-intercepts. What can you conclude about the value of a?

Skills Review 3.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.3 and 1.4.

In Exercises 1–4, solve the quadratic equation by factoring.

1. $2x^2 + 11x - 6 = 0$

2. $5x^2 - 12x - 9 = 0$

3. $3 + x - 2x^2 = 0$

4. $x^2 + 20x + 100 = 0$

In Exercises 5–10, use the Quadratic Formula to solve the quadratic equation.

5. $x^2 - 6x + 4 = 0$

6. $x^2 + 4x + 1 = 0$

7. $2x^2 - 16x + 25 = 0$

8. $3x^2 + 30x + 74 = 0$

9. $x^2 + 3x + 1 = 0$

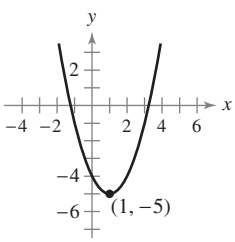
10. $x^2 + 3x - 3 = 0$

Exercises 3.1

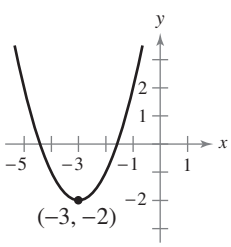
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, match the quadratic function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

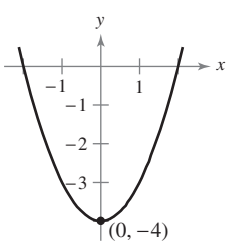
(a)



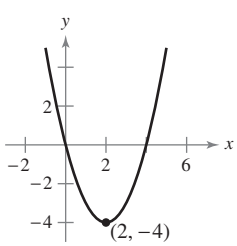
(b)



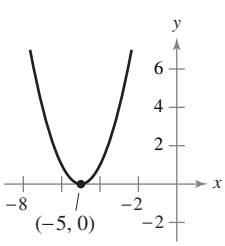
(c)



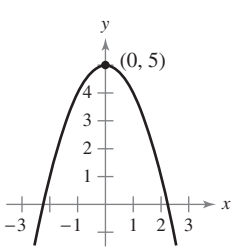
(d)



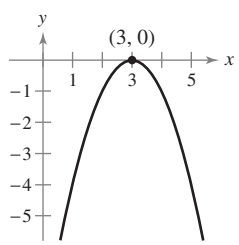
(e)



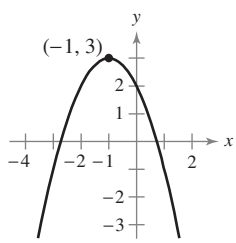
(f)



(g)



(h)



1. $f(x) = -(x - 3)^2$

2. $f(x) = (x + 5)^2$

3. $f(x) = x^2 - 4$

4. $f(x) = 5 - x^2$

5. $f(x) = (x + 3)^2 - 2$

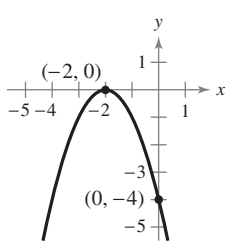
6. $f(x) = (x - 1)^2 - 5$

7. $f(x) = -(x + 1)^2 + 3$

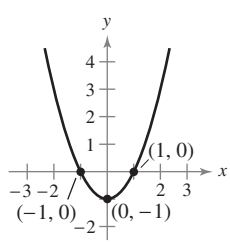
8. $f(x) = (x - 2)^2 - 4$

In Exercises 9–14, find an equation of the parabola.

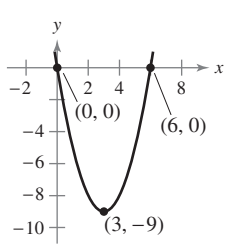
9.



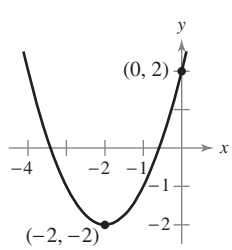
10.



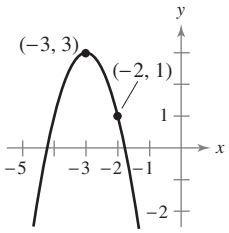
11.



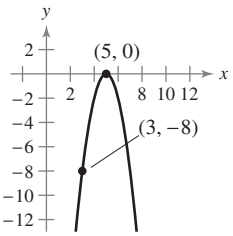
12.



13.

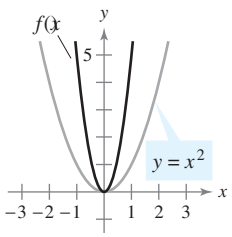


14.

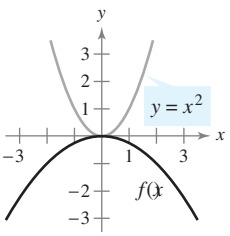


In Exercises 15–18, compare the graph of the quadratic function with the graph of $y = x^2$.

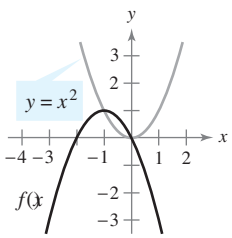
15. $f(x) = 5x^2$



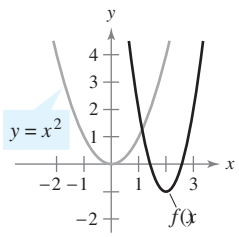
16. $f(x) = -\frac{1}{4}x^2$



17. $f(x) = -(x + 1)^2 + 1$



18. $f(x) = 3(x - 2)^2 - 1$



In Exercises 19–36, sketch the graph of the quadratic function. Identify the vertex and intercepts.

19. $f(x) = 3x^2$

20. $f(x) = -2x^2$

21. $f(x) = 16 - x^2$

22. $h(x) = x^2 - 9$

23. $f(x) = (x + 5)^2 - 6$

24. $f(x) = (x - 6)^2 + 3$

25. $g(x) = x^2 + 2x + 1$

26. $h(x) = x^2 - 4x + 2$

27. $f(x) = -(x^2 + 2x - 3)$

28. $f(x) = -(x^2 + 6x - 3)$

29. $f(x) = x^2 - x + \frac{5}{4}$

30. $f(x) = x^2 + 3x + \frac{1}{4}$

31. $f(x) = -x^2 + 2x + 5$

32. $f(x) = -x^2 - 4x + 1$

33. $h(x) = 4x^2 - 4x + 21$

34. $f(x) = 2x^2 - x + 1$

35. $f(x) = \frac{1}{4}(x^2 - 16x + 32)$

36. $g(x) = \frac{1}{2}(x^2 + 4x - 2)$

In Exercises 37–40, find an equation of the parabola that has the indicated vertex and whose graph passes through the given point.

37. Vertex: $(2, -1)$; point: $(4, -3)$

38. Vertex: $(-3, 5)$; point: $(-6, -1)$

39. Vertex: $(5, 12)$; point: $(7, 15)$

40. Vertex: $(-2, -2)$; point: $(-1, 0)$

In Exercises 41–46, find two quadratic functions whose graphs have the given x-intercepts. Find one function whose graph opens upward and another whose graph opens downward. (There are many correct answers.)

41. $(2, 0), (-1, 0)$

42. $(-4, 0), (0, 0)$

43. $(0, 0), (10, 0)$

44. $(4, 0), (8, 0)$

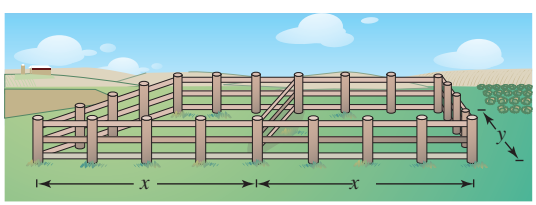
45. $(-3, 0), \left(-\frac{1}{2}, 0\right)$

46. $\left(-\frac{5}{2}, 0\right), (2, 0)$

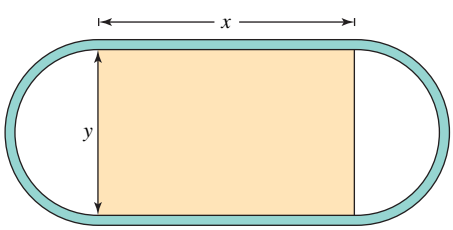
47. Optimal Area The perimeter of a rectangle is 200 feet. Let x represent the width of the rectangle. Write a quadratic function for the area of the rectangle in terms of its width. Find the vertex of the graph of the quadratic function and interpret its meaning in the context of the problem.

48. Optimal Area The perimeter of a rectangle is 540 feet. Let x represent the width of the rectangle. Write a quadratic function for the area of the rectangle in terms of its width. Find the vertex of the graph of the quadratic function and interpret its meaning in the context of the problem.

49. Optimal Area A rancher has 1200 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What measurements will produce a maximum enclosed area?



50. Optimal Area An indoor physical-fitness room consists of a rectangular region with a semicircle on each end (see figure). The perimeter of the room is to be a 200-meter running track. What measurements will produce a maximum area of the rectangle?



Optimal Revenue In Exercises 51 and 52, find the number of units that produces a maximum revenue. The revenue R is measured in dollars and x is the number of units produced.

51. $R = 1000x - 0.02x^2$

52. $R = 80x - 0.0001x^2$

53. Optimal Cost A manufacturer of lighting fixtures has daily production costs C (in dollars per unit) of

$$C(x) = 800 - 10x + 0.25x^2$$

where x is the number of units produced. How many fixtures should be produced each day to yield a minimum cost per unit?

54. Optimal Profit The profit P (in dollars) for a manufacturer of sound systems is given by

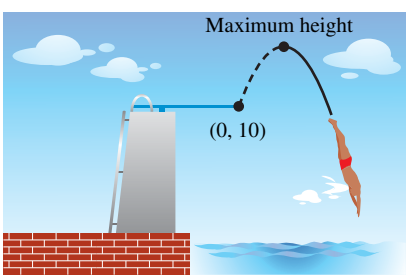
$$P(x) = -0.0003x^2 + 150x - 375{,}000$$

where x is the number of units produced. What production level will yield a maximum profit?

55. Maximum Height of a Diver The path of a diver is given by

$$y = -\frac{4}{9}x^2 + \frac{24}{9}x + 10$$

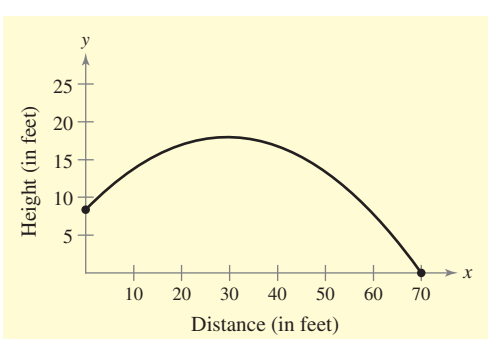
where y is the height (in feet) and x is the horizontal distance from the end of the diving board (in feet) (see figure). Use a graphing utility and the *trace* or *maximum* feature to find the maximum height of the diver.



56. Maximum Height The winning men's shot put in the 2004 Summer Olympics was recorded by Yuriy Belonog of Ukraine. The path of his winning toss is approximately given by

$$y = -0.011x^2 + 0.65x + 8.3$$

where y is the height of the shot (in feet) and x is the horizontal distance (in feet). Use a graphing utility and the *trace* or *maximum* feature to find the length of the winning toss and the maximum height of the shot.



57. Cable TV Subscribers The table shows the average numbers S (in millions) of basic cable subscribers for the years 1995 to 2005. *(Source: Kagan Research, LLC)*

Year	1995	1996	1997	1998
Subscribers, S	60.6	62.3	63.6	64.7

Year	1999	2000	2001	2002
Subscribers, S	65.5	66.3	66.7	66.5

Year	2003	2004	2005
Subscribers, S	66.1	65.7	65.3

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model from part (b) in the same viewing window as the scatter plot.

(d) Use the graph of the model from part (c) to estimate when the number of basic cable subscribers was the greatest. Does this result agree with the actual data?

58. Price of Gold The table shows the average annual prices P (in dollars) of gold for the years 1996 to 2005. *(Source: World Gold Council)*

Year	1996	1997	1998	1999
Price of gold, P	387.82	330.98	294.12	278.55

Year	2000	2001	2002	2003
Price of gold, P	279.10	272.67	309.66	362.91

Year	2004	2005
Price of gold, P	409.17	444.47

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model from part (b) in the same viewing window as the scatter plot.

(d) Use the graph of the model from part (c) to estimate when the price of gold was the lowest. Does this result agree with the actual data?

59. Tuition and Fees The table shows the average values of tuition and fees F (in dollars) for in-state students at public institutions of higher education in the years 1996 to 2005. *(Source: U.S. National Center for Educational Statistics)*

Year	1996	1997	1998	1999	2000
Tuition and fees, F	2179	2271	2360	2430	2506

Year	2001	2002	2003	2004	2005
Tuition and fees, F	2562	2700	2903	3319	3638

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model from part (b) in the same viewing window as the scatter plot of the data.

(d) Use the graph of the model from part (c) to predict the average value of tuition and fees in 2008.

60. Liver Transplants The table shows the numbers L of liver transplant procedures performed in the United States in the years 1995 to 2005. *(Source: U.S. Department of Health and Human Services)*

Year	1995	1996	1997	1998
Transplants, L	3818	3918	4005	4356

Year	1999	2000	2001	2002
Transplants, L	4586	4816	5177	5326

Year	2003	2004	2005
Transplants, L	5671	6168	6444

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model from part (b) in the same viewing window as the scatter plot of the data.

(d) Use the graph of the model from part (c) to predict the number of liver transplant procedures performed in 2008.

61. Regression Problem Let x be the number of units (in tens of thousands) that a computer company produces and let $p(x)$ be the profit (in hundreds of thousands of dollars). The table shows the profits for different levels of production.

Units, x	2	4	6	8	10
Profit, $p(x)$	270.5	307.8	320.1	329.2	325.0

Units, x	12	14	16	18	20
Profit, $p(x)$	311.2	287.8	254.8	212.2	160.0

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for $p(x)$.

(c) Use a graphing utility to graph your model for $p(x)$ with the scatter plot of the data.

(d) Find the vertex of the graph of the model from part (c). Interpret its meaning in the context of the problem.

(e) With these data and this model, the profit begins to decrease. Discuss how it is possible for production to increase and profit to decrease.

62. Regression Problem Let x be the angle (in degrees) at which a baseball is hit with no spin at an initial speed of 40 meters per second and let $d(x)$ be the distance (in meters) the ball travels. The table shows the distances for the different angles at which the ball is hit. *(Source: The Physics of Sports)*

Angle, x	10	15	30	36	42
Distance, $d(x)$	58.3	79.7	126.9	136.6	140.6

Angle, x	44	45	48	54	60
Distance, $d(x)$	140.9	140.9	139.3	132.5	120.5

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for $d(x)$.

(c) Use a graphing utility to graph your model for $d(x)$ with the scatter plot of the data.

(d) Find the vertex of the graph of the model from part (c). Interpret its meaning in the context of the problem.

63. Write the quadratic function

$$f(x) = ax^2 + bx + c$$

in standard form to verify that the vertex occurs at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Section 3.2

Polynomial Functions of Higher Degree

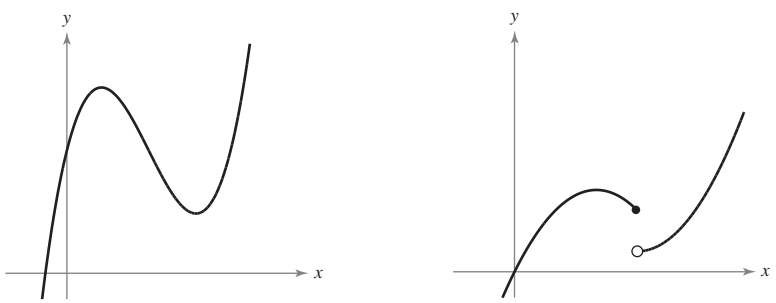
- Sketch a transformation of a monomial function.
- Determine right-hand and left-hand behavior of graphs of polynomial functions.
- Find the real zeros of a polynomial function.
- Sketch the graph of a polynomial function.
- Use a polynomial model to solve an application problem.

Graphs of Polynomial Functions

In this section, you will study basic characteristics of the graphs of polynomial functions. The first characteristic is that the graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, as shown in Figure 3.11(a). Functions with graphs that are not continuous are not polynomial functions, as shown in Figure 3.11(b).

STUDY TIP

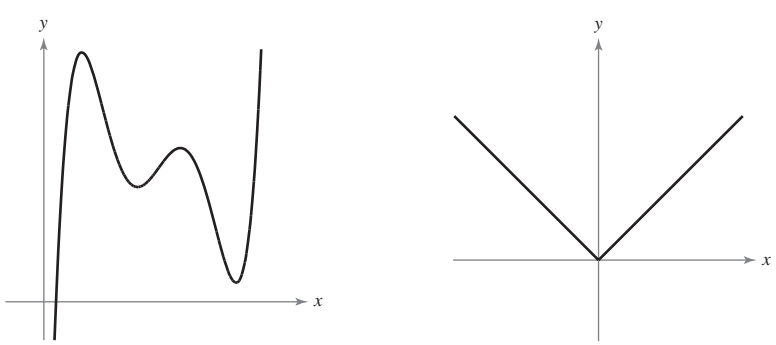
The graphs of polynomial functions of degree greater than 2 are more complicated than those of degree 0, 1, or 2. However, using the characteristics presented in this section, together with point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand.* Of course, if you have a graphing utility, the task is easier.



(a) Continuous (b) Not continuous

FIGURE 3.11

The second characteristic is that the graph of a polynomial function has only smooth, rounded turns, as shown in Figure 3.12(a). A polynomial function cannot have a sharp turn, as shown in Figure 3.12(b).

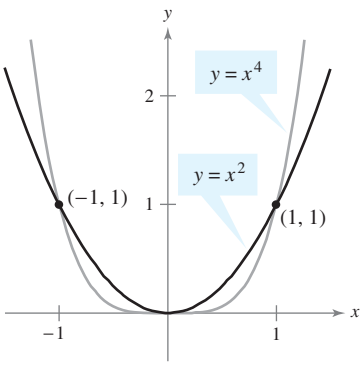


(a) Polynomial functions have smooth, rounded turns.

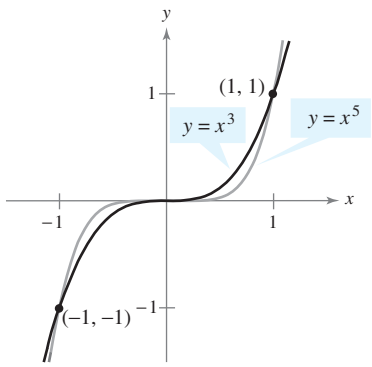
(b) Polynomial functions cannot have sharp turns.

FIGURE 3.12

The polynomial functions that have the simplest graphs are monomial functions of the form $f(x) = x^n$, where n is an integer greater than zero. From Figure 3.13, you can see that when n is *even*, the graph is similar to the graph of $f(x) = x^2$, and when n is *odd*, the graph is similar to the graph of $f(x) = x^3$. Moreover, the greater the value of n, the flatter the graph near the origin.



(a) If n is even, the graph of $y = x^n$ touches the axis at the x-intercept.



(b) If n is odd, the graph of $y = x^n$ crosses the axis at the x-intercept.

FIGURE 3.13

Example 1 Sketching Transformations of Monomial Functions

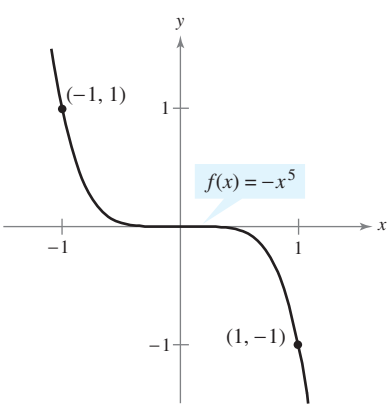
Sketch the graph of each function.

a. $f(x) = -x^5$ **b.** $h(x) = (x + 1)^4$

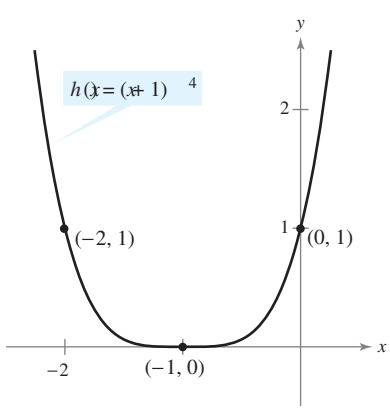
SOLUTION

a. Because the degree of $f(x) = -x^5$ is odd, its graph is similar to the graph of $y = x^3$. In Figure 3.14(a), note that the negative coefficient has the effect of reflecting the graph about the x-axis.

b. The graph of $h(x) = (x + 1)^4$ is a left shift, by one unit, of the graph of $y = x^4$, as shown in Figure 3.14(b).



(a)



(b)

FIGURE 3.14

✓CHECKPOINT 1

Sketch the graph of $f(x) = (x - 3)^3$. ■

The Leading Coefficient Test

In Example 1, note that both graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial function eventually rises or falls can be determined by the function's degree (even or odd) and by its leading coefficient (positive or negative), as indicated in the **Leading Coefficient Test.**

DISCOVERY

For each function below, identify the degree of the function and whether it is even or odd. Identify the leading coefficient, and whether the leading coefficient is positive or negative. Use a graphing utility to graph each function. Describe the relationship between the function's degree and the sign of its leading coefficient and the right-hand and left-hand behavior of the graph of the function.

a. $y = x^3 - 2x^2 - x + 1$

b. $y = 2x^5 + 2x^2 - 5x + 1$

c. $y = -2x^5 - x^2 + 5x + 3$

d. $y = -x^3 + 5x - 2$

e. $y = 2x^2 + 3x - 4$

f. $y = x^4 - 3x^2 + 2x - 1$

g. $y = x^2 + 3x + 2$

h. $y = -x^6 - x^2 - 5x + 4$

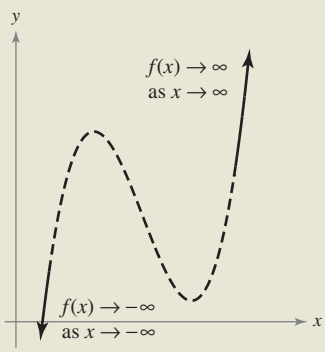
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function given by

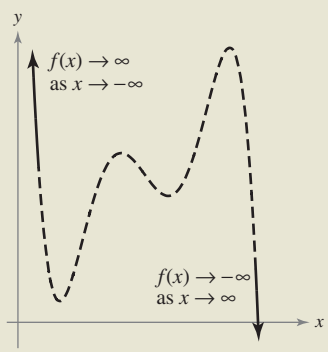
$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

eventually rises or falls in the following manner.

1. When n is *odd:*

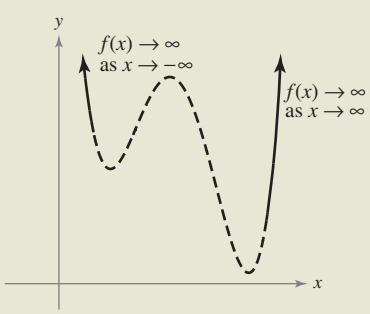


If the leading coefficient is positive ($a_n > 0$), the graph falls to the left and rises to the right.

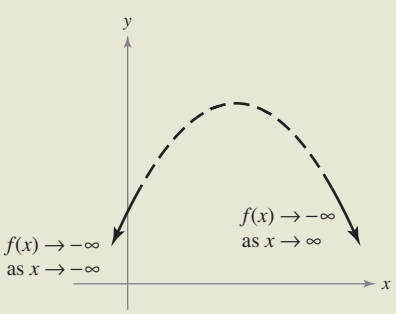


If the leading coefficient is negative ($a_n < 0$), the graph rises to the left and falls to the right.

2. When n is *even:*



If the leading coefficient is positive ($a_n > 0$), the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), the graph falls to the left and right.

The dashed portions of the graphs indicate that the test determines *only* the right-hand and left-hand behavior of the graph.

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

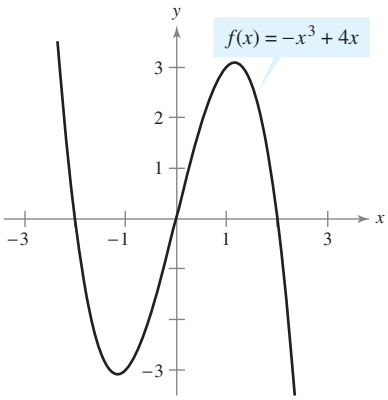


FIGURE 3.15

Example 2 Applying the Leading Coefficient Test

Describe the right-hand and left-hand behavior of the graph of $f(x) = -x^3 + 4x$.

SOLUTION Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 3.15.

✓CHECKPOINT 2

Describe the right-hand and left-hand behavior of the graph of $f(x) = -2x^4 + x$. ■

In Example 2, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts, relative minima, and relative maxima, must be determined by other tests. For example, later you will use the number of real zeros of a polynomial function to determine how many times the graph of the function crosses the x-axis.

STUDY TIP

The function in Example 3 part (a) is a fourth-degree polynomial function. This can also be referred to as a *quartic* function.

Example 3 Applying the Leading Coefficient Test

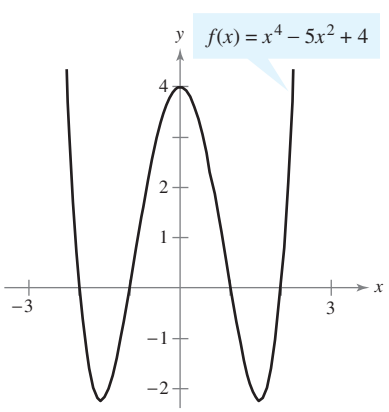
Describe the right-hand and left-hand behavior of the graph of each function.

a. $f(x) = x^4 - 5x^2 + 4$ **b.** $f(x) = x^5 - x$

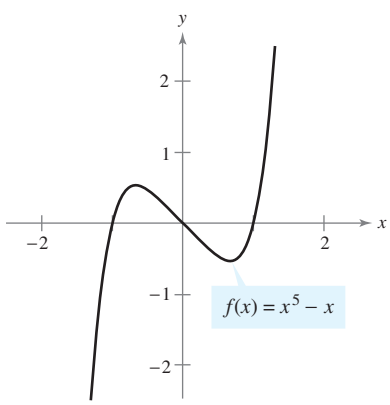
SOLUTION

a. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 3.16(a).

b. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 3.16(b).



(a)



(b)

FIGURE 3.16

✓CHECKPOINT 3

Describe the right-hand and left-hand behavior of the graph of $f(x) = x^4 - 4x^2$. ■

Real Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true. Remember that the **zeros** of a function are the x-values for which the function is zero.

1. The graph of f has, at most, $n - 1$ *turning points*. Turning points are points at which the graph changes from increasing to decreasing or vice versa. For instance, the graph of $f(x) = x^4 - 1$ has at most $4 - 1 = 3$ turning points.
2. The function f has, at most, n real zeros. For instance, the function given by $f(x) = x^4 - 1$ has at most $n = 4$ real zeros. (You will study this result in detail in Section 3.6 on the Fundamental Theorem of Algebra.)

Finding the zeros of polynomial functions is one of the most important problems in algebra. There is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros, and in other cases you can use information about the zeros of a function to help sketch its graph.

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, then the following statements are equivalent.

1. $x = a$ is a zero of the function f.
2. $x = a$ is a solution of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a factor of the polynomial $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of f.

In the equivalent statements above, notice that finding zeros of polynomial functions is closely related to factoring and finding x-intercepts.

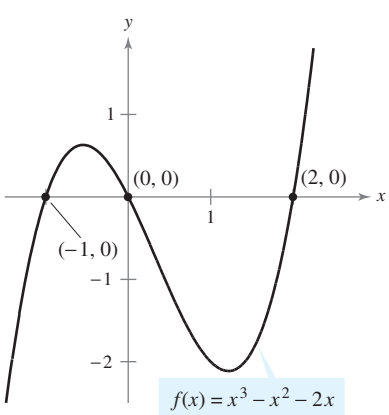


FIGURE 3.17

Example 4 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

SOLUTION By factoring, you obtain the following.

$$\begin{aligned} f(x) &= x^3 - x^2 - 2x && \text{Write original function.} \\ &= x(x^2 - x - 2) && \text{Remove common monomial factor.} \\ &= x(x - 2)(x + 1) && \text{Factor completely.} \end{aligned}$$

So, the real zeros are $x = 0$, $x = 2$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(2, 0)$, and $(-1, 0)$, as shown in Figure 3.17. Note that the graph in the figure has two turning points. This is consistent with the fact that the graph of a third-degree polynomial function can have *at most* $3 - 1 = 2$ turning points.

✓CHECKPOINT 4

Find all real zeros of $f(x) = x^2 - 4$. ■

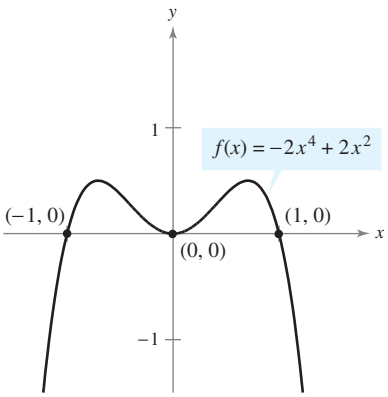


FIGURE 3.18

Example 5 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = -2x^4 + 2x^2$.

SOLUTION In this case, the polynomial factors as follows.

$$f(x) = -2x^2(x^2 - 1) = -2x^2(x - 1)(x + 1)$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 3.18. Note that the graph in the figure has three turning points, which is consistent with the fact that the graph of a fourth-degree polynomial function can have *at most* three turning points.

✓CHECKPOINT 5

Find all real zeros of $f(x) = x^3 - x$. ■

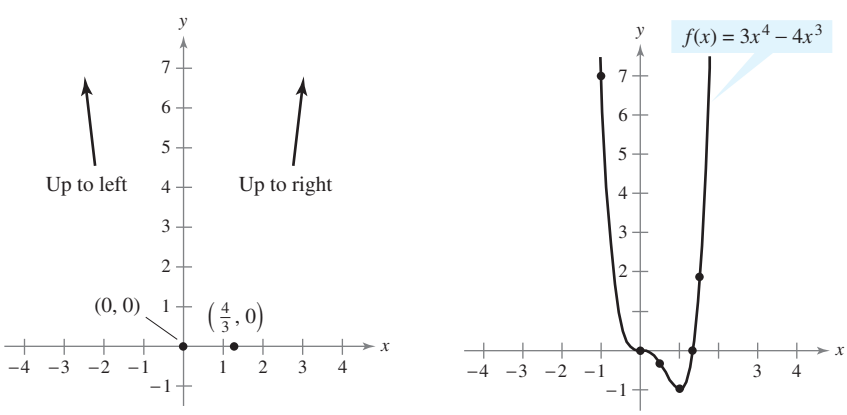
In Example 5, the real zero arising from $-2x^2 = 0$ is called a **repeated zero.** In general, a factor $(x - a)^k$ yields a repeated zero $x = a$ of **multiplicity** k. If k is odd, the graph *crosses* the x-axis at $x = a$. If k is even, the graph *touches* (but does not cross) the x-axis at $x = a$. This is illustrated in Figure 3.18.

Example 6 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = 3x^4 - 4x^3$.

SOLUTION Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and right, as shown in Figure 3.19(a). By factoring $f(x) = 3x^4 - 4x^3$ as $f(x) = x^3(3x - 4)$, you can see that the zeros of f are $x = 0$ and $x = \frac{4}{3}$ (both of odd multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. To sketch the graph by hand, find a few additional points, as shown in the table. Then plot the points and draw a continuous curve through the points to complete the graph, as shown in Figure 3.19(b). If you are unsure of the shape of a portion of a graph, plot some additional points.

x	$f(x)$
-1	7
0.5	-0.3125
1	-1
1.5	1.6875



(a) (b)

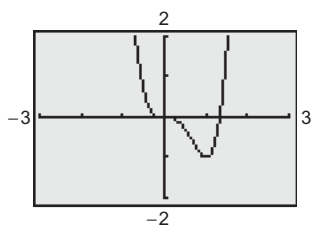
FIGURE 3.19

✓CHECKPOINT 6

Sketch the graph of $f(x) = 2x^3 - 3x^2$. ■

TECHNOLOGY

Example 6 uses an algebraic approach to describe the graph of the function. A graphing utility is a valuable complement to this approach. Remember that when using a graphing utility, it is important that you find a viewing window that shows all important parts of the graph. For instance, the graph below shows the important parts of the graph of the function in Example 6.



Application

Example 7 **Charitable Contributions Revisited**

Example 7 in Section 3.1 discussed the model

$$P(x) = 0.0014x^2 - 0.1529x + 5.855, \quad 5 \le x \le 100$$

where P is the percent of annual income given to charities and x is the annual income (in thousands of dollars). Note that this model gives the charitable contributions as a *percent* of annual income. To find the *amount* that a family gives to charity, you can multiply the given model by the income $1000x$ (and divide by 100 to change from percent to decimal form) to obtain

$$A(x) = 0.014x^3 - 1.529x^2 + 58.55x, \quad 5 \le x \le 100$$

where A represents the amount of charitable contributions (in dollars). Sketch the graph of this function and use the graph to estimate the annual salary of a family that gives \$1000 a year to charities.

SOLUTION Because the leading coefficient is positive and the degree is odd, you know that the graph eventually falls to the left and rises to the right. To sketch the graph by hand, find a few points, as shown in the table. Then plot the points and complete the graph, as shown in Figure 3.20.

x	5	25	45	65	86	100
$A(x)$	256.28	726.88	814.28	1190.48	2527.48	4565.00

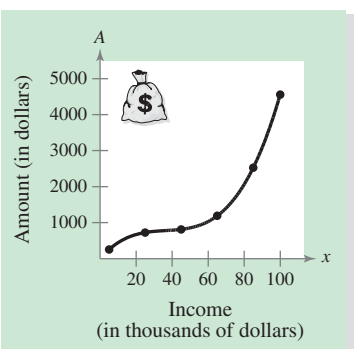


FIGURE 3.20

From the graph you can see that an annual contribution of \$1000 corresponds to an annual income of about \$59,000.

✓CHECKPOINT 7

The median prices P (in thousands of dollars) of new privately owned homes in housing developments from 1998 to 2008 can be approximated by the model

$$P(t) = 0.139t^3 - 4.42t^2 + 51.1t - 39$$

where t represents the year, with $t = 8$ corresponding to 1998. Sketch the graph of this function and use the graph to estimate the year in which the median price of a new privately owned home was about \$195,000. ■

CONCEPT CHECK

1. Write a function whose graph is a downward shift, by one unit, and a reflection in the x-axis, of the graph of $y = x^4$.
2. The graph of a fifth-degree polynomial function rises to the left. Describe the right-hand behavior of the graph.
3. Name a zero of the function f given that $(x - 5)$ is a factor of the polynomial $f(x)$.
4. Does the graph of every function with real zeros cross the x-axis? Explain.

Skills Review 3.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.6, 1.3, 1.4, and 1.5.

In Exercises 1–6, factor the expression completely.

1. $12x^2 + 7x - 10$

2. $25x^3 - 60x^2 + 36x$

3. $12z^4 + 17z^3 + 5z^2$

4. $y^3 + 125$

5. $x^3 + 3x^2 - 4x - 12$

6. $x^3 + 2x^2 + 3x + 6$

In Exercises 7–10, find all real solutions of the equation.

7. $5x^2 + 8 = 0$

8. $x^2 - 6x + 4 = 0$

9. $4x^2 + 4x - 11 = 0$

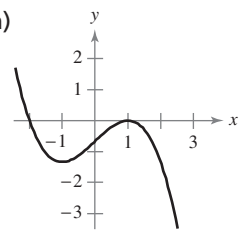
10. $x^4 - 18x^2 + 81 = 0$

Exercises 3.2

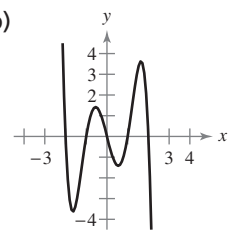
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, match the polynomial function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

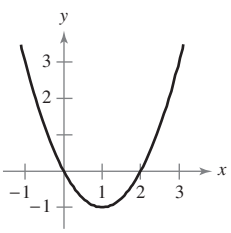
(a)



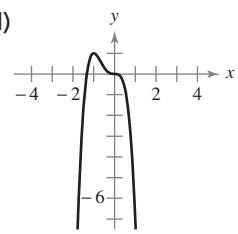
(b)



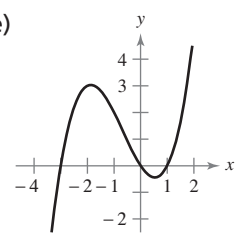
(c)



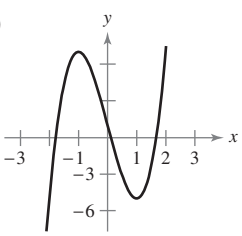
(d)



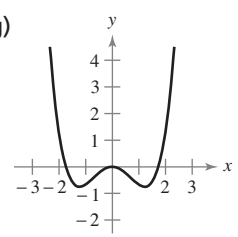
(e)



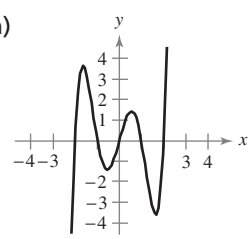
(f)



(g)



(h)



1. $f(x) = \frac{1}{2}(x^3 + 2x^2 - 3x)$

2. $f(x) = x^2 - 2x$

3. $f(x) = \frac{1}{3}x^4 - x^2$

4. $f(x) = -3x^4 - 4x^3$

5. $f(x) = 3x^3 - 9x + 1$

6. $f(x) = x^5 - 5x^3 + 4x$

7. $f(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$

8. $f(x) = -x^5 + 5x^3 - 4x$

In Exercises 9–12, use the graph of $y = x^3$ to sketch the graph of the function.

9. $f(x) = x^3 - 2$

10. $f(x) = (x + 3)^3$

11. $f(x) = (x + 1)^3 - 4$

12. $f(x) = -(x - 2)^3 + 2$

In Exercises 13–16, use the graph of $y = x^4$ to sketch the graph of the function.

13. $f(x) = (x + 3)^4$

14. $f(x) = x^4 - 4$

15. $f(x) = 3 - x^4$

16. $f(x) = \frac{1}{2}(x - 1)^4$

In Exercises 17–26, describe the right-hand and left-hand behavior of the graph of the polynomial function.

17. $f(x) = -x^3 + 1$

18. $f(x) = \frac{1}{3}x^3 + 5x$

19. $g(x) = 6 - 4x^2 + x - 3x^5$

20. $f(x) = 2x^5 - 5x + 7.5$

21. $f(x) = 4x^8 - 2$

22. $h(x) = 1 - x^6$

23. $f(x) = 2 + 5x - x^2 - x^3 + 2x^4$

24. $f(x) = \dfrac{3x^4 - 2x + 5}{4}$

25. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$

26. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

In Exercises 27–30, determine (a) the maximum number of turning points of the graph of the function and (b) the maximum number of real zeros of the function.

27. $f(x) = x^2 - 4x + 1$ **28.** $f(x) = -3x^4 + 1$

29. $f(x) = -x^5 + x^4 - x$ **30.** $f(x) = 2x^3 + x^2 + 1$

Algebraic and Graphical Approaches In Exercises 31–46, find all real zeros of the function algebraically. Then use a graphing utility to confirm your results.

31. $f(x) = 9 - x^2$ **32.** $f(x) = x^2 - 25$

33. $h(t) = t^2 + 8t + 16$ **34.** $f(x) = x^2 - 12x + 36$

35. $f(x) = \frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}$ **36.** $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$

37. $f(x) = 2x^2 + 4x + 6$

38. $g(x) = -5(x^2 + 2x - 4)$

39. $f(t) = t^3 - 4t^2 + 4t$

40. $f(x) = x^4 - x^3 - 20x^2$

41. $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$

42. $f(x) = \frac{1}{3} - \frac{1}{3}x^2$

43. $f(x) = 2x^4 - 2x^2 - 40$

44. $g(t) = t^5 - 6t^3 + 9t$

45. $f(x) = x^3 - 3x^2 + 2x - 6$

46. $f(x) = x^3 - 4x^2 - 25x + 100$

Analyzing a Graph In Exercises 47–58, analyze the graph of the function algebraically and use the results to sketch the graph *by hand*. Then use a graphing utility to confirm your sketch.

47. $f(x) = \frac{2}{3}x + 5$ **48.** $h(x) = -\frac{3}{4}x + 2$

49. $f(t) = \frac{1}{2}(t^2 - 4t - 1)$

50. $g(x) = -x^2 + 10x - 16$

51. $f(x) = 4x^2 - x^3$ **52.** $f(x) = 1 - x^3$

53. $f(x) = x^3 - 9x$ **54.** $f(x) = \frac{1}{4}x^4 - 2x^2$

55. $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)^2$

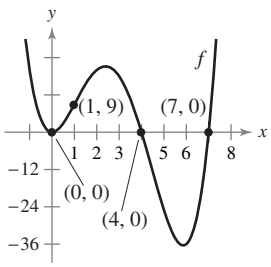
56. $f(x) = x(x - 2)^2(x + 1)$

57. $f(x) = 1 - x^6$ **58.** $g(x) = 1 - (x + 1)^6$

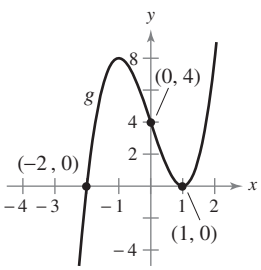
59. Modeling Polynomials Sketch the graph of a polynomial function that is of fourth degree, has a zero of multiplicity 2, and has a negative leading coefficient. Sketch another graph under the same conditions but with a positive leading coefficient.

60. Modeling Polynomials Sketch the graph of a polynomial function that is of fifth degree, has a zero of multiplicity 2, and has a negative leading coefficient. Sketch another graph under the same conditions but with a positive leading coefficient.

61. Modeling Polynomials Determine the equation of the fourth-degree polynomial function f whose graph is shown.



62. Modeling Polynomials Determine the equation of the third-degree polynomial function g whose graph is shown.



63. Credit Cards The numbers of active American Express cards C (in millions) in the years 1997 to 2006 are shown in the table. *(Source: American Express)*

Year	1997	1998	1999	2000	2001
Cards, C	42.7	42.7	46.0	51.7	55.2

Year	2002	2003	2004	2005	2006
Cards, C	57.3	60.5	65.4	71.0	78.0

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use what you know about end behavior and the scatter plot from part (a) to predict the sign of the leading coefficient of a quartic model for C.

(c) Use the *regression* feature of a graphing utility to find a quartic model for C. Does your model agree with your answer from part (b)?

(d) Use a graphing utility to graph the model from part (c). Use the graph to predict the year in which the number of active American Express cards would be about 92 million. Is your prediction reasonable?

64. Population The immigrant population P (in millions) living in the United States at the beginning of each decade from 1900 to 2000 is shown in the table. *(Source: Center of Immigration Studies)*

Year	1900	1910	1920	1930
Population, P	10.3	13.5	13.9	14.2

Year	1940	1950	1960	1970
Population, P	11.6	10.3	9.7	9.6

Year	1980	1990	2000
Population, P	14.1	19.8	30.0

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 0$ correspond to 1900.

(b) Use what you know about end behavior and the scatter plot from part (a) to predict the sign of the leading coefficient of a cubic model for P.

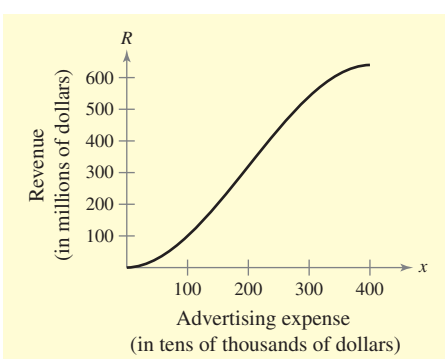
(c) Use the *regression* feature of a graphing utility to find a cubic model for P. Does your model agree with your answer from part (b)?

(d) Use a graphing utility to graph the model from part (c). Use the graph to predict the year in which the immigrant population will be about 45 million. Is your prediction reasonable?

65. Advertising Expenses The total revenue R (in millions of dollars) for a soft-drink company is related to its advertising expenses by the function

$$R = \frac{1}{50{,}000}(-x^3 + 600x^2), \quad 0 \le x \le 400$$

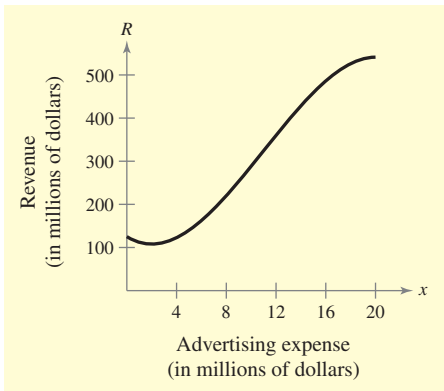
where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of R to estimate the point on the graph at which the function is increasing most rapidly. This point is called the *point of diminishing returns* because any expenditure above this amount will yield less return per dollar invested in advertising.



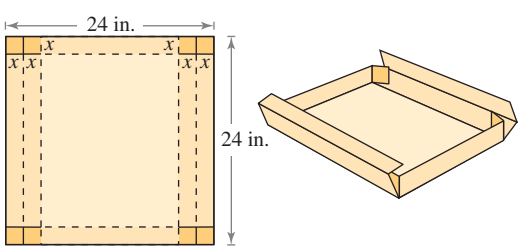
66. Advertising Expenses The total revenue R (in millions of dollars) for a hotel corporation is related to its advertising expenses by the function

$$R = -0.148x^3 + 4.889x^2 - 17.778x + 125.185,$$
$$0 \le x \le 20$$

where x is the amount spent on advertising (in millions of dollars). Use the graph of R to estimate the point on the graph at which the function is increasing most rapidly. This point is called the *point of diminishing returns* because any expenditure above this amount will yield less return per dollar invested in advertising.



67. Maximum Value An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is to be done by cutting equal squares from the corners and folding along the dashed lines shown in the figure. Verify that the volume of the box is given by the function $V(x) = 8x(6 - x)(12 - x)$. Determine the domain of the function V. Then sketch a graph of the function and estimate the value of x for which $V(x)$ is maximum.



68. Comparing Graphs Use a graphing utility to graph the functions given by $f(x) = x^2$, $g(x) = x^4$, and $h(x) = x^6$. Do the three functions have a common shape? Are their graphs identical? Why or why not?

69. Comparing Graphs Use a graphing utility to graph the functions given by $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$. Do the three functions have a common shape? Are their graphs identical? Why or why not?

Section 3.3

Polynomial Division

- Divide one polynomial by a second polynomial using long division.
- Simplify a rational expression using long division.
- Use synthetic division to divide two polynomials.
- Use the Remainder Theorem and synthetic division to evaluate a polynomial.
- Use the Factor Theorem to factor a polynomial.
- Use polynomial division to solve an application problem.

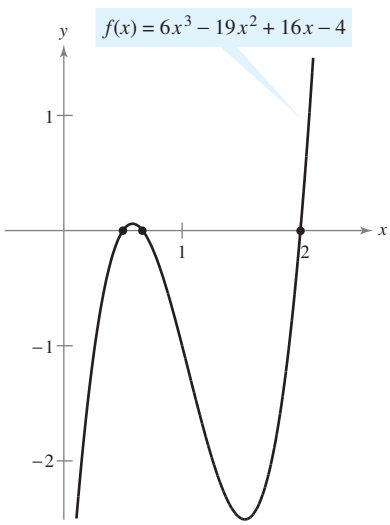


FIGURE 3.21

Long Division of Polynomials

In this section, you will study two procedures for *dividing* polynomials. These procedures are especially valuable in factoring polynomials and finding the zeros of polynomial functions. To begin, suppose you are given the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice that a zero of f occurs at $x = 2$, as shown in Figure 3.21. Because $x = 2$ is a zero of the polynomial function f, you know that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 2) \cdot q(x).$$

To find $q(x)$, you can use **long division,** as illustrated in Example 1.

Example 1 Long Division of Polynomials

Divide the polynomial $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

SOLUTION

$$\begin{array}{rl}
 & 6x^2 - 7x + 2 \\
x - 2 \,\overline{)}\; & 6x^3 - 19x^2 + 16x - 4 \\
 & \underline{6x^3 - 12x^2} \qquad \text{Multiply: } 6x^2 \text{ by } x - 2. \\
 & -7x^2 + 16x \qquad \text{Subtract and bring down } 16x. \\
 & \underline{-7x^2 + 14x} \qquad \text{Multiply: } -7x \text{ by } x - 2 \\
 & 2x - 4 \qquad \text{Subtract and bring down } -4. \\
 & \underline{2x - 4} \qquad \text{Multiply: } 2 \text{ by } x - 2. \\
 & 0 \qquad \text{Subtract.}
\end{array}$$

From this division, you can conclude that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

and by factoring the quadratic $6x^2 - 7x + 2$, you have

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2).$$

✓CHECKPOINT 1

Divide $x^3 - 6x^2 + 5x + 12$ by $x - 4$, and use the result to factor the polynomial completely. ■

Note that the factorization shown in Example 1 agrees with the graph shown in Figure 3.21 in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

In Example 1, $x - 2$ is a factor of the polynomial $6x^3 - 19x^2 + 16x - 4$, and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, when you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$\begin{array}{rl} & x + 2 \quad \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow \; x + 1 \;\overline{)}\; & x^2 + 3x + 5 \quad \leftarrow \text{Dividend} \\ & \underline{x^2 + \;\; x} \\ & \quad 2x + 5 \\ & \underline{2x + 2} \\ & \quad\;\; 3 \quad \leftarrow \text{Remainder} \end{array}$$

In fractional form, you can write this result as follows.

$$\underbrace{\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}}}_{} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overbrace{3}^{\text{Remainder}}}{\underbrace{x + 1}_{\text{Divisor}}}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following well-known theorem called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

Dividend = $f(x)$, Divisor = $d(x)$, Quotient = $q(x)$, Remainder = $r(x)$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ **divides evenly** into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Example 2 Long Division of Polynomials

Divide $x^3 - 1$ by $x - 1$.

SOLUTION Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$\begin{array}{rl} & x^2 + x + 1 \\ x - 1 \,\overline{)}\, & x^3 + 0x^2 + 0x - 1 \\ & \underline{x^3 - x^2} \\ & x^2 + 0x \\ & \underline{x^2 - x} \\ & x - 1 \\ & \underline{x - 1} \\ & 0 \end{array}$$

Insert $0x^2$ and $0x$.
Multiply x^2 by $x - 1$.
Subtract and bring down $0x$.
Multiply x by $x - 1$.
Subtract and bring down -1.
Multiply 1 by $x - 1$.
Subtract.

So, $x - 1$ divides evenly into $x^3 - 1$ and you can write

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1.$$

✓CHECKPOINT 2

Divide $x^3 + 8$ by $x + 2$. ■

You can check the result of a division problem by multiplying. For instance, in Example 2, try checking that $(x - 1)(x^2 + x + 1) = x^3 - 1$.

Example 3 Long Division of Polynomials

Divide $2x^4 + 4x^3 - 5x^2 + 3x - 2$ by $x^2 + 2x - 3$.

SOLUTION

$$\begin{array}{rl} & 2x^2 + 1 \\ x^2 + 2x - 3 \,\overline{)}\, & 2x^4 + 4x^3 - 5x^2 + 3x - 2 \\ & \underline{2x^4 + 4x^3 - 6x^2} \\ & x^2 + 3x - 2 \\ & \underline{x^2 + 2x - 3} \\ & x + 1 \end{array}$$

Multiply $2x^2$ by $x^2 + 2x - 3$.
Subtract and bring down $3x - 2$.
Multiply 1 by $x^2 + 2x - 3$.
Subtract.

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. So, you can write

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

✓CHECKPOINT 3

Divide

$$5x^4 + 10x^3 - 7x^2 + 28x - 39$$

by

$$x^2 + 2x - 4.$$ ■

Synthetic Division

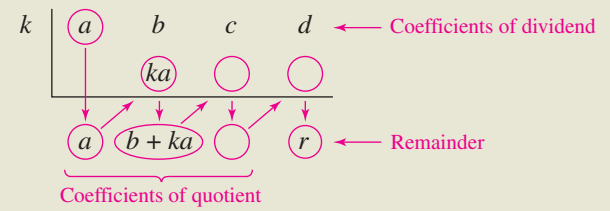
There is a nice shortcut for long division of polynomials when dividing by divisors of the form $x - k$. This shortcut is called **synthetic division.** We summarize the pattern for synthetic division of a cubic polynomial as follows. (The pattern for higher-degree polynomials is similar.)

STUDY TIP

This algorithm for synthetic division works only for divisors of the form $x - k$. Remember that $x + k = x - (-k)$.

Synthetic Division (for a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.

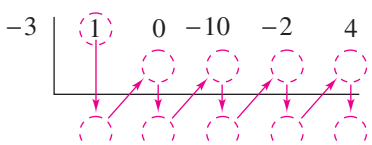


Vertical pattern: Add terms in columns.
Diagonal pattern: Multiply results by k.

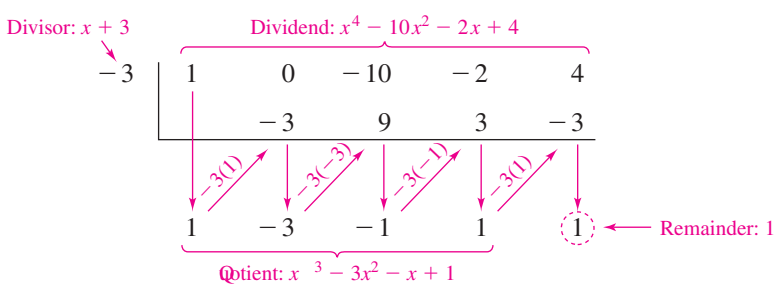
Example 4 Using Synthetic Division

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

SOLUTION You should set up the array as follows. Note that a zero is included for the missing x^3-term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.



So, you have $\dfrac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \dfrac{1}{x + 3}$.

✓CHECKPOINT 4

Use synthetic division to divide $2x^3 - 7x^2 - 80$ by $x - 5$. ■

Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the Remainder Theorem.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function f at $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$, as illustrated in Example 5.

TECHNOLOGY

Remember, you can also evaluate a function with your graphing utility by entering the function in the equation editor and using the *table* feature in ASK mode. For instructions on how to use the *table* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied*.

Example 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function when $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

SOLUTION Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9.$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

CHECK

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 = -9 \end{aligned}$$

✓CHECKPOINT 5

Use the Remainder Theorem to evaluate $f(x) = 4x^3 + 6x^2 + 4x + 5$ when $x = -1$. ■

Another important theorem is the Factor Theorem, which is stated below.

Factor Theorem

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

You can think of the Factor Theorem as stating that if $(x - k)$ is a factor of $f(x)$, then $f(k) = 0$. Conversely, if $f(k) = 0$, then $(x - k)$ is a factor of $f(x)$.

Example 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of the polynomial

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

SOLUTION Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array}$$

0 remainder, so $f(2) = 0$ and $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

Quadratic: $2x^2 + 5x + 3$

0 remainder, so $f(-3) = 0$ and $(x + 3)$ is a factor.

Because the resulting quadratic expression factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Note that this factorization implies that f has four real zeros:

$$2, -3, -\tfrac{3}{2}, \text{ and } -1.$$

This is confirmed by the graph of f, which is shown in Figure 3.22.

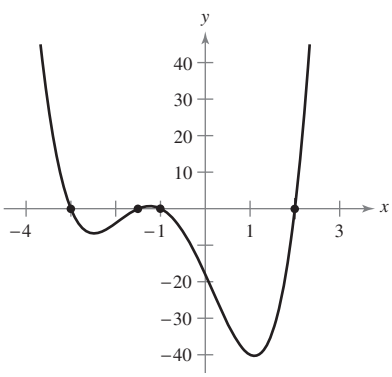


FIGURE 3.22

✓CHECKPOINT 6

Show that $(x + 2)$ and $(x + 4)$ are factors of the polynomial $f(x) = x^4 + 6x^3 + 7x^2 - 6x - 8$. Then find the remaining factors of $f(x)$. ■

Uses of The Remainder in Synthetic Division

The remainder r obtained in the synthetic division of $f(x)$ by $x - k$ provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, if you find that $x - k$ divides evenly into $f(x)$ (with no remainder), try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

Application

Example 7 Tax Liability

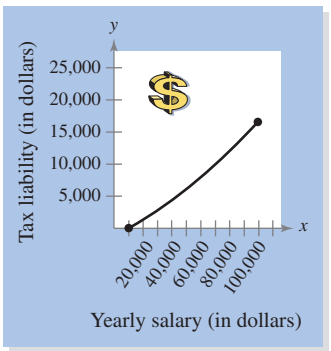


FIGURE 3.23

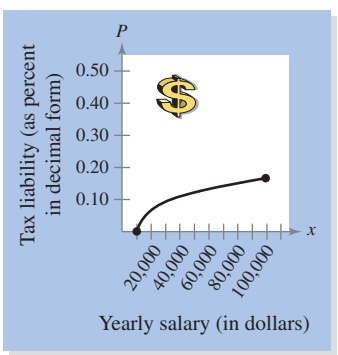


FIGURE 3.24

The 2005 federal income tax liability for an employee who was single and claimed no dependents is given by the function

$$y = 0.00000066x^2 + 0.113x - 1183, \quad 10{,}000 \le x \le 100{,}000$$

where y represents the tax liability (in dollars) and x represents the employee's yearly salary (in dollars) (see Figure 3.23). *(Source: U.S. Department of the Treasury)*

a. Find a function that gives the tax liability as a *percent* of the yearly salary.

b. Graph the function from part (a). What conclusions can you make from the graph?

SOLUTION

a. Because the yearly salary is given by x and the tax liability is given by y, the percent (in decimal form) of yearly salary that the person owes in federal income tax is

$$P = \frac{y}{x}$$

$$= \frac{0.00000066x^2 + 0.113x - 1183}{x}$$

$$= 0.00000066x + 0.113 - \frac{1183}{x}.$$

b. The graph of the function P is shown in Figure 3.24. From the graph you can see that as a person's yearly salary increases, the percent that he or she must pay in federal income tax also increases.

✓CHECKPOINT 7

Using the function P from part (a) of Example 7, what percent of a \$39,000 yearly salary does a person owe in federal income tax? ■

CONCEPT CHECK

1. How should you write the dividend $x^5 - 3x + 10$ to apply the Division Algorithm?
2. Describe and correct the error in using synthetic division to divide $x^3 + 4x^2 - x - 4$ by $x + 4$.

$$\begin{array}{r|cccc} 4 & 1 & 4 & -1 & -4 \\ & & 4 & 32 & 124 \\ \hline & 1 & 8 & 31 & 120 \end{array}$$

3. A factor of the polynomial $f(x)$ is $(x - 3)$. What is the value of $f(3)$?
4. A fourth-degree polynomial is divided by a first-degree polynomial. What is the degree of the quotient?

Skills Review 3.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.5 and 0.6.

In Exercises 1–4, write the expression in standard polynomial form.

1. $(x - 1)(x^2 + 2) + 5$

2. $(x^2 - 3)(2x + 4) + 8$

3. $(x^2 + 1)(x^2 - 2x + 3) - 10$

4. $(x + 6)(2x^3 - 3x) - 5$

In Exercises 5–10, factor the polynomial.

5. $x^2 - 4x + 3$

6. $8x^2 - 24x - 80$

7. $3x^2 + 2x - 5$

8. $9x^2 - 24x + 16$

9. $4x^3 - 10x^2 + 6x$

10. $6x^3 + 7x^2 + 2x$

Exercises 3.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–18, use long division to divide.

	Dividend	*Divisor*
1.	$3x^2 - 7x + 4$	$x - 1$
2.	$5x^2 - 17x - 12$	$x - 4$
3.	$2x^2 + 10x + 12$	$x + 3$
4.	$2x^2 + x - 11$	$x + 5$
5.	$2x^3 + 6x^2 - x - 3$	$2x^2 - 1$
6.	$3x^3 - 12x^2 - 2x + 8$	$3x^2 - 2$
7.	$x^4 + 5x^3 + 6x^2 - x - 2$	$x + 2$
8.	$x^4 + 2x^3 - 3x^2 - 8x - 4$	$x^2 - 4$
9.	$7x + 3$	$x + 4$
10.	$8x - 5$	$2x + 3$
11.	$6x^3 + 10x^2 + x + 8$	$2x^2 + 1$
12.	$2x^3 - 8x^2 + 3x - 9$	$x - 4$
13.	$x^3 - 27$	$x^2 - 1$
14.	$x^3 - 9$	$x^2 + 1$
15.	$x^3 - 4x^2 + 5x - 2$	$x + 2$
16.	$x^3 - x^2 + 2x - 8$	$x - 2$
17.	$2x^5 - 8x^3 + 4x - 1$	$x^2 - 2x + 1$
18.	$x^5 + 7$	$x^3 - 1$

In Exercises 19–36, use synthetic division to divide.

	Dividend	*Divisor*
19.	$2x^3 + 5x^2 - 7x + 20$	$x + 4$
20.	$3x^3 - 23x^2 - 12x + 32$	$x - 8$
21.	$4x^3 - 9x + 8x^2 - 18$	$x + 2$
22.	$9x^3 - 16x - 18x^2 + 32$	$x - 2$
23.	$-x^3 + 75x - 250$	$x + 10$
24.	$3x^3 - 16x^2 - 72$	$x - 6$
25.	$x^4 - 4x^3 - 7x^2 + 22x + 24$	$x + 3$
26.	$6x^4 - 15x^3 - 11x$	$x + 2$
27.	$10x^4 - 50x^3 - 800$	$x - 6$
28.	$x^5 - 13x^4 - 120x + 80$	$x + 3$
29.	$2x^5 - 30x^3 - 37x + 13$	$x - 4$
30.	$5x^3$	$x + 3$
31.	$-3x^4$	$x - 2$
32.	$2x^5$	$x + 3$
33.	$5 - 3x + 2x^2 - x^3$	$x + 1$
34.	$180x - x^4$	$x - 6$
35.	$4x^3 + 16x^2 - 23x - 15$	$x + \frac{1}{2}$
36.	$3x^3 - 4x^2 + 5$	$x - \frac{3}{2}$

In Exercises 37–44, write the function in the form

$f(x) = (x - k)q(x) + r$

for the given value of k, and demonstrate that $f(k) = r$.

37. $f(x) = x^3 + x^2 - 12x + 20, \quad k = 3$

38. $f(x) = x^3 - 2x^2 - 15x + 7, \quad k = -4$

39. $f(x) = 3x^3 + 2x^2 + 5x - 2, \quad k = \frac{1}{3}$

40. $f(x) = 4x^4 + 6x^3 + 4x^2 - 5x + 13, \quad k = -\frac{1}{2}$

41. $f(x) = x^3 + 2x^2 - 3x - 12, \quad k = \sqrt{3}$

42. $f(x) = x^3 + 3x^2 - 7x - 6, \quad k = -\sqrt{2}$

43. $f(x) = 2x^3 + x^2 - 14x - 10, \quad k = 1 + \sqrt{3}$

44. $f(x) = 3x^3 - 19x^2 + 27x - 7, \quad k = 3 - \sqrt{2}$

In Exercises 45–50, use synthetic division to find each function value.

45. $f(x) = 2x^5 - 3x^2 - 4x - 1$

(a) $f(-2)$ (b) $f(-4)$

(c) $f(1)$ (d) $f(3)$

46. $g(x) = x^6 - 4x^4 + 3x^2 + 2$

(a) $g(2)$ (b) $g(-4)$

(c) $g(7)$ (d) $g(-1)$

47. $f(x) = 2x^3 - 3x^2 + 8x - 14$

(a) $f(2)$ (b) $f(-1)$

(c) $f(1.1)$ (d) $f(3)$

48. $f(x) = 3x^4 - 7x^3 + 5x - 12$

(a) $f(1)$ (b) $f(4)$

(c) $f(-3)$ (d) $f(-1.2)$

49. $f(x) = 1.2x^3 - 0.5x^2 - 2.1x - 2.4$

(a) $f(2)$ (b) $f(-6)$

(c) $f\left(\frac{2}{3}\right)$ (d) $f(1)$

50. $f(x) = 0.4x^4 - 1.6x^3 + 0.7x^2 - 2$

(a) $f(1)$ (b) $f(-2)$

(c) $f(5)$ (d) $f(-10)$

In Exercises 51–56, (a) verify the given factors of $f(x)$, (b) find the remaining factor of $f(x)$, (c) use your results to write the complete factorization of $f(x)$, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.

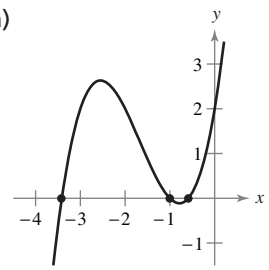
	Function	*Factors*
51.	$f(x) = x^3 - 12x - 16$	$(x + 2), (x - 4)$
52.	$f(x) = x^3 - 28x - 48$	$(x + 4), (x - 6)$
53.	$f(x) = 3x^3 + 10x^2 - 27x - 10$	$(3x + 1), (x - 2)$
54.	$f(x) = 5x^3 - 11x^2 - 38x + 8$	$(5x - 1), (x - 4)$
55.	$f(x) = x^3 + 2x^2 - 3x - 6$	$\left(x - \sqrt{3}\right), (x + 2)$
56.	$f(x) = x^3 + 2x^2 - 2x - 4$	$\left(x - \sqrt{2}\right), (x + 2)$

57. You divide a polynomial by another polynomial. The remainder is zero. What conclusion(s) can you make?

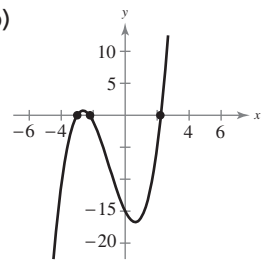
58. Suppose that the remainder obtained in a polynomial division by $x - k$ is zero. How is the divisor related to the graph of the dividend?

In Exercises 59–64, match the function with its graph and use the result to find all real solutions of $f(x) = 0$. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

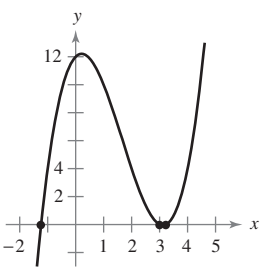
(a)



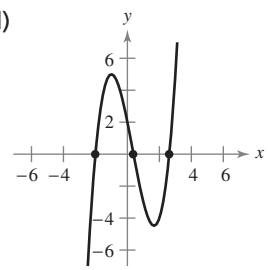
(b)



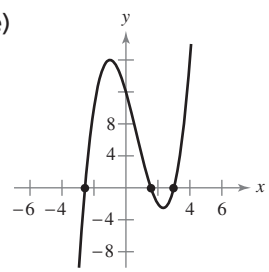
(c)



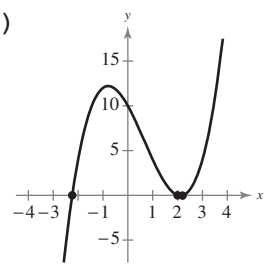
(d)



(e)



(f)



59. $f(x) = x^3 - 2x^2 - 7x + 12$

60. $f(x) = x^3 - x^2 - 5x + 2$

61. $f(x) = x^3 + 5x^2 + 6x + 2$

62. $f(x) = x^3 - 5x^2 + 2x + 12$

63. $f(x) = x^3 + 3x^2 - 5x - 15$

64. $f(x) = x^3 - 2x^2 - 5x + 10$

65. Modeling Polynomials A third-degree polynomial function f has real zeros -1, 2, and $\frac{10}{3}$. Find two different polynomial functions, one with a positive leading coefficient and one with a negative leading coefficient, that could be f. How many different polynomial functions are possible for f?

66. Modeling Polynomials A fourth-degree polynomial function g has real zeros -2, 0, 1, and 5. Find two different polynomial functions, one with a positive leading coefficient and one with a negative leading coefficient, that could be g. How many different polynomial functions are possible for g?

In Exercises 67–74, simplify the rational expression.

67. $\dfrac{x^3 - 10x^2 + 31x - 30}{x - 3}$

68. $\dfrac{x^3 + 15x^2 + 68x + 96}{x + 4}$

69. $\dfrac{6x^3 + x^2 - 21x - 10}{2x + 1}$

70. $\dfrac{3x^3 - 5x^2 - 34x + 24}{3x - 2}$

71. $\dfrac{x^4 - 5x^3 + 14x^2 - 120x}{x^2 + x + 20}$

72. $\dfrac{x^4 + x^3 + 3x^2 + 10x}{x^2 - x + 5}$

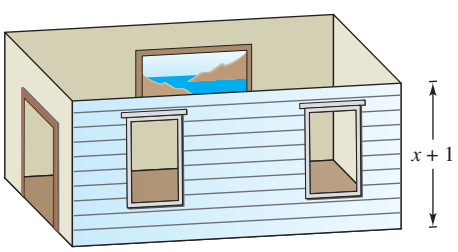
73. $\dfrac{x^4 + 4x^3 - 6x^2 - 36x - 27}{x^2 - 9}$

74. $\dfrac{x^4 + x^3 - 13x^2 - x + 12}{x^2 + x - 12}$

75. Examination Room A rectangular examination room in a veterinary clinic has a volume of

$x^3 + 11x^2 + 34x + 24$

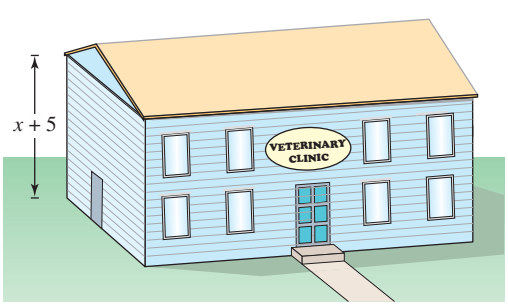
cubic feet. The height of the room is $x + 1$ feet (see figure). Find the number of square feet of floor space in the examination room.



76. Veterinary Clinic A rectangular veterinary clinic has a volume of

$x^3 + 55x^2 + 650x + 2000$

cubic feet (the space in the attic is not counted). The height of the clinic is $x + 5$ feet (see figure). Find the number of square feet of floor space *on the first floor* of the clinic.

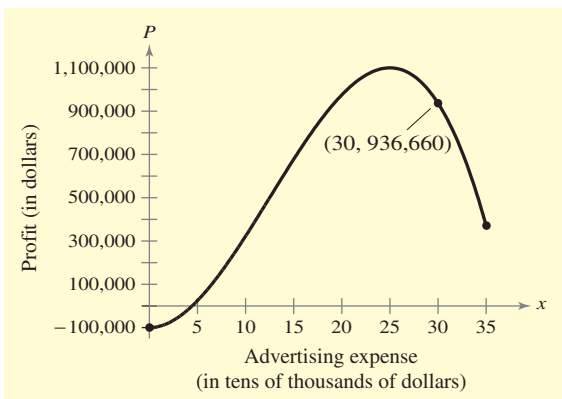


77. Profit A company making fishing poles estimated that the profit P (in dollars) from selling a particular fishing pole was

$P = -140.75x^3 + 5348.3x^2 - 76{,}560, \quad 0 \le x \le 35$

where x was the advertising expense (in tens of thousands of dollars). For this fishing pole, the advertising expense was \$300,000 ($x = 30$) and the profit was \$936,660.

(a) From the graph shown in the figure, it appears that the company could have obtained the same profit by spending less on advertising. Use the graph to estimate another amount the company could have spent on advertising that would have produced the same profit.



(b) Use synthetic division to confirm the result of part (a) algebraically.

78. Profit A company that produces calculators estimated that the profit P (in dollars) from selling a particular model of calculator was

$P = -152x^3 + 7545x^2 - 169{,}625, \quad 0 \le x \le 45$

where x was the advertising expense (in tens of thousands of dollars). For this model of calculator, the advertising expense was \$400,000 ($x = 40$) and the profit was \$2,174,375.

(a) Use a graphing utility to graph the profit function.

(b) Could the company have obtained the same profit by spending less on advertising? Explain your reasoning.

79. Writing Briefly explain what it means for a divisor to divide evenly into a dividend.

80. Writing Briefly explain how to check polynomial division, and justify your answer. Give an example.

Exploration In Exercises 81 and 82, find the constant c such that the denominator will divide evenly into the numerator.

81. $\dfrac{x^3 + 4x^2 - 3x + c}{x - 5}$ **82.** $\dfrac{x^5 - 2x^2 + x + c}{x + 2}$

Section 3.4

Real Zeros of Polynomial Functions

- Find all possible rational zeros of a function using the Rational Zero Test.
- Find all real zeros of a function.
- Approximate the real zeros of a polynomial function using the Intermediate Value Theorem.
- Approximate the real zeros of a polynomial function using a graphing utility.
- Apply techniques for approximating real zeros to solve an application problem.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial function (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

STUDY TIP

When the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

The Rational Zero Test

If the polynomial function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

has *integer* coefficients, then every rational zero of f has the form

$$\text{Rational zeros} = \frac{\text{a factor of the constant term } a_0}{\text{a factor of the leading coefficient } a_n} = \frac{p}{q}$$

where p and q have no common factors other than 1.

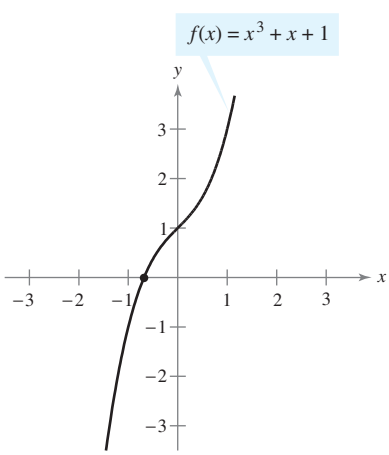


FIGURE 3.25

Make a list of *possible rational zeros*. Then use a trial-and-error method to determine which, if any, are actual zeros of the polynomial function.

Example 1 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^3 + x + 1$.

SOLUTION Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term, 1 and -1. By testing these possible zeros, you can see that neither checks.

$$f(1) = (1)^3 + 1 + 1 = 3 \qquad f(-1) = (-1)^3 + (-1) + 1 = -1$$

So, you can conclude that the given function has *no* rational zeros. Note from the graph of f in Figure 3.25 that f does have one real zero (between -1 and 0). By the Rational Zero Test, you know that this real zero is *not* a rational number.

✓CHECKPOINT 1

Find the rational zeros of $f(x) = x^3 + 2x^2 + 1$. ■

Example 2 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of

$$f(x) = x^4 - x^3 + x^2 - 3x - 6.$$

SOLUTION Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

Test each possible rational zero. The test shows $x = -1$ and $x = 2$ are the only two rational zeros of the function.

✓CHECKPOINT 2

Find the rational zeros of $f(x) = x^4 + 2x^3 + x^2 - 4$. ■

If the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways: (1) a programmable calculator can be used to speed up the calculations; (2) a graph, created either by hand or with a graphing utility, can give a good estimate of the locations of the zeros; and (3) synthetic division can be used to test the possible rational zeros.

TECHNOLOGY

There are several ways to use your graphing utility to locate the zeros of a polynomial function after listing the possible rational zeros. You can use the *table* feature by setting the increments of x to the smallest difference between possible rational zeros, or use the *table* feature in ASK mode. In either case the value in the function column will be 0 when x is a zero of the function. Another way to locate zeros is to graph the function. Be sure that your viewing window contains all the possible rational zeros.

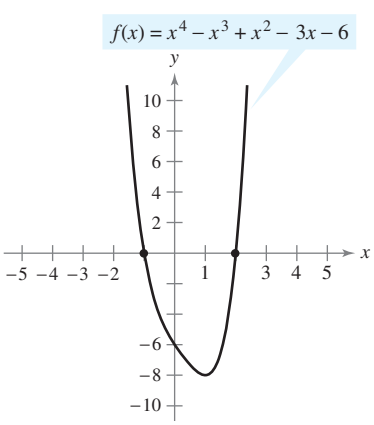


FIGURE 3.26

To see how to use synthetic division to test the possible rational zeros, let's take another look at the function given by

$$f(x) = x^4 - x^3 + x^2 - 3x - 6$$

from Example 2. To test that $x = -1$ and $x = 2$ are zeros of f, you can apply synthetic division, as follows.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & 0 \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

So, you have

$$f(x) = (x + 1)(x - 2)(x^2 + 3).$$

Because the factor $(x^2 + 3)$ produces no real zeros, you can conclude that $x = -1$ and $x = 2$ are the only *real* zeros of f. This is verified in the graph of f shown in Figure 3.26.

Finding the first zero is often the hardest part. After that, the search is simplified by using the lower-degree polynomial obtained in synthetic division. Once the lower-degree polynomial is quadratic, either factoring or the Quadratic Formula can be used to find the remaining zeros.

Example 3 Using the Rational Zero Test

Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

SOLUTION The leading coefficient is 2 and the constant term is 3.

$$\textit{Possible rational zeros: } \frac{\text{Factors of 3}}{\text{Factors of 2}} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

✓CHECKPOINT 3

Find the rational zeros of $f(x) = 2x^3 + 5x^2 + x - 2$. ■

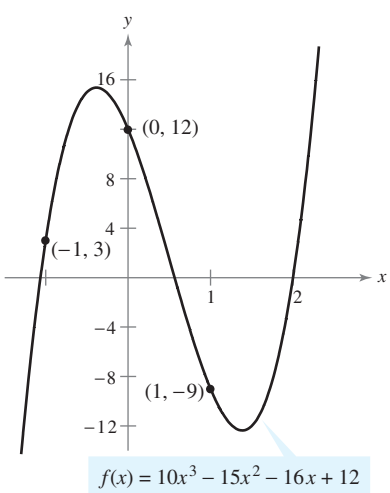


FIGURE 3.27

Example 4 Using the Rational Zero Test

Find all the real zeros of $f(x) = 10x^3 - 15x^2 - 16x + 12$.

SOLUTION The leading coefficient is 10 and the constant term is 12.

$$\textit{Possible rational zeros: } \frac{\text{Factors of 12}}{\text{Factors of 10}} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

With so many possibilities (32, in fact), it is worth your time to stop and sketch a graph. From Figure 3.27, it looks like three reasonable choices would be $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$. Testing these by synthetic division shows that only $x = 2$ checks. So, you have

$$f(x) = (x - 2)(10x^2 + 5x - 6).$$

Using the Quadratic Formula, you find that the two additional zeros are irrational numbers.

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.5639 \quad \text{and} \quad x = \frac{-5 - \sqrt{265}}{20} \approx -1.0639$$

You can conclude that the real zeros of f are $x = 2$, $x \approx 0.5639$, and $x \approx -1.0639$.

✓CHECKPOINT 4

Find the rational zero of $f(x) = 3x^3 + 2x^2 - 5x + 6$. ■

The Intermediate Value Theorem

The next theorem, called the **Intermediate Value Theorem,** tells you of the existence of real zeros of polynomial functions. The theorem implies that if $(a, f(a))$ and $(b, f(b))$ are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See Figure 3.28.)

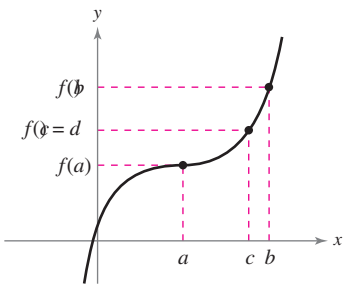


FIGURE 3.28

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

The Intermediate Value Theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ where a polynomial function is positive, and another value $x = b$ where it is negative, you can conclude that the function has at least one real zero between these two values. For example, the function given by

$$f(x) = x^3 + x^2 + 1$$

is negative when $x = -2$ and positive when $x = -1$. So, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1, as shown in Figure 3.29.

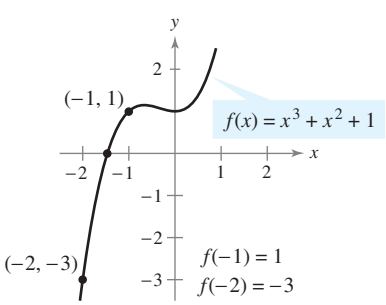


FIGURE 3.29

By continuing this line of reasoning, you can approximate any real zeros of a polynomial function to any desired level of accuracy. This concept is further demonstrated in Example 5.

Example 5 **Approximating a Zero of a Polynomial Function**

Use the Intermediate Value Theorem to approximate a real zero of

$$f(x) = x^3 - x^2 + 1.$$

SOLUTION Begin by computing a few function values, as follows.

x	-2	-1	0	1
$f(x)$	-11	-1	1	1

Because $f(-1)$ is negative and $f(0)$ is positive, you can apply the Intermediate Value Theorem to conclude that the function has a zero between -1 and 0. To pinpoint this zero more closely, divide the interval $[-1, 0]$ into tenths and evaluate the function at each point. When you do this, you will find that

$$f(-0.8) = -0.152$$

and

$$f(-0.7) = 0.167.$$

So, f must have a zero between -0.8 and -0.7, as shown in Figure 3.30. By continuing this process, you can approximate this zero to any desired level of accuracy.

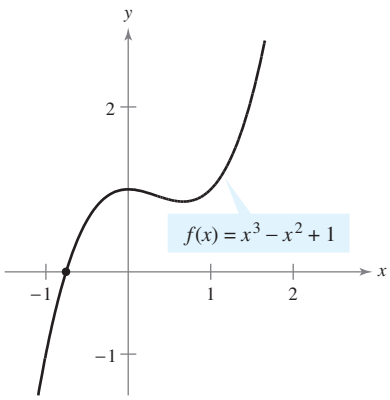


FIGURE 3.30 f has a zero between -8 and -0.7.

✓CHECKPOINT 5

Use the Intermediate Value Theorem to approximate a real zero of $f(x) = x^3 + x - 4$. ■

Approximating Zeros of Polynomial Functions

There are several different techniques for approximating the zeros of a polynomial function. All such techniques are better suited to computers or graphing utilities than they are to "hand calculations." In this section, you will study two techniques that can be used with a graphing utility. The first is called the **zoom-and-trace** technique.

STUDY TIP

To help you visually determine when you have zoomed in enough times to reach the desired level of accuracy, set the Xscale of the viewing window to the accuracy you need and zoom in repeatedly. For instance, to approximate the zero to the nearest hundredth, set the Xscale to 0.01.

Zoom-and-Trace Technique

To approximate a real zero of a function with a graphing utility, use the following steps.

1. Graph the function so that the real zero you want to approximate appears as an x-intercept on the screen.
2. Move the cursor near the x-intercept and use the *zoom* feature to zoom in to get a better look at the intercept.
3. Use the *trace* feature to find the x-values that occur just before and just after the x-intercept. If the difference between these values is sufficiently small, use their average as the approximation. If not, continue zooming in until the approximation reaches the desired level of accuracy.

The amount that a graphing utility zooms in is determined by the *zoom factor*. The zoom factor is a positive number greater than or equal to 1 that gives the ratio of the larger screen to the smaller screen. For instance, if you zoom in with a zoom factor of 2, you will obtain a screen in which the x- and y-values are half their original values. This text uses a zoom factor of 4.

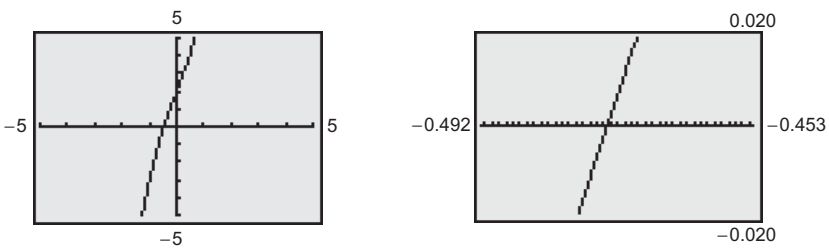
Example 6 Approximating a Zero of a Polynomial Function

Approximate a real zero of

$$f(x) = x^3 + 4x + 2$$

to the nearest thousandth.

SOLUTION To begin, use a graphing utility to graph the function, as shown in Figure 3.31(a). Set the Xscale to 0.001 and zoom in several times until the tick marks on the x-axis become visible. The final screen should be similar to the one shown in Figure 3.31(b).



(a) (b)

FIGURE 3.31

At this point, you can use the *trace* feature to determine that the x-values just to the left and right of the x-intercept are

$$x \approx -0.4735 \quad \text{and} \quad x \approx -0.4733.$$

So, to the nearest thousandth, you can approximate the zero of the function to be

$$x \approx -0.473.$$

To check this, try substituting -0.473 into the function. You should obtain a result that is approximately zero.

✓CHECKPOINT 6

Approximate a real zero of $f(x) = 2x^3 - x + 3$ to the nearest thousandth. ■

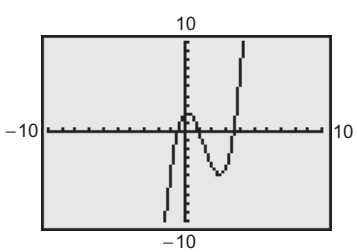


FIGURE 3.32

In Example 6, the cubic polynomial function has only one real zero. Remember that functions can have two or more real zeros. In such cases, you can use the zoom-and-trace technique for each zero separately. For instance, the function given by

$$f(x) = x^3 - 4x^2 + x + 2$$

has three real zeros, as shown in Figure 3.32. Using a zoom-and-trace approach for each real zero, you can approximate the real zeros to be

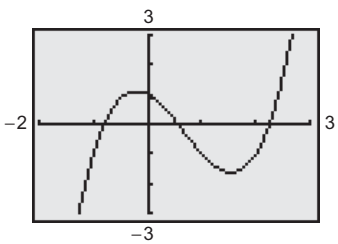
$$x \approx -0.562, \quad x = 1.000, \quad \text{and} \quad x \approx 3.562.$$

The second technique that can be used with some graphing utilities is to employ the graphing utility's *zero* or *root* feature. The name of this feature differs with different calculators. Consult your user's guide to determine if this feature is available.

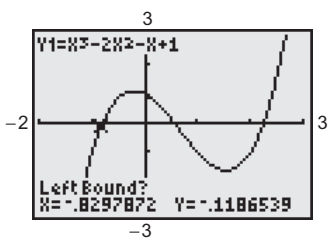
Example 7 Approximating the Zeros of a Polynomial Function

Approximate the real zeros of $f(x) = x^3 - 2x^2 - x + 1$.

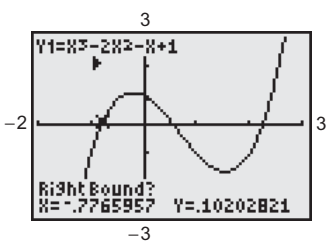
SOLUTION To begin, use a graphing utility to graph the function, as shown in the first screen in Figure 3.33. Notice that the graph has three x-intercepts. To approximate the leftmost intercept, find an appropriate viewing window and use the zero feature, as shown below. The calculator should display an approximation of $x \approx -0.8019377$, which is accurate to seven decimal places.



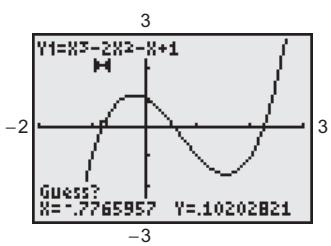
Find an appropriate viewing window, then use the *zero* feature.



Move the cursor to the left of the intercept and press "Enter."



Move the cursor to the right of the intercept and press "Enter."



Move the cursor near the intercept and press "Enter."

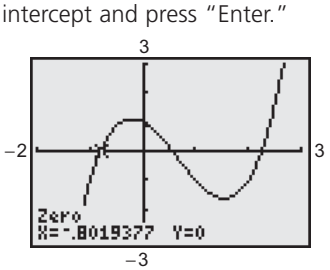


FIGURE 3.33

By repeating this process, you can determine that the other two zeros are $x \approx 0.555$ and $x \approx 2.247$.

TECHNOLOGY

For instructions on how to use the *zoom*, *trace*, *zero*, and *root* features, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

✓CHECKPOINT 7

Approximate the real zeros of $f(x) = x^3 - 4x^2 + 3x + 1$. ■

You may be wondering why so much time is spent in algebra trying to find the zeros of a function. The reason is that if you have a technique that will enable you to solve the equation $f(x) = 0$, you can use the same technique to solve the more general equation

$$f(x) = c$$

where c is any real number. This procedure is demonstrated in Example 8.

Example 8 Solving the Equation $f(x) = c$

Find a value of x such that $f(x) = 30$ for the function given by

$$f(x) = x^3 - 4x + 4.$$

SOLUTION The graph of

$$f(x) = x^3 - 4x + 4$$

is shown in Figure 3.34. Note from the graph that $f(x) = 30$ when x is about 3.5. To use the zoom-and-trace technique to approximate this x-value more closely, consider the equation

$$x^3 - 4x + 4 = 30$$

$$x^3 - 4x - 26 = 0.$$

So, the *solutions* of the equation $f(x) = 30$ are precisely the same x-values as the *zeros* of

$$g(x) = x^3 - 4x - 26.$$

Using the graph of g, as shown in Figure 3.35, you can approximate the zero of g to be

$$x \approx 3.41.$$

You can check this value by substituting $x = 3.41$ into the original function.

$$f(3.41) = (3.41)^3 - 4(3.41) + 4$$

$$\approx 30.01 \checkmark$$

Remember that with decimal approximations, a check usually will not produce an exact value.

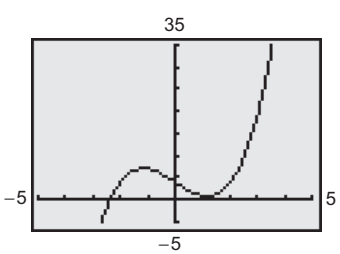


FIGURE 3.34

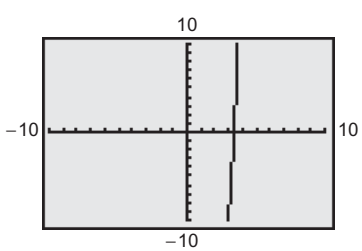


FIGURE 3.35

✓CHECKPOINT 8

Find a value of x such that $f(x) = 20$ for the function given by $f(x) = x^3 + 4x^2 - 1$. ■

Application

Example 9
MAKE A DECISION **Profit and Advertising Expenses**

A company that produces sports clothes estimates that the profit from selling a particular line of sportswear is given by

$$P = -0.014x^3 + 0.752x^2 - 40, \quad 0 \le x \le 50$$

where P is the profit (in tens of thousands of dollars) and x is the advertising expense (in tens of thousands of dollars). According to this model, how much money should the company spend on advertising to obtain a profit of $2,750,000?

SOLUTION From Figure 3.36, it appears that there are two different values of x between 0 and 50 that will produce a profit of $2,750,000. However, because of the context of the problem, it is clear that the better answer is the smaller of the two numbers. So, to solve the equation

$$-0.014x^3 + 0.752x^2 - 40 = 275$$

$$-0.014x^3 + 0.752x^2 - 315 = 0$$

find the zeros of the function

$$g(x) = -0.014x^3 + 0.752x^2 - 315.$$

Using the zoom-and-trace technique, you can find that the leftmost zero is

$$x \approx 32.8.$$

You can check this solution by substituting

$$x = 32.8$$

into the original function.

$$P = -0.014(32.8)^3 + 0.752(32.8)^2 - 40$$
$$\approx 275$$

The company should spend about $328,000 on advertising for the line of sportswear.

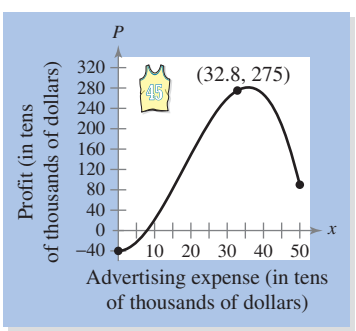


FIGURE 3.36

✓CHECKPOINT 9

In Example 9, how much should the company spend on advertising to obtain a profit of $2,500,000? ■

CONCEPT CHECK

1. Use the Rational Zero Test to explain why $\frac{3}{2}$ is not a possible rational zero of $f(x) = 3x^2 - x + 2$.
2. Can you use the *zero* feature of a graphing utility to find rational zeros of a function? Irrational zeros? Imaginary zeros? Explain your reasoning.
3. Is it possible for a polynomial function to have no real zeros? Explain your reasoning.
4. Explain how to use the Intermediate Value Theorem to approximate the real zeros of a function.

Skills Review 3.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.5 and 3.3.

In Exercises 1 and 2, find a polynomial function with integer coefficients having the given zeros.

1. $-1, \frac{2}{3}, 3$

2. $-2, 0, \frac{3}{4}, 2$

In Exercises 3 and 4, use synthetic division to divide.

3. $\dfrac{x^5 - 9x^3 + 5x + 18}{x + 3}$

4. $\dfrac{3x^4 + 17x^3 + 10x^2 - 9x - 8}{x + \frac{2}{3}}$

In Exercises 5–8, use the given zero to find all the real zeros of f.

5. $f(x) = 2x^3 + 11x^2 + 2x - 4,\ x = \frac{1}{2}$

6. $f(x) = 6x^3 - 47x^2 - 124x - 60,\ x = 10$

7. $f(x) = 4x^3 - 13x^2 - 4x + 6,\ x = -\frac{3}{4}$

8. $f(x) = 10x^3 + 51x^2 + 48x - 28,\ x = \frac{2}{5}$

In Exercises 9 and 10, find all real solutions of the equation.

9. $x^4 - 3x^2 + 2 = 0$

10. $x^4 - 7x^2 + 12 = 0$

Exercises 3.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

(T) In Exercises 1 and 2, use the Rational Zero Test to list all possible rational zeros of f. Then use a graphing utility to graph the function. Use the graph to help determine which of the possible rational zeros are actual zeros of the function.

1. $f(x) = x^3 + x^2 - 4x - 4$

2. $f(x) = 2x^4 - x^2 - 6$

In Exercises 3–6, find the rational zeros of the polynomial function.

3. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6$

4. $f(x) = x^3 + 3x^2 - x - 3$

5. $f(x) = 4x^4 - 17x^2 + 4$

6. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

In Exercises 7–14, find all real zeros of the function.

7. $f(x) = x^3 - 6x^2 + 11x - 6$

8. $g(x) = x^3 - 4x^2 - x + 4$

9. $h(t) = t^3 + 12t^2 + 21t + 10$

10. $f(x) = x^3 - 4x^2 + 5x - 2$

11. $C(x) = 2x^3 + 3x^2 - 1$

12. $f(x) = 3x^3 - 19x^2 + 33x - 9$

13. $f(x) = x^4 - 11x^2 + 18$

14. $P(t) = t^4 - 19t^2 + 48$

In Exercises 15–20, find all real solutions of the polynomial equation.

15. $z^4 - z^3 - 2z - 4 = 0$

16. $x^4 - 13x^2 - 12x = 0$

17. $2y^4 + 7y^3 - 26y^2 + 23y - 6 = 0$

18. $2x^4 - 11x^3 - 6x^2 + 64x + 32 = 0$

19. $x^5 - x^4 - 3x^3 + 5x^2 - 2x = 0$

20. $x^5 - 7x^4 + 10x^3 + 14x^2 - 24x = 0$

In Exercises 21 and 22, (a) list the possible rational zeros of f, (b) sketch the graph of f so that some of the possible zeros in part (a) can be discarded, and (c) determine all real zeros of f.

21. $f(x) = 32x^3 - 52x^2 + 17x + 3$

22. $f(x) = 4x^3 + 7x^2 - 11x - 18$

In Exercises 23–26, use the Intermediate Value Theorem to show that the function has at least one zero in the interval $[a, b]$. (You do not have to approximate the zero.)

23. $f(x) = x^3 + 2x - 5, \quad [1, 2]$

24. $f(x) = x^5 - 3x + 3, \quad [-2, -1]$

25. $f(x) = x^4 - 3x^2 - 10, \quad [2, 3]$

26. $f(x) = -x^3 + 2x^2 + 7x - 3, \quad [3, 4]$

In Exercises 27–30, use the Intermediate Value Theorem to approximate the zero of f in the interval $[a, b]$. Give your approximation to the nearest tenth. (If you have a graphing utility, use it to help you approximate the zero.)

27. $f(x) = x^3 + x - 1, \quad [0, 1]$

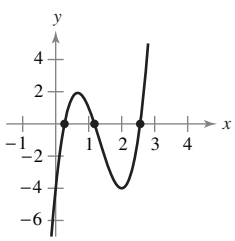
28. $f(x) = x^5 + x + 1, \quad [-1, 0]$

29. $f(x) = x^4 - 10x^2 - 11, \quad [3, 4]$

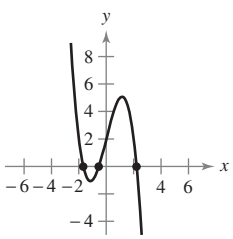
30. $f(x) = -x^3 + 3x^2 + 9x - 2, \quad [4, 5]$

In Exercises 31–36, match the function with its graph. Then approximate the real zeros of the function to three decimal places. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

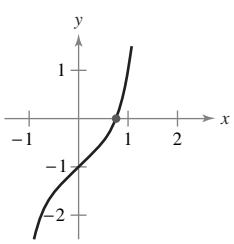
(a)



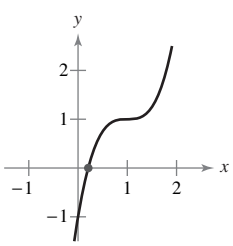
(b)



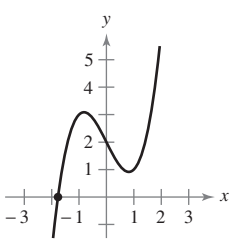
(c)



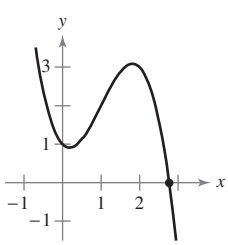
(d)



(e)



(f)



31. $f(x) = x^3 - 2x + 2$ **32.** $f(x) = x^5 + x - 1$

33. $f(x) = 2x^3 - 6x^2 + 6x - 1$

34. $f(x) = 5x^3 - 20x^2 + 20x - 4$

35. $f(x) = -x^3 + 3x^2 - x + 1$

36. $f(x) = -x^3 + 4x + 2$

In Exercises 37–40, use the *zoom* and *trace* features of a graphing utility to approximate the real zeros of f. Give your approximations to the nearest thousandth.

37. $f(x) = x^4 - x - 3$ **38.** $f(x) = 4x^3 + 14x - 8$

39. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

40. $f(x) = -x^3 + 2x^2 + 4x + 5$

In Exercises 41–44, use the *zero* or *root* feature of a graphing utility to approximate the real zeros of f. Give your approximations to the nearest thousandth.

41. $f(x) = x^4 + x - 3$

42. $f(x) = -x^4 + 2x^3 + 4$

43. $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$

44. $f(x) = 3x^4 - 12x^3 + 27x^2 + 4x - 4$

In Exercises 45–48, match the cubic function with the numbers of rational and irrational zeros.

(a) Rational zeros: 0; Irrational zeros: 1

(b) Rational zeros: 3; Irrational zeros: 0

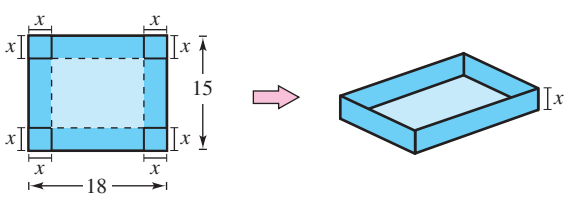
(c) Rational zeros: 1; Irrational zeros: 2

(d) Rational zeros: 1; Irrational zeros: 0

45. $f(x) = x^3 - 1$ **46.** $f(x) = x^3 - 2$

47. $f(x) = x^3 - x$ **48.** $f(x) = x^3 - 2x$

49. Dimensions of a Box An open box is to be made from a rectangular piece of material, 18 inches by 15 inches, by cutting equal squares from the corners and turning up the sides (see figure).



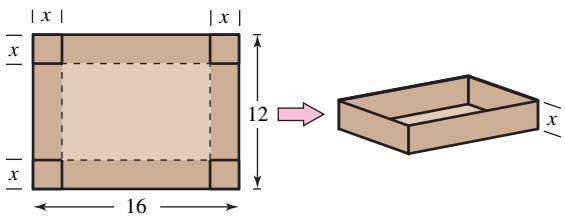
(a) Write the volume V of the box as a function of x. Determine the domain of the function.

(b) Sketch the graph of the function and approximate the dimensions of the box that yield a maximum volume.

(c) Find values of x such that $V = 108$. Which of these values is a physical impossibility in the construction of the box? Explain.

(d) What value of x should you use to make the tallest possible box with a volume of 108 cubic inches?

50. Dimensions of a Box An open box is to be made from a rectangular piece of material, 16 inches by 12 inches, by cutting equal squares from the corners and turning up the sides (see figure).



(a) Write the volume V of the box as a function of x. Determine the domain of the function.

(b) Sketch the graph of the function and approximate the dimensions of the box that yield a maximum volume.

(c) Find values of x such that $V = 120$. Which of these values is a physical impossibility in the construction of the box? Explain.

(d) What value of x should you use to make the tallest possible box with a volume of 120 cubic inches?

51. Dimensions of a Terrarium A rectangular terrarium with a square cross section has a combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the terrarium, given that the volume is 11,664 cubic inches.

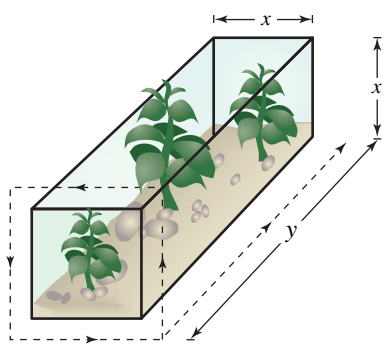


Figure for 51 and 52

52. Dimensions of a Terrarium A rectangular terrarium has a combined length and girth (perimeter of a cross section) of 120 inches (see figure). Find the dimensions of the terrarium, given that the volume is 16,000 cubic inches.

53. Geometry A bulk food storage bin with dimensions 2 feet by 3 feet by 4 feet needs to be increased in size to hold five times as much food as the current bin. (Assume each dimension is increased by the same amount.)

(a) Write a function that represents the volume V of the new bin.

(b) Find the dimensions of the new bin.

54. Geometry A rancher wants to enlarge an existing rectangular corral such that the total area of the new corral is 1.5 times that of the original corral. The current corral's dimensions are 250 feet by 160 feet. The rancher wants to increase each dimension by the same amount.

(a) Write a function that represents the area A of the new corral.

(b) Find the dimensions of the new corral.

(c) A rancher wants to add a length to the sides of the corral that are 160 feet, and twice the length to the sides that are 250 feet, such that the total area of the new corral is 1.5 times that of the original corral. Repeat parts (a) and (b). Explain your results.

55. Medicine The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t \geq 0.$$

The concentration is greatest when

$$3t^4 + 2t^3 - 300t - 50 = 0.$$

Approximate this time to the nearest hundredth of an hour.

56. Transportation Cost The transportation cost C (in thousands of dollars) of the components used in manufacturing prefabricated homes is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where x is the order size (in hundreds). The cost is a minimum when

$$3x^3 - 40x^2 - 2400x - 36{,}000 = 0.$$

Approximate the optimal order size to the nearest unit.

57. Online Sales The revenues per share R (in dollars) for Amazon.com for the years 1996 to 2005 are shown in the table. *(Source: Amazon.com)*

Year	Revenue per share, R	Year	Revenue per share, R
1996	0.07	2001	8.37
1997	0.51	2002	10.14
1998	1.92	2003	13.05
1999	4.75	2004	17.16
2000	7.73	2005	20.41

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a linear model, a quadratic model, a cubic model, and a quartic model for the data.

(c) Use a graphing utility to graph each model separately with the data in the same viewing window. How well does each model fit the data?

(d) Use each model to predict the year in which the revenue per share is about \$37. Explain any differences in the predictions.

58. Population The numbers P (in millions) of people age 18 and over in the United States for the years 1996 to 2005 are shown in the table. *(Source: U.S. Census Bureau)*

Year	Population, P	Year	Population, P
1996	199.2	2001	212.5
1997	201.7	2002	215.1
1998	204.4	2003	217.8
1999	207.1	2004	220.4
2000	209.1	2005	222.9

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 6$ correspond to 1996.

(b) Use the *regression* feature of a graphing utility to find a linear model, a quadratic model, and a cubic model for the data.

(c) Use a graphing utility to graph each model separately with the data in the same viewing window. How well does each model fit the data?

(d) Use each model to predict the year in which the population is about 231,000,000. Explain any differences in the predictions.

59. Cost of Dental Care The amount that \$100 worth of dental care at 19821984 prices would cost in a different year is given by a CPI (Consumer Price Index). The CPIs for dental care in the United States for the years 1996 to 2005 are shown in the table. *(Source: U.S. Bureau of Labor Statistics)*

Year	CPI	Year	CPI
1996	216.5	2001	269.0
1997	226.6	2002	281.0
1998	236.2	2003	292.5
1999	247.2	2004	306.9
2000	258.5	2005	324.0

(a) Use a spreadsheet software program to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a spreadsheet software program to find a linear model, a quadratic model, a cubic model, and a quartic model for the data.

(c) Use each model to predict the year in which the CPI for dental care will be about \$400. Then discuss the appropriateness of each model for predicting future values.

60. Solar Energy Photovoltaic cells convert light energy into electricity. The photovoltaic cell and module domestic shipments S (in peak kilowatts) for the years 1996 to 2005 are shown in the table. *(Source: Energy Information Administration)*

Year	Shipments, S	Year	Shipments, S
1996	13,016	2001	36,310
1997	12,561	2002	45,313
1998	15,069	2003	48,664
1999	21,225	2004	78,346
2000	19,838	2005	134,465

(a) Use a spreadsheet software program to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a spreadsheet software program to find a cubic model and a quartic model for the data.

(c) Use each model to predict the year in which the shipments will be about 1,000,000 peak kilowatts. Then discuss the appropriateness of each model for predicting future values.

61. Advertising Cost A company that produces video games estimates that the profit P (in dollars) from selling a new game is given by

$$P = -82x^3 + 7250x^2 - 450{,}000, \quad 0 \le x \le 80$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, how much should the company spend on advertising to obtain a profit of \$5,900,000?

62. Advertising Cost A company that manufactures hydroponic gardening systems estimates that the profit P (in dollars) from selling a new system is given by

$$P = -35x^3 + 2700x^2 - 300{,}000, \quad 0 \le x \le 70$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, how much should the company spend on advertising to obtain a profit of \$1,800,000?

63. ***MAKE A DECISION: DEMAND FUNCTION*** A company that produces cell phones estimates that the demand D for a new model of phone is given by

$$D = -x^3 + 54x^2 - 140x - 3000, \quad 10 \le x \le 50$$

where x is the price of the phone (in dollars).

(a) Use a graphing utility to graph D. Use the *trace* feature to determine the values of x for which the demand is 14,400 phones.

(b) You may also determine the values of x for which the demand is 14,400 phones by setting D equal to 14,400 and solving for x with a graphing utility. Discuss this alternative solution method. Of the solutions that lie within the given interval, what price would you recommend the company charge for the phones?

64. ***MAKE A DECISION: DEMAND FUNCTION*** A company that produces hand-held organizers estimates that the demand D for a new model of organizer is given by

$$D = -0.005x^3 + 2.65x^2 - 70x - 2500, \quad 50 \le x \le 500$$

where x is the price of the organizer (in dollars).

(a) Use a graphing utility to graph D. Use the *trace* feature to determine the values of x for which the demand will be 80,000 organizers.

(b) You may also determine the values of x for which the demand will be 80,000 organizers by setting D equal to 80,000 and solving for x with a graphing utility. Discuss this alternative solution method. Of the solutions that lie within the given interval, what price would you recommend the company charge for the new organizers?

65. Height of a Baseball A baseball is launched upward from ground level with an initial velocity of 48 feet per second, and its height h (in feet) is

$$h(t) = -16t^2 + 48t, \quad 0 \le t \le 3$$

where t is the time (in seconds). You are told the ball reaches a height of 64 feet. Is this possible?

66. Exploration Use a graphing utility to graph the function

$$f(x) = x^4 - 4x^2 + k$$

for different values of k. Find the values of k such that the zeros of f satisfy the specified characteristics. (Some parts do not have unique answers.)

(a) Four real zeros

(b) Two real zeros and two complex roots

67. Reasoning Is it possible that a second-degree polynomial function with integer coefficients has one rational zero and one irrational zero? If so, give an example.

68. Reasoning Is it possible that a third-degree polynomial function with integer coefficients has one rational zero and two irrational zeros? If so, give an example.

69. Use the information in the table.

Interval	Value of $f(x)$
$(-\infty, -2)$	Positive
$(-2, 1)$	Negative
$(1, 4)$	Negative
$(4, \infty)$	Positive

(a) What are the three real zeros of the polynomial function f?

(b) What can be said about the behavior of the graph of f at $x = 1$?

(c) What is the least possible degree of f? Explain. Can the degree of f ever be odd? Explain.

(d) Is the leading coefficient of f positive or negative? Explain.

(e) Write an equation for f. (There are many correct answers.)

(f) Sketch a graph of the equation you wrote in part (e).

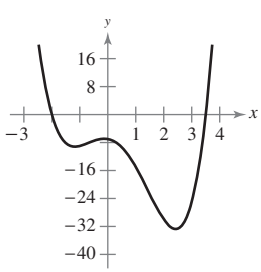
70. Graphical Reasoning The graph of one of the following functions is shown below. Identify the function shown in the graph. Explain why each of the others is not the correct function. Use a graphing utility to verify your result.

(a) $f(x) = x^2(x + 2)(x - 3.5)$

(b) $g(x) = (x + 2)(x - 3.5)$

(c) $h(x) = (x + 2)(x - 3.5)(x^2 + 1)$

(d) $k(x) = (x + 1)(x + 2)(x - 3.5)$



71. Extended Application To work an extended application analyzing the sales per share of Best Buy, visit this text's website at *college.hmco.com*. *(Source: Best Buy)*

Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, sketch the graph of the quadratic function. Identify the vertex and the intercepts.

1. $f(x) = (x + 1)^2 - 2$

2. $f(x) = 25 - x^2$

In Exercises 3 and 4, describe the right-hand and left-hand behavior of the graph of the polynomial function. Verify with a graphing utility.

3. $f(x) = -2x^3 + 7x^2 - 9$

4. $f(x) = x^4 + 7x^2 - 8$

5. Use synthetic division to evaluate $f(x) = 2x^4 + x^3 + 18x^2 - 4$ when $x = -3$.

In Exercises 6 and 7, write the function in the form $f(x) = (x - k)q(x) + r$ for the given value of k, and demonstrate that $f(k) = r$.

6. $f(x) = x^4 - 5x^2 + 4, \quad k = 1$

7. $f(x) = x^3 + 5x^2 - 2x - 24, \quad k = -3$

8. Simplify $\dfrac{2x^4 + 9x^3 - 32x^2 - 99x + 180}{x^2 + 2x - 15}$.

In Exercises 9–12, find the real zeros of the function.

9. $f(x) = -2x^3 - 7x^2 + 10x + 35$

10. $f(x) = 4x^4 - 37x^2 + 9$

11. $f(x) = 3x^4 + 4x^3 - 3x - 4$

12. $f(x) = 2x^3 - 3x^2 + 2x - 3$

13. The profit P (in dollars) for a clothing company is

$$P = -95x^3 + 5650x^2 - 250{,}000, \quad 0 \le x \le 55$$

where x is the advertising expense (in tens of thousands of dollars). What is the profit for an advertising expense of $450,000? Use a graphing utility to approximate another advertising expense that would yield the same profit.

14. **Crops** The worldwide land areas A (in millions of hectares) of transgenic crops for the years 1996 to 2006 are shown in the table. *(Source: International Service for the Acquisition of Agri-Biotech Applications)*

Year	Area, A
1996	1.7
1997	11.0
1998	27.8
1999	39.9
2000	44.2
2001	52.6
2002	58.7
2003	67.7
2004	81.0
2005	90.0
2006	102.0

Table for 14

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a linear model, a quadratic model, a cubic model, and a quartic model for the data.

(c) Use a graphing utility to graph each model separately with the data in the same viewing window. How well does each model fit the data?

(d) Use each model to predict the year in which the land area will be about 150 million hectares. Explain any differences in the predictions.

Section 3.5

Complex Numbers

- Perform operations with complex numbers and write the results in standard form.
- Find the complex conjugate of a complex number.
- Solve a polynomial equation that has complex solutions.
- Plot a complex number in the complex plane.
- Determine whether a complex number is in the Mandelbrot Set.

The Imaginary Unit *i*

Some quadratic equations have no real solutions. For instance, the quadratic equations

$$x^2 + 1 = 0 \quad \text{and} \quad x^2 = -5$$ Equations with no real solutions

have no real solutions because there is no real number x that can be squared to produce a negative number. To overcome this deficiency, mathematicians utilized an expanded system of numbers that used the **imaginary unit *i*,** which is defined as

$$i = \sqrt{-1}$$ Imaginary unit

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, we obtain the set of **complex numbers.** Each complex number can be written in the **standard form** $a + bi$.

Complex numbers

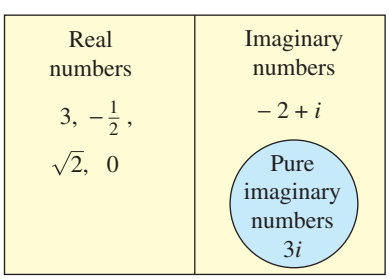


FIGURE 3.37

Definition of a Complex Number

If a and b are real numbers, the number $a + bi$ is called a **complex number,** and it is said to be written in **standard form.** If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 3.37. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, we can write $a = a + 0i$.

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$ written in standard form are **equal** to each other,

$$a + bi = c + di$$ Equality of two complex numbers

if and only if $a = c$ and $b = d$.

Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi. \qquad \text{Additive inverse}$$

So, you have

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

Example 1 Adding and Subtracting Complex Numbers

Perform the operation(s) and write each result in standard form.

a. $(3 - i) + (2 + 3i)$ **b.** $2i + (-4 - 2i)$ **c.** $3 - (-2 + 3i) + (-5 + i)$

SOLUTION

a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$ Remove parentheses.

$= 3 + 2 - i + 3i$ Group like terms.

$= (3 + 2) + (-1 + 3)i$

$= 5 + 2i$ Write in standard form.

b. $2i + (-4 - 2i) = 2i - 4 - 2i$ Remove parentheses.

$= -4 + 2i - 2i$ Group like terms.

$= -4$ Write in standard form.

c. $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$

$= 3 + 2 - 5 - 3i + i$

$= 0 - 2i$

$= -2i$

✓CHECKPOINT 1

Perform the operation(s) and write each result in standard form.

a. $(4 + 7i) + (1 - 6i)$ **b.** $3i - (-2 + 3i) - (2 + 5i)$ ■

Note in Example 1(b) that the sum of two imaginary numbers can be a real number.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication

Commutative Properties of Addition and Multiplication

Distributive Property of Multiplication Over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di) && \text{Distributive Property}\\ &= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive Property}\\ &= ac + (ad)i + (bc)i + (bd)(-1) && i^2 = -1\\ &= ac - bd + (ad)i + (bc)i && \text{Commutative Property}\\ &= (ac - bd) + (ad + bc)i && \text{Associative Property}\end{aligned}$$

Rather than trying to memorize this multiplication rule, you should simply remember how the Distributive Property is used to multiply two complex numbers. The procedure is similar to multiplying two binomials and combining like terms (as in the FOIL Method).

DISCOVERY

Fill in the blanks:

$i^1 = i$ $\quad i^5 =$ $\quad i^9 =$

$i^2 = -1$ $\quad i^6 =$ $\quad i^{10} =$

$i^3 = -i$ $\quad i^7 =$ $\quad i^{11} =$

$i^4 = 1$ $\quad i^8 =$ $\quad i^{12} =$

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

Example 2 Multiplying Complex Numbers

Find each product.

a. $(i)(-3i)$ **b.** $(2 - i)(4 + 3i)$ **c.** $(3 + 2i)(3 - 2i)$ **d.** $(3 + 2i)^2$

SOLUTION

a. $(i)(-3i) = -3i^2$ Multiply.

$= -3(-1)$ $i^2 = -1$

$= 3$ Simplify.

b. $(2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2$ Distributive Property

$= 8 + 6i - 4i - 3(-1)$ $i^2 = -1$

$= 8 + 3 + 6i - 4i$ Group like terms.

$= 11 + 2i$ Write in standard form.

c. $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2$ Distributive Property

$= 9 - 4(-1)$ $i^2 = -1$

$= 9 + 4$ Simplify.

$= 13$ Write in standard form.

d. $(3 + 2i)^2 = 9 + 6i + 6i + 4i^2$ Distributive Property

$= 9 + 4(-1) + 12i$ $i^2 = -1$

$= 9 - 4 + 12i$ Simplify.

$= 5 + 12i$ Write in standard form.

✓CHECKPOINT 2

Find each product.

a. $4(-2 + 3i)$

b. $(5 - 3i)^2$ ■

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates.** In general, the product of two complex conjugates can be written as follows.

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - b^2(-1) = a^2 + b^2\end{aligned}$$

Complex conjugates can be used to write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero. To do this, multiply the numerator and denominator by the complex conjugate of the denominator to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

TECHNOLOGY

Some graphing utilities can perform operations with complex numbers. For specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

Example 3 **Writing Quotients of Complex Numbers in Standard Form**

Write each quotient in standard form.

a. $\dfrac{1}{1 + i}$ **b.** $\dfrac{2 + 3i}{4 - 2i}$

SOLUTION

a.
$$\begin{aligned}\frac{1}{1 + i} &= \frac{1}{1 + i}\left(\frac{1 - i}{1 - i}\right) && \text{Multiply numerator and denominator by complex conjugate of denominator.} \\ &= \frac{1 - i}{1^2 - i^2} && \text{Expand.} \\ &= \frac{1 - i}{1 - (-1)} && i^2 = -1 \\ &= \frac{1 - i}{2} && \text{Simplify.} \\ &= \frac{1}{2} - \frac{1}{2}i && \text{Write in standard form.}\end{aligned}$$

b.
$$\begin{aligned}\frac{2 + 3i}{4 - 2i} &= \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right) && \text{Multiply numerator and denominator by complex conjugate of denominator.} \\ &= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} && \text{Expand.} \\ &= \frac{8 - 6 + 16i}{16 + 4} && i^2 = -1 \\ &= \frac{2 + 16i}{20} && \text{Simplify.} \\ &= \frac{1}{10} + \frac{4}{5}i && \text{Write in standard form.}\end{aligned}$$

✓CHECKPOINT 3

Write $\dfrac{6 - 7i}{1 - 2i}$ in standard form. ■

Complex Solutions

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}\,i$$

The number $\sqrt{3}\,i$ is called the *principal square root* of -3.

STUDY TIP

The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid if *both* a and b are negative. For example,

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

whereas

$$\sqrt{-5}\sqrt{-5} = 5i^2 = -5$$

To avoid problems with multiplying square roots of negative numbers, be sure to convert complex numbers to standard form *before* multiplying.

Principal Square Root of a Negative Number

If a is a positive number, the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}\,i.$$

Example 4 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}\,i\sqrt{12}\,i = \sqrt{36}\,i^2 = 6(-1) = -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}\,i - \sqrt{27}\,i = 4\sqrt{3}\,i - 3\sqrt{3}\,i = \sqrt{3}\,i$

c. $\left(-1 + \sqrt{-3}\right)^2 = \left(-1 + \sqrt{3}\,i\right)^2$

$$= (-1)^2 - 2\sqrt{3}\,i + \left(\sqrt{3}\right)^2(i^2)$$

$$= 1 - 2\sqrt{3}\,i + 3(-1)$$

$$= -2 - 2\sqrt{3}\,i$$

✓CHECKPOINT 4

Write $4 + \sqrt{-9}$ in standard form. ■

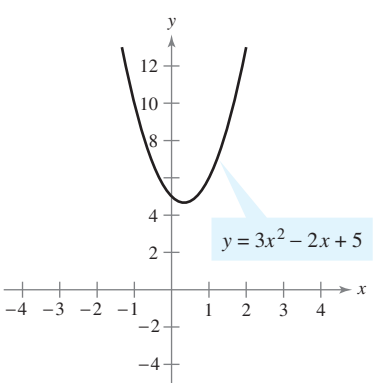


FIGURE 3.38

✓CHECKPOINT 5

Solve $x^2 + 3x + 4 = 0$. ■

Example 5 Complex Solutions of a Quadratic Equation

Solve $3x^2 - 2x + 5 = 0$.

SOLUTION

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$$ Quadratic Formula

$$= \frac{2 \pm \sqrt{-56}}{6}$$ Simplify.

$$= \frac{2 \pm 2\sqrt{14}\,i}{6}$$ Write in i-form.

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}\,i$$ Write in standard form.

The graph of $f(x) = 3x^2 - 2x + 5$, shown in Figure 3.38, does not touch or cross the x-axis. This confirms that the equation in Example 5 has no real solution.

Applications

Most applications involving complex numbers are either theoretical (see the next section) or very technical, and so are not appropriate for inclusion in this text. However, to give you some idea of how complex numbers can be used in applications, a general description of their use in **fractal geometry** is presented.

To begin, consider a coordinate system called the **complex plane.** Just as every real number corresponds to a point on the real number line, every complex number corresponds to a point in the complex plane, as shown in Figure 3.39. In this figure, note that the vertical axis is the **imaginary axis** and the horizontal axis is the real axis. The point that corresponds to the complex number $a + bi$ is (a, b).

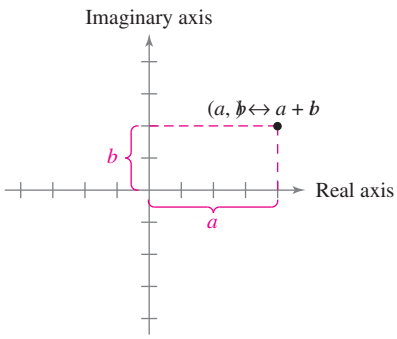


FIGURE 3.39

Complex number		Ordered pair
$a + bi$	⇨	(a, b)

From Figure 3.39, you can see that i is called the imaginary unit because it is located one unit from the origin on the imaginary axis of the complex plane.

Example 6 Plotting Complex Numbers in the Complex Plane

Plot each complex number in the complex plane.

a. $2 + 3i$ **b.** $-1 + 2i$ **c.** 4

SOLUTION

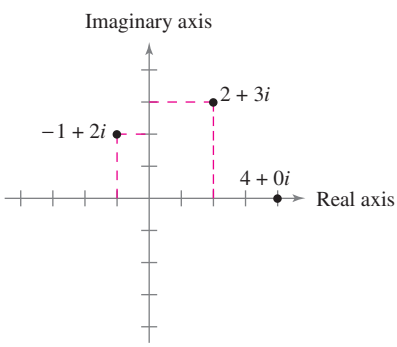


FIGURE 3.40

a. To plot the complex number $2 + 3i$, move (from the origin) two units to the right on the real axis and then three units upward. See Figure 3.40. In other words, plotting the complex number $2 + 3i$ in the complex plane is comparable to plotting the point $(2, 3)$ in the Cartesian plane.

b. The complex number $-1 + 2i$ corresponds to the point $(-1, 2)$. See Figure 3.40.

c. The complex number 4 corresponds to the point $(4, 0)$. See Figure 3.40.

✓CHECKPOINT 6

Plot $-3i$ in the complex plane. ■

In the hands of a person who understands "fractal geometry," the complex plane can become an easel on which stunning pictures, called **fractals,** can be drawn. The most famous such picture is called the **Mandelbrot Set,** named after the Polish-born mathematician Benoit Mandelbrot. To draw the Mandelbrot Set, consider the following sequence of numbers.

$$c, \quad c^2 + c, \quad (c^2 + c)^2 + c, \quad [(c^2 + c)^2 + c]^2 + c, \quad \ldots$$

The behavior of this sequence depends on the value of the complex number c. For some values of c, this sequence is **bounded,** which means that the absolute value of each number $\left(|a + bi| = \sqrt{a^2 + b^2}\right)$ in the sequence is less than some fixed number N. For other values of c, this sequence is **unbounded,** which means that the absolute values of the terms of the sequence become infinitely large. If the sequence is bounded, the complex number c is in the Mandelbrot Set, and if the sequence is unbounded, the complex number c is not in the Mandelbrot Set.

Example 7
MAKE A DECISION **Members of the Mandelbrot Set**

Decide whether each complex number is a member of the Mandelbrot Set.

a. -2 **b.** i **c.** $1 + i$

SOLUTION

a. For $c = -2$, the corresponding Mandelbrot sequence is

$$-2, \quad 2, \quad 2, \quad 2, \quad 2, \quad 2, \ldots$$

Because the sequence is bounded, the complex number -2 is in the Mandelbrot Set.

b. For $c = i$, the corresponding Mandelbrot sequence is

$$i, \quad -1 + i, \quad -i, \quad -1 + i, \quad -i, \quad -1 + i, \ldots$$

Because the sequence is bounded, the complex number i is in the Mandelbrot Set.

c. For $c = 1 + i$, the corresponding Mandelbrot sequence is

$$1 + i, \quad 1 + 3i, \quad -7 + 7i, \quad 1 - 97i, \quad -9407 - 193i, \quad 88454401 + 3631103i, \ldots$$

Because the sequence is unbounded, the complex number $1 + i$ is *not* in the Mandelbrot Set.

✓CHECKPOINT 7

Decide whether -3 is in the Mandelbrot Set. Explain your reasoning. ■

With this definition, a picture of the Mandelbrot Set would have only two colors: one color for points that are in the set (the sequence is bounded) and one color for points that are outside the set (the sequence is unbounded). Figure 3.41 shows a black and yellow picture of the Mandelbrot Set. The points that are black are in the Mandelbrot Set and the points that are yellow are not.

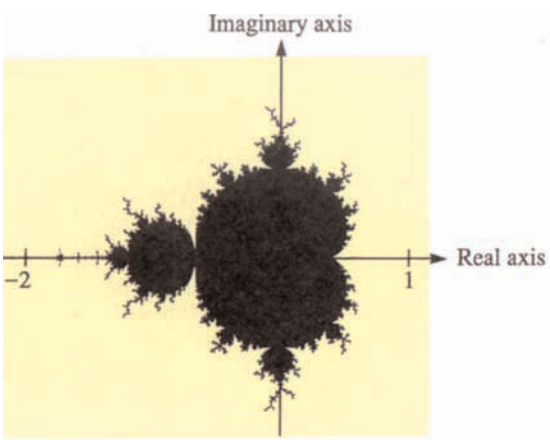
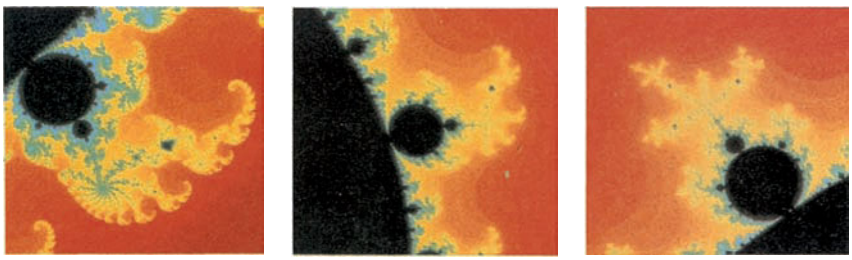


FIGURE 3.41 Mandelbrot Set

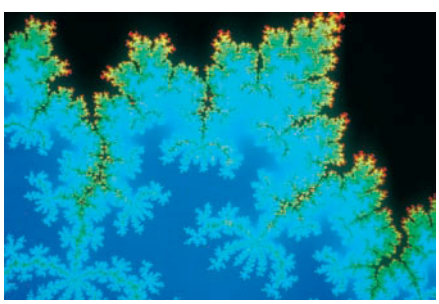
To add more interest to the picture, computer scientists discovered that the points that are not in the Mandelbrot Set can be assigned a variety of colors, depending on how quickly their sequences diverge. Figure 3.42 shows three different appendages of the Mandelbrot Set using a spectrum of colors. (The colored portions of the picture represent points that are *not* in the Mandelbrot Set.)



American Mathematical Society

FIGURE 3.42

Figures 3.43, 3.44, and 3.45 show other types of fractal sets. From these pictures, you can see why fractals have fascinated people since their discovery (around 1980).



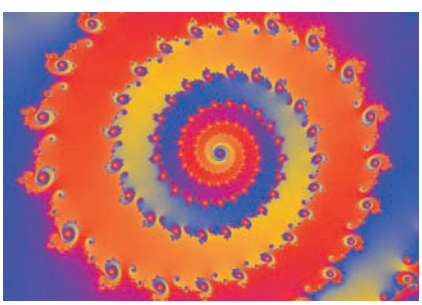
Fred Espenak/Photo Researchers, Inc.

FIGURE 3.43



Gregory Sams/Photo Researchers, Inc.

FIGURE 3.44



Francoise Sauze/Photo Researchers, Inc.

FIGURE 3.45

CONCEPT CHECK

1. Is $3 + \sqrt{-4}$ written in standard form? Explain.
2. Is $-(m + ni)$ the complex conjugate of $(m + ni)$? Use multiplication to justify your answer.
3. Is $-2\sqrt{2}$ the principal square root of -8? Explain.
4. Can the difference of two imaginary numbers be a real number? Justify your answer with an example.

Skills Review 3.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4 and 1.4.

In Exercises 1–8, simplify the expression.

1. $\sqrt{12}$

2. $\sqrt{500}$

3. $\sqrt{20} - \sqrt{5}$

4. $\sqrt{27} - \sqrt{243}$

5. $\sqrt{24}\sqrt{6}$

6. $2\sqrt{18}\sqrt{32}$

7. $\dfrac{1}{\sqrt{3}}$

8. $\dfrac{2}{\sqrt{2}}$

In Exercises 9 and 10, solve the quadratic equation.

9. $x^2 + x - 1 = 0$

10. $x^2 + 2x - 1 = 0$

Exercises 3.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Write out the first 16 positive integer powers of i ($i, i^2, i^3, \ldots, i^{16}$), and write each as i, $-i$, 1, or -1. What pattern do you observe?

2. Use the pattern you found in Exercise 1 to help you write each power of i as i, $-i$, 1, or -1.

(a) i^{28} (b) i^{37} (c) i^{127} (d) i^{82}

In Exercises 3–6, find the real numbers a and b such that the equation is true.

3. $a + bi = 7 + 12i$

4. $a + bi = -2 - 5i$

5. $(a + 3) + (b - 1)i = 7 - 4i$

6. $(a + 6) + 2bi = 6 - 5i$

In Exercises 7–18, write the complex number in standard form and find its complex conjugate.

7. $9 + \sqrt{-16}$

8. $2 + \sqrt{-25}$

9. $-3 - \sqrt{-12}$

10. $1 + \sqrt{-8}$

11. -21

12. 45

13. $-6i + i^2$

14. $4i^2 - 2i^3$

15. $-5i^5$

16. $(-i)^3$

17. $\left(\sqrt{-6}\right)^2 + 3$

18. $\left(\sqrt{-4}\right)^2 - 5$

In Exercises 19–44, perform the indicated operation and write the result in standard form.

19. $(-4 + 3i) + (6 - 2i)$

20. $(13 - 2i) + (-5 + 6i)$

21. $(12 + 5i) - (7 - i)$

22. $(3 + 2i) - (6 + 13i)$

23. $\left(-2 + \sqrt{-8}\right) + \left(5 - \sqrt{-50}\right)$

24. $\left(5 + \sqrt{-18}\right) - \left(3 + \sqrt{-32}\right)$

25. $-\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$

26. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

27. $(3 + 4i)^2 + (3 - 4i)^2$

28. $(2 - 5i)^2 - (2 + 5i)^2$

29. $\sqrt{-3} \cdot \sqrt{-8}$

30. $\sqrt{-5} \cdot \sqrt{-10}$

31. $\left(\sqrt{-10}\right)^2$

32. $\left(\sqrt{-75}\right)^3$

33. $(2 + 3i)(1 - i)$

34. $(6 - 5i)(1 + i)$

35. $(3 + 4i)(3 - 4i)$

36. $(8 + 3i)(8 - 3i)$

37. $5i(4 - 6i)$

38. $-2i(7 + 9i)$

39. $(5 + 6i)^2$

40. $(3 - 7i)^2$

41. $\left(\sqrt{5} - \sqrt{3}i\right)\left(\sqrt{5} + \sqrt{3}i\right)$

42. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

43. $\left(2 - \sqrt{-8}\right)\left(8 + \sqrt{-6}\right)$

44. $\left(3 + \sqrt{-5}\right)\left(7 - \sqrt{-10}\right)$

In Exercises 45–56, write the quotient in standard form.

45. $\dfrac{3-i}{3+i}$ **46.** $\dfrac{8-5i}{1-3i}$

47. $\dfrac{5}{4-2i}$ **48.** $\dfrac{3}{1+2i}$

49. $\dfrac{7+10i}{i}$ **50.** $\dfrac{8+15i}{3i}$

51. $\dfrac{1}{(2i)^3}$ **52.** $\dfrac{1}{(3i)^3}$

53. $\dfrac{4}{(1-2i)^3}$ **54.** $\dfrac{3}{(5-2i)^2}$

55. $\dfrac{(21-7i)(4+3i)}{2-5i}$ **56.** $\dfrac{(3-i)(2+5i)}{4+3i}$

Error Analysis In Exercises 57 and 58, a student has handed in the specified problem. Find the error(s) and discuss how to explain the error(s) to the student.

57. Write $\dfrac{5}{3-2i}$ in standard form.

$$\frac{5}{3-2i}\cdot\frac{3+2i}{3+2i}=\frac{15+10i}{9-4}=3+2i$$

58. Multiply $\left(\sqrt{-4}+3\right)\left(i-\sqrt{-3}\right)$.

$$\begin{aligned}\left(\sqrt{-4}+3\right)\left(i-\sqrt{-3}\right)&=i\sqrt{-4}-\sqrt{-4}\sqrt{-3}+3i-3\sqrt{-3}\\&=-2i-\sqrt{12}+3i-3i\sqrt{3}\\&=\left(1-3\sqrt{3}\right)i-2\sqrt{3}\end{aligned}$$

In Exercises 59–66, solve the quadratic equation.

59. $x^2-2x+2=0$ **60.** $x^2+6x+10=0$

61. $4x^2+16x+17=0$ **62.** $9x^2-6x+37=0$

63. $4x^2+16x+15=0$ **64.** $9x^2-6x+35=0$

65. $16t^2-4t+3=0$ **66.** $5s^2+6s+3=0$

In Exercises 67–70, solve the quadratic equation and then use a graphing utility to graph the related quadratic function in the standard viewing window. Discuss how the graph of the quadratic function relates to the solutions of the quadratic equation.

	Equation	*Function*
67.	$x^2+x+2=0$	$y=x^2+x+2$
68.	$-x^2+3x-5=0$	$y=-x^2+3x-5$
69.	$x^2+3x-5=0$	$y=x^2+3x-5$
70.	$-x^2-3x+4=0$	$y=-x^2-3x+4$

In Exercises 71–76, plot the complex number.

71. 3 **72.** i

73. $-2+i$ **74.** $-2-3i$

75. $1-2i$ **76.** $-2i$

In Exercises 77–82, decide whether the number is in the Mandelbrot Set. Explain your reasoning.

77. $c=0$ **78.** $c=2$

79. $c=1$ **80.** $c=-1$

81. $c=\frac{1}{2}i$ **82.** $c=-i$

In Exercises 83 and 84, determine whether the statement is *true* or *false*. Explain.

83. There is no complex number that is equal to its conjugate.

84. The conjugate of the sum of two complex numbers is equal to the sum of the conjugates of the two complex numbers.

Business Capsule



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Fractal Graphics, established in 1992, built a world-class reputation as a leader in application of 3-D visualization technology as applied to the interpretation of complex geoscientific models. In 2002, Fractal Graphics split to form the software development group Fractal Technologies Pty Ltd and the geological consulting group Fractal Geoscience. Fractal Technologies develops dimensional data management and visualization software for the geosciences. One of Fractal Technologies' product suites is FracSIS, which stores geological, geochemical, and geophysical data with an interactive visualization environment.

85. Research Project Use your campus library, the Internet, or some other reference source to find information about a company that uses algorithms to generate 3-D images or gaming software. Write a brief paper about such a company or small business.

Section 3.6

The Fundamental Theorem of Algebra

- Use the Fundamental Theorem of Algebra and the Linear Factorization Theorem to write a polynomial as the product of linear factors.
- Find a polynomial with real coefficients whose zeros are given.
- Factor a polynomial over the rational, real, and complex numbers.
- Find all real and complex zeros of a polynomial function.

The Fundamental Theorem of Algebra

You have been using the fact that an nth-degree polynomial function can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely* n zeros. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the famous German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the following theorem.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n > 0$, then $f(x)$ has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers and a_n is the leading coefficient of $f(x)$.

Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial. Such theorems are called **existence theorems.** To find the zeros of a polynomial function, you still must rely on the techniques developed in the earlier parts of the text.

Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated. Example 1 illustrates several cases.

Example 1 Zeros of Polynomial Functions

Determine the number of zeros of each polynomial function. Then list the zeros.

a. $f(x) = x - 2$ **b.** $f(x) = x^2 - 6x + 9$

c. $f(x) = x^3 + 4x$ **d.** $f(x) = x^4 - 1$

SOLUTION

a. The first-degree polynomial function given by $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function given by

$$f(x) = x^2 - 6x + 9 = (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$.

c. The third-degree polynomial function given by

$$f(x) = x^3 + 4x = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fourth-degree polynomial function given by

$$f(x) = x^4 - 1 = (x - 1)(x + 1)(x - i)(x + i)$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

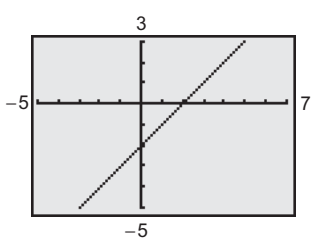
✓CHECKPOINT 1

Determine the number of zeros of $f(x) = x^4 - 36$. Then list the zeros. ■

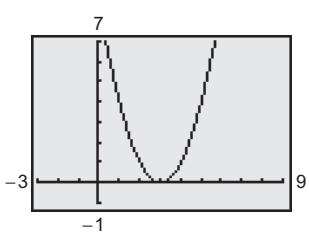
TECHNOLOGY

Remember that when you use a graphing utility to locate the zeros of a function, the only zeros that appear as x-intercepts are the *real zeros*. Compare the graphs below with the four polynomial functions in Example 1. Which zeros appear on the graphs?

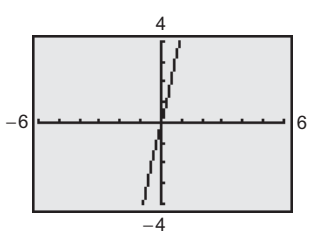
(a)



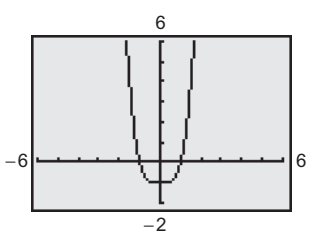
(b)



(c)



(d)



Example 2 shows how you can use the methods described in Sections 3.3 and 3.4 (the Rational Zero Test, synthetic division, and factoring) to find all the zeros of a polynomial function, including the complex zeros.

Example 2 Finding the Zeros of a Polynomial Function

Find all of the zeros of

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$$

and write the polynomial as a product of linear factors.

SOLUTION From the Rational Zero Test, the possible rational zeros are ± 1, ± 2, ± 4, and ± 8. Synthetic division produces the following.

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 1 & 2 & -12 & 8 \\ & & 1 & 1 & 2 & 4 & -8 \\ \hline & 1 & 1 & 2 & 4 & -8 & 0 \end{array} \quad \Rightarrow \text{1 is a zero.}$$

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & 2 & 4 & -8 \\ & & 1 & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 8 & 0 \end{array} \quad \Rightarrow \text{1 is a repeated zero.}$$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 4 & 8 \\ & & -2 & 0 & -8 \\ \hline & 1 & 0 & 4 & 0 \end{array} \quad \Rightarrow -2 \text{ is a zero.}$$

So, you have

$$\begin{aligned} f(x) &= x^5 + x^3 + 2x^2 - 12x + 8 \\ &= (x - 1)(x - 1)(x + 2)(x^2 + 4). \end{aligned}$$

By factoring $x^2 + 4$ as the difference of two squares over the imaginary numbers

$$\begin{aligned} x^2 - (-4) &= \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) \\ &= (x - 2i)(x + 2i) \end{aligned}$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$1, \quad 1, \quad -2, \quad 2i, \quad \text{and} \quad -2i$$

Note from the graph of f shown in Figure 3.46 that the *real* zeros are the only ones that appear as x-intercepts.

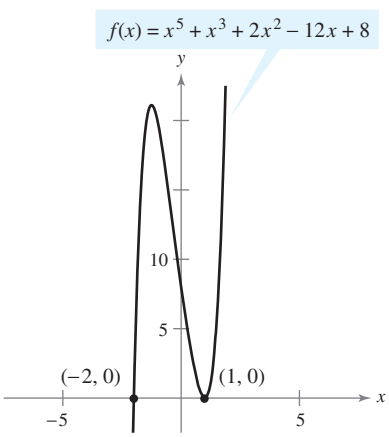


FIGURE 3.46

✓CHECKPOINT 2

Find all the zeros of each function and write the polynomial as the product of linear factors.

a. $f(x) = x^4 + 8x^2 - 9$

b. $g(x) = x^5 - 5x^4 + 4x^3 + 4x^2 + 3x + 9$ ■

DISCOVERY

Use a graphing utility to graph

$$f(x) = x^3 + x^2 - 2x + 1$$

and

$$g(x) = x^3 + x^2 - 2x - 1.$$

How many zeros does f have? How many zeros does g have? Is it possible for an odd-degree polynomial function with real coefficients to have no real zeros (only complex zeros)? Can an even-degree polynomial function with real coefficients have only imaginary zeros? If so, how does the graph of such a polynomial function behave?

Conjugate Pairs

In Example 2, note that the two imaginary zeros are **conjugates.** That is, they are of the form $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let f be a polynomial function that has *real coefficients.* If $a + bi$, where $b \neq 0$, is a zero of the function, then the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only if the polynomial function has *real coefficients.* For instance, the result applies to the function given by $f(x) = x^2 + 1$, but not to the function given by $g(x) = x - i$.

You have been using the Rational Zero Test, synthetic division, and factoring to find the zeros of polynomial functions. The Linear Factorization Theorem enables you to reverse this process and find a polynomial function when its zeros are given.

Example 3 Finding a Polynomial Function with Given Zeros

Find a *fourth-degree* polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

SOLUTION Because $3i$ is a zero *and* the function is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, -1, -1, $3i$, and $-3i$ are the four zeros and from the Linear Factorization Theorem, $f(x)$ can be written as a product of linear factors, as shown.

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i)$$

For simplicity, let $a = 1$. Then multiply the factors with real coefficients to get $(x^2 + 2x + 1)$ and multiply the complex conjugates to get $(x^2 + 9)$. So, you obtain the following fourth-degree polynomial function.

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9 \end{aligned}$$

✓CHECKPOINT 3

Find a *fourth-degree* polynomial function with real coefficients that has -3, 3, and $2i$ as zeros. ■

Factoring a Polynomial

The Linear Factorization Theorem shows that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

However, this result includes the possibility that some of the values of c_i are complex. The following result implies that even if you do not want to get involved with "imaginary factors," you can still write $f(x)$ as the product of linear and/or quadratic factors.

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

A quadratic factor with no real zeros is said to be **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals.* For example, the quadratic

$$x^2 + 1 = (x - i)(x + i)$$

is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

is irreducible over the rationals, but it is *reducible* over the reals.

Example 4 Factoring a Polynomial

Use the polynomial $x^4 - x^2 - 20$ to complete the following.

a. Write the polynomial as the product of factors that are irreducible over the *rationals.*

b. Write the polynomial as the product of linear factors and quadratic factors that are irreducible over the *reals*.

c. Write the polynomial in completely factored form.

d. How many of the zeros are rational, irrational, or imaginary?

SOLUTION

a. Begin by factoring the polynomial into the product of two quadratic polynomials.

$$x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$$

Both of these factors are irreducible over the rationals.

b. By factoring over the reals, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x^2 + 4)$$

where the quadratic factor is irreducible over the reals.

c. In completely factored form, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x - 2i)(x + 2i).$$

d. Using the completely factored form, you can conclude that there are no rational zeros, two irrational zeros $\left(\pm\sqrt{5}\right)$, and two imaginary zeros $(\pm 2i)$.

✓CHECKPOINT 4

In Example 4, complete parts (a)–(d) using the polynomial $x^4 + x^2 - 12$. ■

Example 5 Finding the Zeros of a Polynomial Function

Find all the zeros of $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$, given that $1 + 3i$ is a zero of f.

SOLUTION Because imaginary zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$[x - (1 + 3i)] \quad \text{and} \quad [x - (1 - 3i)]$$

are factors of $f(x)$. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into $f(x)$ to obtain the following.

$$\begin{array}{r} x^2 - x - 6 \\ x^2 - 2x + 10 \overline{)\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\ \underline{x^4 - 2x^3 + 10x^2} \\ -x^3 - 4x^2 + 2x \\ \underline{-x^3 + 2x^2 - 10x} \\ -6x^2 + 12x - 60 \\ \underline{-6x^2 + 12x - 60} \\ 0 \end{array}$$

So, you have

$$\begin{aligned} f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2) \end{aligned}$$

and you can conclude that the zeros of f are $1 + 3i$, $1 - 3i$, 3, and -2.

✓CHECKPOINT 5

Find all the zeros of $f(x) = 3x^3 - 5x^2 + 48x - 80$, given that $4i$ is a zero of f. ■

CONCEPT CHECK

1. Given that $2 + 3i$ is a zero of a polynomial function f with real coefficients, name another zero of f.
2. Explain how to find a second-degree polynomial function with real coefficients that has $-i$ as a zero.
3. Explain the difference between a polynomial that is irreducible over the rationals and a polynomial that is irreducible over the reals. Justify your answer with examples.
4. Does the Fundamental Theorem of Algebra indicate that a cubic function must have at least one imaginary zero? Explain.

Skills Review 3.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 3.5.

In Exercises 1–4, write the complex number in standard form and find its complex conjugate.

1. $4 - \sqrt{-29}$ **2.** $-5 - \sqrt{-144}$ **3.** $-1 + \sqrt{-32}$ **4.** $6 + \sqrt{-1/4}$

In Exercises 5–10, perform the indicated operation and write the result in standard form.

5. $(-3 + 6i) - (10 - 3i)$

6. $(12 - 4i) + 20i$

7. $(4 - 2i)(3 + 7i)$

8. $(2 - 5i)(2 + 5i)$

9. $\dfrac{1 + i}{1 - i}$

10. $(3 + 2i)^3$

Exercises 3.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, determine the number of zeros of the polynomial function.

1. $f(x) = x - 7$

2. $g(x) = x^4 - 256$

3. $h(x) = -x^3 + 2x^2 - 5$

4. $f(t) = -2t^5 - 3t^3 + 1$

5. $f(x) = 6x - x^4$

6. $f(x) = 3 - 7x^2 - 5x^4 + 9x^6$

In Exercises 7–34, find all the zeros of the function and write the polynomial as a product of linear factors.

7. $f(x) = x^2 + 16$

8. $f(x) = x^2 + 36$

9. $h(x) = x^2 - 5x + 5$

10. $g(x) = x^2 + 10x + 23$

11. $f(x) = x^4 - 81$

12. $f(t) = t^4 - 625$

13. $g(x) = x^3 + 5x$

14. $g(x) = x^3 + 7x$

15. $h(x) = x^3 - 11x^2 - 15x + 325$

16. $h(x) = x^3 - 3x^2 + 4x - 2$

17. $g(x) = x^3 - 6x^2 + 13x - 10$

18. $f(x) = x^3 - 2x^2 - 11x + 52$

19. $f(t) = t^3 - 3t^2 - 15t + 125$

20. $f(x) = x^3 + 8x^2 + 20x + 13$

21. $f(x) = x^3 + 24x^2 + 214x + 740$

22. $h(x) = x^3 - x + 6$

23. $h(x) = x^3 + 9x^2 + 27x + 35$

24. $f(s) = 2s^3 - 5s^2 + 12s - 5$

25. $f(x) = 16x^3 - 20x^2 - 4x + 15$

26. $f(x) = 9x^3 - 15x^2 + 11x - 5$

27. $f(x) = 5x^3 - 9x^2 + 28x + 6$

28. $g(x) = 3x^3 - 4x^2 + 8x + 8$

29. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

30. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

31. $f(x) = x^4 + 10x^2 + 9$

32. $f(x) = x^4 + 29x^2 + 100$

33. $f(t) = t^5 + 5t^4 - 7t^3 - 43t^2 - 8t - 48$

34. $g(x) = x^5 - 8x^4 + 28x^3 - 56x^2 + 64x - 32$

In Exercises 35–44, find a polynomial with real coefficients that has the given zeros. (There are many correct answers.)

35. $-2, 3i, -3i$

36. $5, 2i, -2i$

37. $1, 2 + i, 2 - i$

38. $6, -5 + 2i, -5 - 2i$

39. $-4, 3i, -3i, 2i, -2i$

40. $2, 2, 2, 4i, -4i$

41. $-5, -5, 1 + \sqrt{3}\,i$

42. $0, 0, 4, 1 + i$

43. $\frac{2}{3}, -1, 3 + \sqrt{2}\,i$

44. $\frac{3}{4}, -2, -\frac{1}{2} + i$

In Exercises 45–48, write the polynomial (a) as the product of factors that are irreducible over the *rationals,* (b) as the product of linear and quadratic factors that are irreducible over the *reals,* and (c) in completely factored form.

45. $x^4 - 7x^2 - 8$

46. $x^4 - 6x^2 - 72$

47. $x^4 - 5x^3 + 4x^2 + x - 15$
(*Hint:* One factor is $x^2 - 2x + 3$.)

48. $x^4 + x^3 + 8x^2 + 9x - 9$
(*Hint:* One factor is $x^2 + 9$.)

In Exercises 49–58, use the given zero of f to find all the zeros of f.

49. $f(x) = 3x^3 - 7x^2 + 12x - 28, 2i$

50. $f(x) = 3x^3 - x^2 + 27x - 9, 3i$

51. $f(x) = x^4 - 2x^3 + 37x^2 - 72x + 36, \quad 6i$

52. $f(x) = x^3 - 7x^2 - x + 87, \quad 5 + 2i$

53. $f(x) = 4x^3 + 23x^2 + 34x - 10, \quad -3 + i$

54. $f(x) = 3x^3 - 10x^2 + 31x + 26, 2 + 3i$

55. $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22, \quad -3 + \sqrt{2}\,i$

56. $f(x) = 2x^3 - 13x^2 + 34x - 35, 2 - \sqrt{3}\,i$

57. $f(x) = 8x^3 - 14x^2 + 18x - 9, \quad \frac{1}{2}(1 - \sqrt{5}\,i)$

58. $f(x) = 25x^3 - 55x^2 - 54x - 18, \quad \frac{1}{5}(-2 + \sqrt{2}\,i)$

59. Graphic Reasoning Solve $x^4 - 5x^2 + 4 = 0$. Then use a graphing utility to graph

$y = x^4 - 5x^2 + 4.$

What is the connection between the solutions you found and the intercepts of the graph?

60. Graphical Reasoning Solve $x^4 + 5x^2 + 4 = 0$. Then use a graphing utility to graph

$y = x^4 + 5x^2 + 4.$

What is the connection between the solutions you found and the intercepts of the graph?

61. Graphical Analysis Find a fourth-degree polynomial function that has (a) four real zeros, (b) two real zeros, and (c) no real zeros. Use a graphing utility to graph the functions and describe the similarities and differences among them.

62. Graphical Analysis Find a sixth-degree polynomial function that has (a) six real zeros, (b) four real zeros, (c) two real zeros, and (d) no real zeros. Use a graphing utility to graph the functions and describe the similarities and differences among them.

63. Profit The demand and cost equations for a stethoscope are given by

$p = 140 - 0.0001x$

and

$C = 80x + 150{,}000$

where p is the unit price (in dollars), C is the total cost (in dollars), and x is the number of units. The total profit P (in dollars) obtained by producing and selling x units is given by

$P = R - C = xp - C.$

Try to determine a price p that would yield a profit of \$9 million, and then use a graphing utility to explain why this is not possible.

64. Revenue The demand equation for a stethoscope is given by

$p = 140 - 0.0001x$

where p is the unit price (in dollars) and x is the number of units sold. The total revenue R obtained by producing and selling x units is given by

$R = xp.$

Try to determine a price p that would yield a revenue of \$50 million, and then use a graphing utility to explain why this is not possible.

65. Reasoning The imaginary number $2i$ is a zero of

$f(x) = x^3 - 2ix^2 - 4x + 8i$

but the complex conjugate of $2i$ is not a zero of $f(x)$. Is this a contradiction of the conjugate pairs statement on page 317? Explain.

66. Reasoning The imaginary number $1 - 2i$ is a zero of

$f(x) = x^3 - (1 - 2i)x^2 - 9x + 9(1 - 2i)$

but $1 + 2i$ is not a zero of $f(x)$. Is this a contradiction of the conjugate pairs statement on page 317? Explain.

67. Reasoning Let f be a fourth-degree polynomial function with real coefficients. Three of the zeros of f are

3, $1 + i$, and $1 - i$.

Explain how you know that the fourth zero must be a real number.

68. Reasoning Let f be a fourth-degree polynomial function with real coefficients. Three of the zeros of f are

-1, 2, and $3 + 2i$.

What is the fourth zero? Explain.

69. Reasoning Let f be a third-degree polynomial function with real coefficients. Explain how you know that f must have at least one zero that is a real number.

70. Reasoning Let f be a fifth-degree polynomial function with real coefficients. Explain how you know that f must have at least one zero that is a real number.

71. Think About It A student claims that a third-degree polynomial function with real coefficients can have three complex zeros. Describe how you could use a graphing utility and the Leading Coefficient Test (Section 3.2) to convince the student otherwise.

72. Think About It A student claims that the polynomial

$x^4 - 7x^2 + 12$

may be factored over the rational numbers as

$(x - \sqrt{3})(x + \sqrt{3})(x - 2)(x + 2).$

Do you agree with this claim? Explain your answer.

Section 3.7

Rational Functions

- Find the domain of a rational function.
- Find the vertical and horizontal asymptotes of the graph of a rational function.
- Sketch the graph of a rational function.
- Sketch the graph of a rational function that has a slant asymptote.
- Use a rational function model to solve an application problem.

Introduction

A **rational function** is one that can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and $q(x)$ is not the zero polynomial. In this section, assume that $p(x)$ and $q(x)$ have no common factors. Unlike polynomial functions, whose domains consist of all real numbers, rational functions often have restricted domains. In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero.

Example 1 Finding the Domain of a Rational Function

Find the domain of $f(x) = \frac{1}{x}$ and discuss its behavior near any excluded x-values.

SOLUTION The domain of f is all real numbers except $x = 0$. To determine the behavior of f near this x-value, evaluate $f(x)$ to the left and right of $x = 0$.

x approaches 0 from the left

x	-1	-0.5	-0.1	-0.01	-0.001	$\rightarrow 0$
$f(x)$	-1	-2	-10	-100	-1000	$\rightarrow -\infty$

x approaches 0 from the right

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

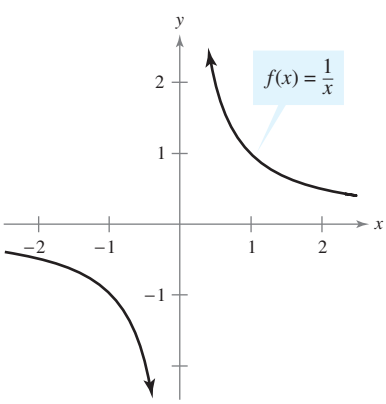


FIGURE 3.47

Note that as x approaches 0 *from the left,* $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right,* $f(x)$ increases without bound. The graph of f is shown in Figure 3.47.

✓CHECKPOINT 1

Find the domain of $f(x) = \frac{1}{x - 1}$ and discuss the behavior of f near any excluded x-values. ■

Horizontal and Vertical Asymptotes

In Example 1, the behavior of f near $x = 0$ is denoted as follows.

$f(x) \to -\infty$ as $x \to 0^-$ — $f(x)$ decreases without bound as x approaches 0 from the left.

$f(x) \to \infty$ as $x \to 0^+$ — $f(x)$ increases without bound as x approaches 0 from the right.

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 3.48. In this figure, note that the graph of f also has a **horizontal asymptote**~~the~~ line $y = 0$. The behavior of f near $y = 0$ is denoted as follows.

$f(x) \to 0$ as $x \to -\infty$ — $f(x)$ approaches 0 as x decreases without bound.

$f(x) \to 0$ as $x \to \infty$ — $f(x)$ approaches 0 as x increases without bound.

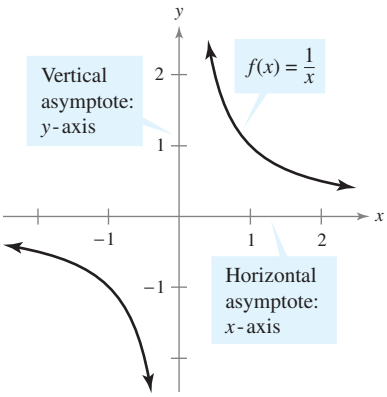
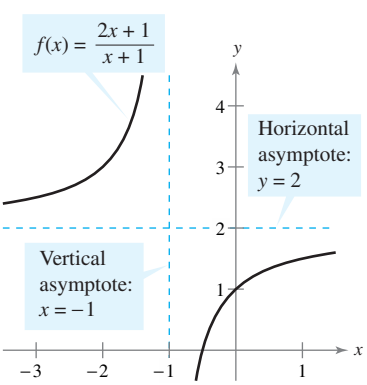


FIGURE 3.48

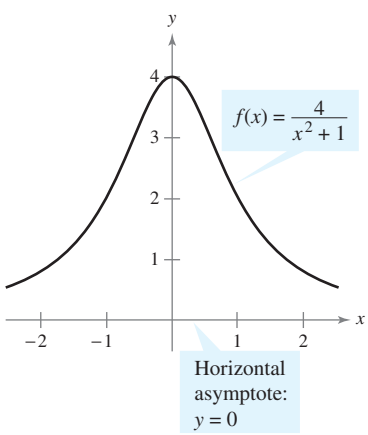
Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if
$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$
as $x \to a$, either from the right or from the left.
2. The line $y = b$ is a **horizontal asymptote** of the graph of f if
$$f(x) \to b$$
as $x \to \infty$ or $x \to -\infty$.

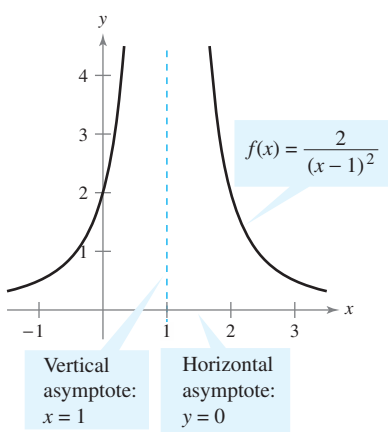
The graph of a rational function can never intersect its vertical asymptote. It may or may not intersect its horizontal asymptote. In either case, the distance between the horizontal asymptote and the points on the graph must approach zero (as $x \to \infty$ or $x \to -\infty$). Figure 3.49 shows the horizontal and vertical asymptotes of the graphs of three rational functions.



(a)



(b)



(c)

FIGURE 3.49

The graphs of $f(x) = 1/x$ in Figure 3.48 and $f(x) = (2x + 1)/(x + 1)$ in Figure 3.49(a) are **hyperbolas.**

Asymptotes of a Rational Function

Let f be the rational function given by

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \quad a_n \neq 0,\ b_m \neq 0.$$

1. The graph of f has *vertical* asymptotes at the zeros of $q(x)$.
2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $p(x)$ and $q(x)$.
 - **a.** If $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
 - **b.** If $n = m$, the graph of f has the line $y = a_n/b_m$ (ratio of the leading coefficients) as a horizontal asymptote.
 - **c.** If $n > m$, the graph of f has no horizontal asymptote.

Example 2 Finding Horizontal and Vertical Asymptotes

Find all horizontal and vertical asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x}{3x^2 + 1}$ **b.** $f(x) = \dfrac{2x^2}{x^2 - 1}$

SOLUTION

a. For this rational function, the degree of the numerator is *less than* the degree of the denominator, so the graph has the line $y = 0$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x. Because the equation $3x^2 + 1 = 0$ has no real solutions, you can conclude that the graph has no vertical asymptote. The graph of the function is shown in Figure 3.50(a).

b. For this rational function, the degree of the numerator is *equal to* the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$x^2 - 1 = 0 \qquad \text{Set denominator equal to zero.}$$
$$(x + 1)(x - 1) = 0 \qquad \text{Factor.}$$
$$x + 1 = 0 \Rightarrow x = -1 \qquad \text{Set 1st factor equal to 0.}$$
$$x - 1 = 0 \Rightarrow x = 1 \qquad \text{Set 2nd factor equal to 0.}$$

This equation has two real solutions $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes. The graph of the function is shown in Figure 3.50(b).

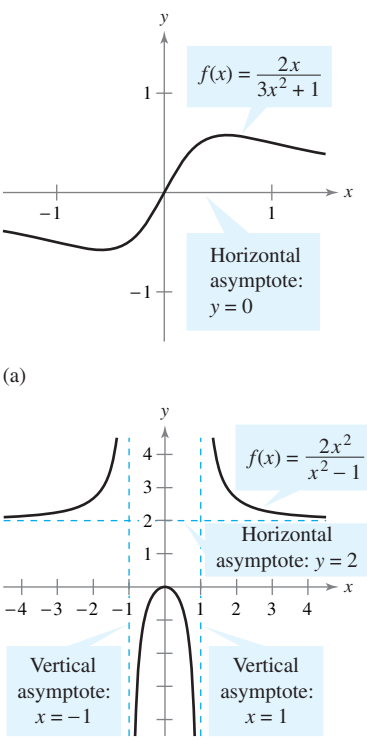


FIGURE 3.50

✓CHECKPOINT 2

Find all horizontal and vertical asymptotes of the graph of $f(x) = \dfrac{x^2}{x^2 + 1}$. ■

Sketching the Graph of a Rational Function

DISCOVERY

Consider the rational function

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

Is $x = 2$ in the domain of f? Graph f on a graphing utility. Is there a vertical asymptote at $x = 2$? Describe the graph of f. Factor the numerator and reduce the rational function. Describe the resulting function. Under what conditions will a rational function have no vertical asymptote?

Guidelines for Graphing Rational Functions

Let $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with no common factors.

1. Find and plot the y-intercept (if any) by evaluating $f(0)$.
2. Find the zeros of the numerator (if any) by solving the equation $p(x) = 0$. Then plot the corresponding x-intercepts.
3. Find the zeros of the denominator (if any) by solving the equation $q(x) = 0$. Then sketch the corresponding vertical asymptotes.
4. Find and sketch the horizontal asymptote (if any) by using the rule for finding the horizontal asymptote of a rational function.
5. Test for symmetry.
6. Plot at least one point both *between and beyond* each x-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.

Testing for symmetry can be useful, especially for simple rational functions. For example, the graph of $f(x) = 1/x$ is symmetric with respect to the origin, and the graph of $g(x) = 1/x^2$ is symmetric with respect to the y-axis.

Example 3 Sketching the Graph of a Rational Function

Sketch the graph of $g(x) = \dfrac{3}{x - 2}$.

SOLUTION Begin by noting that the numerator and denominator have no common factors.

y-intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$

x-intercept: None, numerator has no zeros.

Vertical asymptote: $x = 2$, zero of denominator

Horizontal asymptote: $y = 0$, degree of $p(x) <$ degree of $q(x)$

Additional points:

x	-4	1	2	3	5
$g(x)$	-0.5	-3	Undefined	3	1

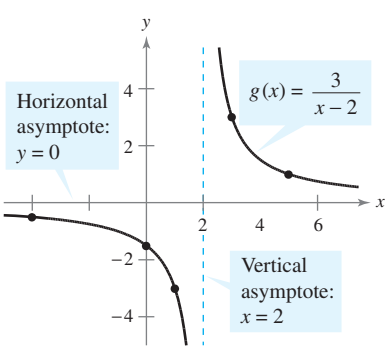


FIGURE 3.51

By plotting the intercepts, asymptotes, and a few additional points, you can obtain the graph shown in Figure 3.51. In the figure, note that the graph of g is a vertical stretch and a right shift of the graph of $y = 1/x$.

✓CHECKPOINT 3

Sketch the graph of $f(x) = \dfrac{1}{x + 2}$. ■

Note that in the examples in this section, the vertical asymptotes are included in the table of additional points. This is done to emphasize numerically the behavior of the graph of the function.

Example 4 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x}{x^2 - x - 2}$.

SOLUTION Factor the denominator to determine more easily the zeros of the denominator.

$$f(x) = \frac{x}{x^2 - x - 2} = \frac{x}{(x + 1)(x - 2)}$$

y-intercept: $(0, 0)$, because $f(0) = 0$

x-intercept: $(0, 0)$

Vertical asymptotes: $x = -1, x = 2$, zeros of denominator

Horizontal asymptote: $y = 0$, degree of $p(x) <$ degree of $q(x)$

Additional points:

x	-3	-1	-0.5	1	2	3
$f(x)$	-0.3	Undefined	0.4	-0.5	Undefined	0.75

The graph is shown in Figure 3.52. Confirm the graph with your graphing utility.

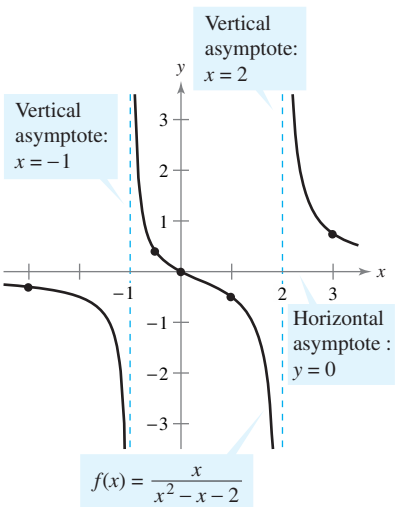


FIGURE 3.52

✓CHECKPOINT 4

Sketch the graph of $f(x) = \dfrac{3x}{x^2 + x - 6}$. ■

Example 5 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$.

SOLUTION By factoring the numerator and denominator, you have

$$f(x) = \frac{2(x^2 - 9)}{x^2 - 4} = \frac{2(x - 3)(x + 3)}{(x - 2)(x + 2)}.$$

y-intercept: $\left(0, \frac{9}{2}\right)$, because $f(0) = \frac{9}{2}$

x-intercepts: $(-3, 0)$ and $(3, 0)$

Vertical asymptotes: $x = -2, x = 2$, zeros of denominator

Horizontal asymptote: $y = 2$, degree of $p(x) =$ degree of $q(x)$

Symmetry: With respect to y-axis, because $f(-x) = f(x)$

Additional points:

x	-2	0.5	2	2.5	6
$f(x)$	Undefined	4.67	Undefined	-2.44	1.6875

The graph is shown in Figure 3.53.

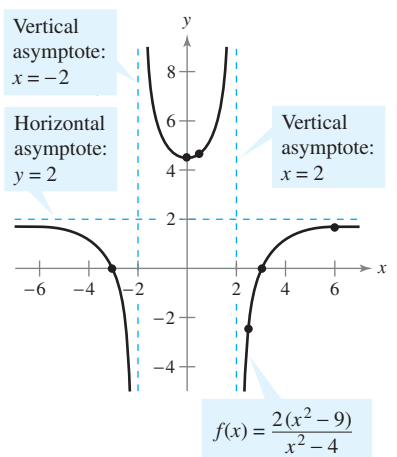


FIGURE 3.53

✓CHECKPOINT 5

Sketch the graph of

$$f(x) = \frac{5(x^2 - 1)}{x^2 - 9}.$$ ■

Slant Asymptotes

Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 3.54. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you have

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\text{Slant asymptote } (y = x - 2)} + \frac{2}{x + 1}.$$

As x increases or decreases without bound, the remainder term $2/(x + 1)$ approaches 0, so the graph of f approaches the line $y = x - 2$, as shown in Figure 3.54.

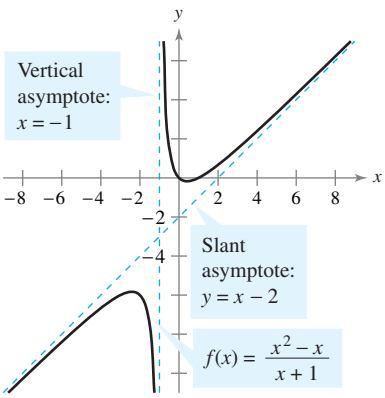


FIGURE 3.54

Example 6 A Rational Function with a Slant Asymptote

Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

SOLUTION First write f in two different ways. Factoring the numerator enables you to recognize the x-intercepts.

$$f(x) = \frac{x^2 - x - 2}{x - 1} = \frac{(x - 2)(x + 1)}{x - 1}$$

Then long division enables you to recognize that the line $y = x$ is a slant asymptote of the graph.

$$f(x) = \frac{x^2 - x - 2}{x - 1} = x - \frac{2}{x - 1}$$

y-intercept: $(0, 2)$, because $f(0) = 2$

x-intercepts: $(-1, 0)$ and $(2, 0)$

Vertical asymptote: $x = 1$, zero of denominator

Horizontal asymptote: None; degree of $p(x)$ > degree of $q(x)$

Slant asymptote: $y = x$

Additional points:

x	-2	0.5	1	1.5	3
$f(x)$	$-1.\overline{3}$	4.5	Undefined	-2.5	2

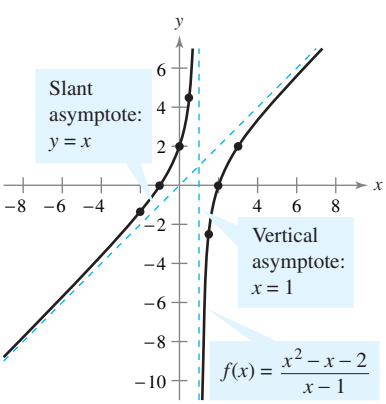


FIGURE 3.55

The graph is shown in Figure 3.55.

✓CHECKPOINT 6

Sketch the graph of $f(x) = \dfrac{x^2 + 3x + 2}{x - 1}$.

Applications

TECHNOLOGY

Most graphing utilities do not produce good graphs of rational functions (the presence of vertical asymptotes is a problem). To obtain a reasonable graph, you should set the utility to *dot* mode. For specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

There are many examples of asymptotic behavior in business and biology. For instance, the following example describes the asymptotic behavior related to the cost of removing smokestack emissions.

Example 7 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost of removing a certain *percent* of the pollutants from the stack emissions is typically not a linear function. That is, if it costs C dollars to remove 25% of the pollutants, it would cost more than $2C$ dollars to remove 50% of the pollutants. As the percent of pollutants removed approaches 100%, the cost tends to become prohibitive. The cost C (in dollars) of removing p percent of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}.$$

Suppose that you are a member of a state legislature that is considering a law that will require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal.

a. How much additional expense is the new law asking the utility company to incur?

b. According to the model, would it be possible to remove 100% of the pollutants?

SOLUTION

a. The graph of this function is shown in Figure 3.56. Note that the graph has a vertical asymptote at $p = 100$. Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \qquad \text{Substitute 85 for } p.$$

$$\approx \$453{,}333. \qquad \text{Use a calculator.}$$

If the new law increases the percent removal to 90%, the cost to the utility company will be

$$C = \frac{80{,}000(90)}{100 - 90} \qquad \text{Substitute 90 for } p.$$

$$= \$720{,}000. \qquad \text{Use a calculator.}$$

The new law would require the utility company to spend an additional

$$\$720{,}000 - \$453{,}333 = \$266{,}667.$$

b. From Figure 3.56, you can see that the graph has a vertical asymptote at $p = 100$. Because the graph of a rational function can never intersect its vertical asymptote, you can conclude that it is not possible for the company to remove 100% of the pollutants from the stack emissions.

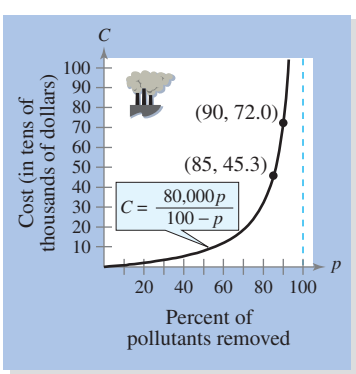


FIGURE 3.56

✓CHECKPOINT 7

In Example 7, suppose the new law will require utility companies to remove 95% of the pollutants. Find the additional cost to the utility company.

Example 8 Per Capita Land Area

A model for the population P (in millions) of the United States from 1960 to 2005 is $P = 2.5049t + 179.214$, where t represents the year, with $t = 0$ corresponding to 1960. A model for the land area A (in millions of acres) of the United States from 1960 to 2005 is $A = 2263.960$. Construct a rational function for per capita land area L (in acres per person). Sketch a graph of the rational function. Use the model to predict the per capita land area in 2013. *(Source: U.S. Census Bureau)*

SOLUTION The rational function for the per capita land area L is

$$L = \frac{A}{P} = \frac{2263.960}{2.5049t + 179.214}.$$

The graph of the function is shown in Figure 3.57. To find the per capita land area in 2013, substitute $t = 53$ into L.

$$L = \frac{2263.960}{2.5049t + 179.214} = \frac{2263.960}{2.5049(53) + 179.214} \approx \frac{2263.960}{311.974} \approx 7.26$$

The per capita land area will be approximately 7.3 acres per person in 2013.

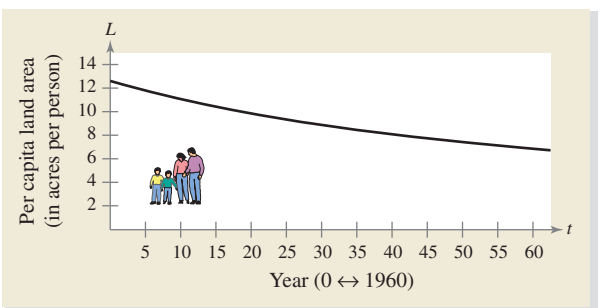


FIGURE 3.57

✓ CHECKPOINT 8

In Example 8, use the model to predict the per capita land area in 2020. ■

CONCEPT CHECK

In Exercises 1–4, determine whether the statement is true or false. Justify your answer.

1. The domain of $f(x) = \dfrac{x^2 + 2x - 8}{x^2 - 9}$ is all real numbers except $x = -3$ and $x = 3$.
2. The graph of $g(x) = \dfrac{x^2 - 1}{x^2 + 4x + 4}$ has vertical asymptotes $x = -1$ and $x = 1$.
3. The graph of every rational function has a horizontal asymptote.
4. A rational function f has a numerator of degree n. The graph of f has a slant asymptote. So, the denominator has degree n.

Skills Review 3.7

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.6 and 2.2.

In Exercises 1–6, factor the polynomial.

1. $x^2 - 4x$

2. $2x^3 - 6x$

3. $x^2 - 3x - 10$

4. $x^2 - 7x + 10$

5. $x^3 + 4x^2 + 3x$

6. $x^3 - 4x^2 - 2x + 8$

In Exercises 7–10, sketch the graph of the equation.

7. $y = 2$

8. $x = -1$

9. $y = x - 2$

10. $y = -x + 1$

Exercises 3.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find the domain of the function and identify any horizontal and vertical asymptotes.

1. $f(x) = \dfrac{3x}{x + 1}$

2. $f(x) = \dfrac{x}{x - 2}$

3. $f(x) = \dfrac{x - 7}{5 - x}$

4. $f(x) = \dfrac{1 - 5x}{1 + 2x}$

5. $f(x) = \dfrac{3x^2 + 1}{x^2 + 9}$

6. $f(x) = \dfrac{3x^2 + x - 5}{x^2 + 1}$

7. $f(x) = \dfrac{5}{(x + 4)^2}$

8. $f(x) = \dfrac{1}{(x - 1)^2}$

In Exercises 9–12, find any (a) vertical, (b) horizontal, and (c) slant asymptotes of the graph of the function. Then sketch the graph of f.

9. $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$

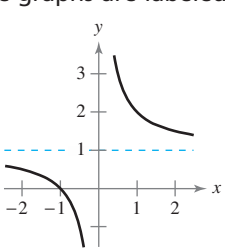
10. $f(x) = \dfrac{x + 3}{x^2 - 9}$

11. $f(x) = \dfrac{x^2}{x + 1}$

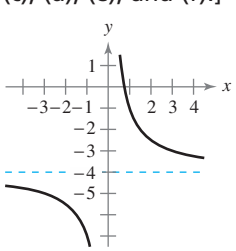
12. $f(x) = \dfrac{x^3 + x}{x^2 - 1}$

In Exercises 13–18, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

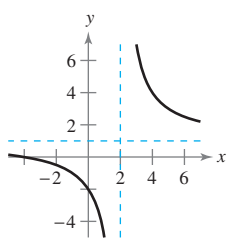
(a)



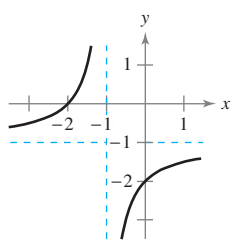
(b)



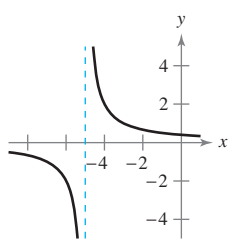
(c)



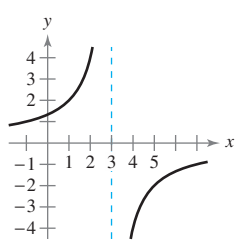
(d)



(e)



(f)



13. $f(x) = -\dfrac{4}{x - 3}$

14. $f(x) = \dfrac{2}{x + 5}$

15. $f(x) = \dfrac{x + 1}{x}$

16. $f(x) = \dfrac{3 - 4x}{x}$

17. $f(x) = \dfrac{x + 4}{x - 2}$

18. $f(x) = -\dfrac{x + 2}{x + 1}$

In Exercises 19–22, compare the graph of $f(x) = 1/x$ with the graph of g.

19. $g(x) = f(x) - 2 = \dfrac{1}{x} - 2$

20. $g(x) = f(x + 1) = \dfrac{1}{x + 1}$

21. $g(x) = -f(x) = -\dfrac{1}{x}$

22. $g(x) = -f(x+1) = -\dfrac{1}{x+1}$

In Exercises 23–26, compare the graph of $f(x) = 4/x^2$ with the graph of g.

23. $g(x) = f(x) + 3 = \dfrac{4}{x^2} + 3$

24. $g(x) = f(x-1) = \dfrac{4}{(x-1)^2}$

25. $g(x) = -f(x) = -\dfrac{4}{x^2}$

26. $g(x) = \dfrac{1}{8}f(x) = \dfrac{1}{2x^2}$

In Exercises 27–30, compare the graph of $f(x) = 8/x^3$ with the graph of g.

27. $g(x) = f(x) + 5 = \dfrac{8}{x^3} + 5$

28. $g(x) = f(x-3) = \dfrac{8}{(x-3)^3}$

29. $g(x) = -f(x) = -\dfrac{8}{x^3}$

30. $g(x) = \dfrac{1}{4}f(x) = \dfrac{2}{x^3}$

In Exercises 31–56, sketch the graph of the rational function. To aid in sketching the graphs, check for intercepts, symmetry, vertical asymptotes, and horizontal asymptotes.

31. $f(x) = \dfrac{1}{x+3}$

32. $f(x) = \dfrac{1}{x-3}$

33. $f(x) = \dfrac{1}{x-4}$

34. $f(x) = \dfrac{1}{x+6}$

35. $f(x) = \dfrac{-1}{x+1}$

36. $f(x) = \dfrac{-2}{x-3}$

37. $f(x) = \dfrac{x+4}{x-5}$

38. $f(x) = \dfrac{x-2}{x-3}$

39. $f(x) = \dfrac{2+x}{1-x}$

40. $f(x) = \dfrac{3-x}{2-x}$

41. $f(t) = \dfrac{3t+1}{t}$

42. $f(t) = \dfrac{1-2t}{t}$

43. $C(x) = \dfrac{5+2x}{1+x}$

44. $P(x) = \dfrac{1-3x}{1-x}$

45. $g(x) = \dfrac{1}{x+2} + 2$

46. $h(x) = \dfrac{1}{x-3} + 1$

47. $f(x) = \dfrac{1}{x^2} + 2$

48. $f(x) = 2 - \dfrac{3}{x^2}$

49. $h(x) = \dfrac{x^2}{x^2-9}$

50. $h(t) = \dfrac{3t^2}{t^2-4}$

51. $g(s) = \dfrac{s}{s^2+1}$

52. $g(x) = \dfrac{x}{x^2+3}$

53. $f(x) = \dfrac{x}{x^2-3x-4}$

54. $f(x) = \dfrac{-x}{x^2+x-6}$

55. $f(x) = \dfrac{3x}{x^2-x-2}$

56. $f(x) = \dfrac{2x}{x^2+x-2}$

In Exercises 57–60, write a rational function f that has the specified characteristics. (There are many correct answers.)

57. Vertical asymptote: None
Horizontal asymptote: $y = 2$

58. Vertical asymptotes: $x = 0, x = \frac{5}{2}$
Horizontal asymptote: $y = -3$

59. Vertical asymptotes: $x = -2, x = 1$
Horizontal asymptote: None

60. Vertical asymptote: $x = 3$
Horizontal asymptote: x-axis

In Exercises 61–64, find a counterexample to show that the statement is incorrect.

61. Every rational function has a vertical asymptote.

62. Every rational function has at least one asymptote.

63. A rational function can have only one vertical asymptote.

64. The graph of a rational function with a slant asymptote cannot cross its slant asymptote.

65. Is it possible for a rational function to have all three types of asymptotes (vertical, horizontal, and slant)? Why or why not?

66. Is it possible for a rational function to have more than one horizontal asymptote? Why or why not?

67. ***MAKE A DECISION: SEIZURE OF ILLEGAL DRUGS*** The cost C (in millions of dollars) for the federal government to seize p percent of an illegal drug as it enters the country is

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the cost of seizing 25% of the drug.

(b) Find the cost of seizing 50% of the drug.

(c) Find the cost of seizing 75% of the drug.

(d) According to this model, would it be possible to seize 100% of the drug? Explain.

68. ***MAKE A DECISION: WATER POLLUTION*** The cost C (in millions of dollars) of removing p percent of the industrial and municipal pollutants discharged into a river is

$$C = \frac{255p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the cost of removing 15% of the pollutants.

(b) Find the cost of removing 50% of the pollutants.

(c) Find the cost of removing 80% of the pollutants.

(d) According to the model, would it be possible to remove 100% of the pollutants? Explain.

69. **Population of Deer** The Game Commission introduces 100 deer into newly acquired state game lands. The population N of the herd is given by

$$N = \frac{25(4 + 2t)}{1 + 0.02t}, \quad t \ge 0$$

where t is time (in years).

(a) Find the populations when t is 5, 10, and 25.

(b) What is the limiting size of the herd as time progresses?

70. **Population of Elk** The Game Commission introduces 40 elk into newly acquired state game lands. The population N of the herd is given by

$$N = \frac{10(4 + 2t)}{1 + 0.03t}, \quad t \ge 0$$

where t is time (in years).

(a) Find the populations when t is 5, 10, and 25.

(b) What is the limiting size of the herd as time progresses?

71. **Defense** The table shows the national defense outlays D (in billions of dollars) from 1997 to 2005. The data can be modeled by

$$D = \frac{1.493t^2 - 39.06t + 273.5}{0.0051t^2 - 0.1398t + 1}, \quad 7 \le t \le 15$$

where t is the year, with $t = 7$ corresponding to 1997. *(Source: U.S. Office of Management and Budget)*

Year	Defense outlays
1997	270.5
1998	268.5
1999	274.9
2000	294.5
2001	305.5

Year	Defense outlays
2002	348.6
2003	404.9
2004	455.9
2005	465.9

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model represent the data?

(b) Use the model to predict the national defense outlays for the years 2010, 2015, and 2020. Are the predictions reasonable?

(c) Determine the horizontal asymptote of the graph of the model. What does it represent in the context of the situation?

72. **Average Cost** The cost C (in dollars) of producing x basketballs is $C = 375{,}000 + 4x$. The average cost $\overline{C}$ per basketball is

$$\overline{C} = \frac{C}{x} = \frac{375{,}000 + 4x}{x}, \quad x > 0.$$

(a) Sketch the graph of the average cost function.

(b) Find the average costs of producing 1000, 10,000, and 100,000 basketballs.

(c) Find the horizontal asymptote and explain its meaning in the context of the problem.

73. **Human Memory Model** Psychologists have developed mathematical models to predict memory performance as a function of the number of trials n of a certain task. Consider the learning curve modeled by

$$P = \frac{0.6 + 0.95(n - 1)}{1 + 0.95(n - 1)}, \quad n > 0$$

where P is the percent of correct responses (in decimal form) after n trials.

(a) Complete the table.

n	1	2	3	4	5	6	7	8	9	10
P										

(b) According to this model, what is the limiting percent of correct responses as n increases?

74. Human Memory Model Consider the learning curve modeled by

$$P = \frac{0.55 + 0.87(n - 1)}{1 + 0.87(n - 1)}, \quad n > 0$$

where P is the percent of correct responses (in decimal form) after n trials.

(a) Complete the table.

n	1	2	3	4	5	6	7	8	9	10
P										

(b) According to this model, what is the limiting percent of correct responses as n increases?

75. Average Recycling Cost The cost C (in dollars) of recycling a waste product is

$$C = 450{,}000 + 6x, \quad x > 0$$

where x is the number of pounds of waste. The average recycling cost $\overline{C}$ per pound is

$$\overline{C} = \frac{C}{x} = \frac{450{,}000 + 6x}{x}, \quad x > 0.$$

(T) (a) Use a graphing utility to graph $\overline{C}$.

(b) Find the average costs of recycling 10,000, 100,000, 1,000,000, and 10,000,000 pounds of waste. What can you conclude?

76. Drug Concentration The concentration C of a medication in the bloodstream t minutes after sublingual (under the tongue) application is given by

$$C(t) = \frac{3t - 1}{2t^2 + 5}, \quad t > 0.$$

(T) (a) Use a graphing utility to graph the function. Estimate when the concentration is greatest.

(b) Does this function have a horizontal asymptote? If so, discuss the meaning of the asymptote in terms of the concentration of the medication.

77. Domestic Demand The U.S. domestic demand D (in millions of barrels) for refined oil products from 1995 to 2005 can be modeled by

$$D = 100.9708t + 6083.999, \quad 5 \le t \le 15$$

where t represents the year, with $t = 5$ corresponding to 1995. The population P (in millions) of the United States from 1995 to 2005 can be modeled by

$$P = 3.0195t + 251.817, \quad 5 \le t \le 15$$

where t represents the year, with $t = 5$ corresponding to 1995. *(Sources: U.S. Energy Information Administration and the U.S. Census Bureau)*

(a) Construct a rational function B to describe the per capita demand for refined oil products.

(T) (b) Use a graphing utility to graph the rational function B.

(c) Use the model to predict the per capita demand for refined oil products in 2010.

78. Health Care Spending The total health care spending H (in millions of dollars) in the United States from 1995 to 2005 can be modeled by

$$H = 6136.36t^2 - 22{,}172.7t + 979{,}909, \quad 5 \le t \le 15$$

where t represents the year, with $t = 5$ corresponding to 1995. The population P (in millions) of the United States from 1995 to 2005 can be modeled by

$$P = 3.0195t + 251.817, \quad 5 \le t \le 15$$

where t represents the year, with $t = 5$ corresponding to 1995. *(Sources: U.S. Centers for Medicare and Medicaid Services and the U.S. Census Bureau)*

(a) Construct a rational function S to describe the per capita health spending.

(T) (b) Use a graphing utility to graph the rational function S.

(c) Use the model to predict the per capita health care spending in 2010.

79. 100-Meter Freestyle The winning times for the men's 100-meter freestyle swim at the Olympics from 1952 to 2004 can be approximated by the quadratic model

$$y = 86.24 - 0.752t + 0.0037t^2, \quad 52 \le t \le 104$$

where y is the winning time (in seconds) and t represents the year, with $t = 52$ corresponding to 1952. *(Sources: The World Almanac and Book of Facts 2005)*

(T) (a) Use a graphing utility to graph the model.

(b) Use the model to predict the winning times in 2008 and 2012.

(c) Does this model have a horizontal asymptote? Do you think that a model for this type of data should have a horizontal asymptote?

80. 3000-Meter Speed Skating The winning times for the women's 3000-meter speed skating race at the Olympics from 1960 to 2006 can be approximated by the quadratic model

$$y = 0.0202t^2 - 5.066t + 550.24, \quad 60 \le t \le 106$$

where y is the winning time (in seconds) and t represents the year, with $t = 60$ corresponding to 1960. *(Sources: World Almanac and Book of Facts 2005 and NBC)*

(a) Use a graphing utility to graph the model.

(T) (b) Use the model to predict the winning times in 2010 and 2014.

(c) Does this model have a horizontal asymptote? Do you think that a model for this type of data should have a horizontal asymptote?

Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 336.
Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 3.1	Review Exercises
■ Sketch the graph of a quadratic function and identify its vertex and intercepts.	*1–4*
■ Find a quadratic function given its vertex and a point on its graph.	*5, 6*
■ Construct and use a quadratic model to solve an application problem.	*7–12*
Section 3.2	
■ Determine right-hand and left-hand behavior of graphs of polynomial functions. When n is odd and the leading coefficient is positive, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$. When n is odd and the leading coefficient is negative, $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$. When n is even and the leading coefficient is positive, $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$. When n is even and the leading coefficient is negative, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.	*13–16*
■ Find the real zeros of a polynomial function.	*17–20*
Section 3.3	
■ Divide one polynomial by a second polynomial using long division.	*21, 22*
■ Simplify a rational expression using long division.	*23, 24*
■ Use synthetic division to divide two polynomials.	*25, 26, 31, 32*
■ Use the Remainder Theorem and synthetic division to evaluate a polynomial.	*27, 28*
■ Use the Factor Theorem to factor a polynomial.	*29, 30*
Section 3.4	
■ Find all possible rational zeros of a function using the Rational Zero Test.	*33, 34*
■ Find all real zeros of a function.	*35–42*
■ Approximate the real zeros of a polynomial function using the Intermediate Value Theorem.	*43, 44*
■ Approximate the real zeros of a polynomial function using a graphing utility.	*45, 46*
■ Apply techniques for approximating real zeros to solve an application problem.	*47, 48*

* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

Section 3.5

	Review Exercises
■ Find the complex conjugate of a complex number.	*49–52*
■ Perform operations with complex numbers and write the results in standard form.	*53–68*

$(a + bi) + (c + di) = (a + c) + (b + d)i$

$(a + bi) - (c + di) = (a - c) + (b - d)i$

$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

■ Solve a polynomial equation that has complex solutions.	*69–72*
■ Plot a complex number in the complex plane.	*73, 74*

Section 3.6

■ Use the Fundamental Theorem of Algebra and the Linear Factorization Theorem to write a polynomial as the product of linear factors.	*75–80*
■ Find a polynomial with real coefficients whose zeros are given.	*81, 82*
■ Factor a polynomial over the rational, real, and complex numbers.	*83, 84*
■ Find all real and complex zeros of a polynomial function.	*85–88*

Section 3.7

■ Find the domain of a rational function.	*89–92*
■ Find the vertical and horizontal asymptotes of the graph of a rational function.	*89–92*

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \quad a_n \neq 0, b_m \neq 0.$

1. The graph of f has *vertical* asymptotes at the zeros of $q(x)$.

2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $p(x)$ and $q(x)$.

a. If $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.

b. If $n = m$, the graph of f has the line $y = a_n/b_m$ (ratio of the leading coefficients) as a horizontal asymptote.

c. If $n > m$, the graph of f has no horizontal asymptote.

■ Sketch the graph of a rational function, including graphs with slant asymptotes.	*93–98*
■ Use a rational function model to solve an application problem.	*99–103*

Study Strategies

■ **Use a Graphing Utility** A graphing calculator or graphing software for a computer can help you in this course in two important ways. As an *exploratory device*, a graphing utility allows you to learn concepts by allowing you to compare graphs of functions. For instance, sketching the graphs of $f(x) = x^3$ and $f(x) = -x^3$ helps confirm that the negative coefficient has the effect of reflecting the graph about the x-axis. As a *problem-solving tool*, a graphing utility frees you from some of the difficulty of sketching complicated graphs by hand. The time you can save can be spent using mathematics to solve real-life problems.

■ **Problem-Solving Strategies** If you get stuck when trying to solve a real-life problem, consider the strategies below.

1. *Draw a Diagram.* If feasible, draw a diagram that represents the problem. Label all known values and unknown values on the diagram.

2. *Solve a Simpler Problem.* Simplify the problem, or write several simple examples of the problem. For instance, if you are asked to find the dimensions that will produce a maximum area, try calculating the areas of several examples.

3. *Rewrite the Problem in Your Own Words.* Rewriting a problem can help you understand it better.

4. *Guess and Check.* Try guessing the answer, then check your guess in the statement of the original problem. By refining your guesses, you may be able to think of a general strategy for solving the problem.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, sketch the graph of the quadratic function. Identify the vertex and intercepts.

1. $f(x) = (x + 3)^2 - 5$

2. $g(x) = -(x - 1)^2 + 3$

3. $h(x) = 3x^2 - 12x + 11$

4. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$

In Exercises 5 and 6, find an equation of the parabola that has the indicated vertex and whose graph passes through the given point.

5. Vertex: $(-5, -1)$; point: $(-2, 6)$

6. Vertex: $(2, 5)$; point: $(4, 7)$

7. Optimal Area The perimeter of a rectangular archaeological dig site is 500 feet. Let x represent the width of the dig site. Write a quadratic function for the area of the rectangle in terms of its width. Of all possible dig sites with perimeters of 500 feet, what are the measurements of the one with the greatest area?

8. Optimal Revenue Find the number of units that produces a maximum revenue R (in dollars) for

$$R = 900x - 0.015x^2$$

where x is the number of units produced.

9. Optimal Cost A manufacturer of retinal imaging systems has daily production costs C (in dollars per unit) of

$$C = 25{,}000 - 50x + 0.065x^2$$

where x is the number of units produced.

(a) Use a graphing utility to graph the cost function.

(b) Graphically estimate the number of units that should be produced to yield a minimum cost per unit.

(c) Explain how to confirm the result of part (b) algebraically.

10. Optimal Profit The profit P (in dollars) for an electronics company is given by

$$P = -0.00015x^2 + 155x - 450{,}000$$

where x is the number of units produced.

(a) Use a graphing utility to graph the profit function.

(b) Graphically estimate the number of units that should be produced to yield a maximum profit.

(c) Explain how to confirm the result of part (b) algebraically.

11. Regression Problem Let x be the angle (in degrees) at which a baseball is hit with a 30-hertz backspin at an initial speed of 40 meters per second and let $d(x)$ be the distance (in meters) the ball travels. The table shows the distances traveled for the different angles at which the ball is hit. *(Source: The Physics of Sports)*

x	10	15	30	36	42	43
$d(x)$	61.2	83.0	130.4	139.4	143.2	143.3

x	44	45	48	54	60
$d(x)$	142.8	142.7	140.7	132.8	119.7

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model from part (b) in the same viewing window as the scatter plot of the data.

(d) Find the vertex of the graph of the model from part (c). Interpret its meaning in the context of the problem.

12. Doctorates in Science The numbers of non-U.S. citizens from Thailand with temporary visas that were awarded doctorates in science for the years 2000 to 2005 are shown in the table. *(Source: National Science Foundation)*

Year	2000	2001	2002	2003
Number, D	90	118	130	138

Year	2004	2005
Number, D	128	115

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model from part (b) in the same viewing window as the scatter plot of the data.

(d) Find the vertex of the graph of the model from part (c). Interpret its meaning in the context of the problem.

In Exercises 13–16, describe the right-hand and left-hand behavior of the graph of the polynomial function.

13. $f(x) = \frac{1}{2}x^3 + 2x$

14. $f(x) = 5 + 4x^3 - x^5$

15. $f(x) = -x^6 + 3x^4 - x^2 + 6$

16. $f(x) = \frac{3}{4}(x^4 + 3x^2 + 2)$

In Exercises 17–20, find all real zeros of the function.

17. $f(x) = 16 - x^2$

18. $f(x) = x^4 - 6x^2 + 8$

19. $f(x) = x^3 - 7x^2 + 10x$

20. $f(x) = x^3 - 6x^2 - 3x + 18$

In Exercises 21 and 22, use long division to divide.

	Dividend	*Divisor*
21.	$2x^3 - 5x^2 - x$	$2x + 1$
22.	$x^4 - 5x^3 + 10x^2 - 12$	$x^2 - 2x + 4$

In Exercises 23 and 24, simplify the rational expression.

23. $\dfrac{x^3 + 9x^2 + 2x - 48}{x - 2}$ **24.** $\dfrac{x^4 + 5x^3 - 20x - 16}{x^2 - 4}$

In Exercises 25 and 26, use synthetic division to divide.

	Dividend	*Divisor*
25.	$x^3 - 6x + 9$	$x + 3$
26.	$x^5 - x^4 + x^3 - 13x^2 + x + 6$	$x - 2$

In Exercises 27 and 28, use synthetic division to find each function value.

27. $f(x) = 6 + 2x^2 - 3x^3$ (a) $f(2)$ (b) $f(-1)$

28. $f(x) = 2x^4 + 3x^3 + 6$ (a) $f\left(\frac{1}{2}\right)$ (b) $f(-1)$

In Exercises 29 and 30, (a) verify the given factors of $f(x)$, (b) find the remaining factors of $f(x)$, (c) use your results to write the complete factorization of $f(x)$, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.

	Function	*Factors*
29.	$f(x) = x^3 - 4x^2 - 11x + 30$	$(x - 5), (x + 3)$
30.	$f(x) = 3x^3 + 23x^2 + 37x - 15$	$(3x - 1), (x + 5)$

31. Profit The profit P (in dollars) from selling a motorcycle is given by

$$P = -42x^3 + 3000x^2 - 6000, \quad 0 \le x \le 65$$

where x is the advertising expense (in tens of thousands of dollars). For this motorcycle, the advertising expense was \$600,000 ($x = 60$) and the profit was \$1,722,000.

(a) Use a graphing utility to graph the function and use the result to find another advertising expense that would have produced the same profit.

(b) Use synthetic division to confirm the result of part (a) algebraically.

32. Profit The profit P (in dollars) from selling a novel is given by

$$P = -150x^3 + 7500x^2 - 450{,}000, \quad 0 \le x \le 45$$

where x is the advertising expense (in tens of thousands of dollars). For this novel, the advertising expense was \$400,000 ($x = 40$), and the profit was \$1,950,000.

(a) Use a graphing utility to graph the function and use the result to find another advertising expense that would have produced the same profit.

(b) Use synthetic division to confirm the result of part (a) algebraically.

In Exercises 33 and 34, use the Rational Zero Test to list all possible rational zeros of f. Then use a graphing utility to graph the function. Use the graph to help determine which of the possible rational zeros are actual zeros of the function.

33. $f(x) = -4x^3 + 8x^2 - 3x + 15$

34. $f(x) = 3x^4 + 4x^3 - 5x^2 + 10x - 8$

In Exercises 35–42, find all real zeros of the function.

35. $f(x) = x^3 + 2x^2 - 5x - 6$

36. $g(x) = 2x^3 - 15x^2 + 24x + 16$

37. $h(x) = 3x^4 - 27x^2 + 60$

38. $f(x) = x^5 - 4x^3 + 3x$

39. $B(x) = 6x^3 - 19x^2 + 11x + 6$

40. $C(x) = 3x^4 + 3x^3 - 7x^2 - x + 2$

41. $p(x) = x^4 - x^3 - 2x - 4$

42. $q(x) = x^5 - 2x^4 + 2x^3 - 4x^2 - 3x + 6$

In Exercises 43 and 44, use the Intermediate Value Theorem to approximate the zero of f in the interval $[a, b]$. Give your approximation to the nearest tenth.

43. $f(x) = x^3 - 4x + 3, \quad [-3, -2]$

44. $f(x) = x^5 + 5x^2 + x - 1, \quad [0, 1]$

In Exercises 45 and 46, use a graphing utility to approximate the real zeros of f. Give your approximations to the nearest thousandth.

45. $f(x) = 5x^3 - 11x - 3$

46. $f(x) = 2x^4 - 9x^3 - 5x^2 + 10x + 12$

47. Wholesale Revenue The revenues R (in millions of dollars) for Costco Wholesale for the years 1996 to 2005 are shown in the table. *(Source: Costco Wholesale)*

Year	Revenue, R
1996	19,566
1997	21,874
1998	24,270
1999	27,456
2000	32,164

Year	Revenue, R
2001	34,797
2002	38,762
2003	42,546
2004	48,107
2005	52,935

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a linear model, a quadratic model, a cubic model, and a quartic model for the data.

(c) Use a graphing utility to graph each model separately with the data in the same viewing window. How well does each model fit the data?

(d) Use each model to predict the year in which the revenue will be about \$65 billion. Explain any differences in the predictions.

48. Shoe Sales The sales S (in millions of dollars) for Steve Madden for the years 1996 to 2005 are shown in the table. *(Source: Steve Madden, LTD)*

Year	Sales, S
1996	45.8
1997	59.3
1998	85.8
1999	163.0
2000	205.1

Year	Sales, S
2001	243.4
2002	326.1
2003	324.2
2004	338.1
2005	375.8

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a linear model, a quadratic model, and a quartic model for the data.

(c) Use a graphing utility to graph each model separately with the data in the same viewing window. How well does each model fit the data?

(d) Use each model to predict the year in which the sales will be about \$500 million. Explain any differences in the predictions.

In Exercises 49–52, write the complex number in standard form and find its complex conjugate.

49. $\sqrt{-32}$ **50.** 12

51. $-3 + \sqrt{-16}$ **52.** $2 - \sqrt{-18}$

In Exercises 53–64, perform the indicated operation and write the result in standard form.

53. $(7 - 4i) + (-2 + 5i)$

54. $(14 + 6i) - (-1 - 2i)$

55. $(1 + \sqrt{-12})(5 - \sqrt{-3})$

56. $(3 - \sqrt{-4})(4 - \sqrt{-49})$

57. $(5 + 8i)(5 - 8i)$ **58.** $(\frac{1}{2} + \frac{3}{4}i)(\frac{1}{2} - \frac{3}{4}i)$

59. $-2i(4 - 5i)$ **60.** $-3(-2 + 4i)$

61. $(3 + 4i)^2$ **62.** $(2 - 5i)^2$

63. $(3 + 2i)^2 + (3 - 2i)^2$ **64.** $(1 + i)^2 - (1 - i)^2$

In Exercises 65–68, write the quotient in standard form.

65. $\dfrac{8 - i}{2 + i}$ **66.** $\dfrac{3 - 4i}{1 - 5i}$

67. $\dfrac{4 - 3i}{i}$ **68.** $\dfrac{2}{(1 + i)^2}$

In Exercises 69–72, solve the equation.

69. $2x^2 - x + 3 = 0$

70. $3x^2 + 6x + 11 = 0$

71. $4x^2 + 11x + 3 = 0$

72. $9x^2 - 2x + 5 = 0$

In Exercises 73 and 74, plot the complex number.

73. $-3 + 2i$ **74.** $-1 - 4i$

In Exercises 75–80, find all the zeros of the function and write the polynomial as a product of linear factors.

75. $f(x) = x^4 - 81$

76. $h(x) = 2x^3 - 5x^2 + 4x - 10$

77. $f(t) = t^3 + 5t^2 + 3t + 15$

78. $h(x) = x^4 + 17x^2 + 16$

79. $g(x) = 4x^3 - 8x^2 + 9x - 18$

80. $f(x) = x^5 - 2x^4 + x^3 - x^2 + 2x - 1$

In Exercises 81 and 82, find a polynomial with real coefficients that has the given zeros. (There are many correct answers.)

81. $1, 3i, -3i$

82. $1, -2, 1 - 3i, 1 + 3i$

In Exercises 83 and 84, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

83. $x^4 + 5x^2 - 24$

84. $x^4 - 2x^3 - 2x^2 - 14x - 63$
(*Hint:* One factor is $x^2 + 7$.)

In Exercises 85–88, use the given zero of f to find all the zeros of f.

85. $f(x) = 4x^3 - x^2 + 64x - 16, \quad -4i$

86. $f(x) = 50 - 75x + 2x^2 - 3x^3, \quad 5i$

87. $f(x) = x^4 + 7x^3 + 24x^2 + 58x + 40, \quad -1 + 3i$

88. $f(x) = x^4 + 4x^3 + 8x^2 + 4x + 7, \quad -2 - \sqrt{3}\,i$

In Exercises 89–92, find the domain of the function and identify any horizontal or vertical asymptotes.

89. $f(x) = \dfrac{-3}{x + 2}$

90. $f(x) = \dfrac{3x^2 + 7x - 5}{x^2 + 1}$

91. $f(x) = \dfrac{2x^2}{x^2 - 9}$

92. $f(x) = \dfrac{3x}{x^2 + x - 6}$

In Exercises 93–96, sketch the graph of the rational function. As sketching aids, check for intercepts, symmetry, vertical asymptotes, and horizontal asymptotes.

93. $P(x) = \dfrac{3 - x}{x + 2}$

94. $f(x) = \dfrac{4}{(x - 1)^2}$

95. $g(x) = \dfrac{1}{x^2 - 4} + 2$

96. $h(x) = \dfrac{-3x}{2x^2 + 3x - 5}$

In Exercises 97 and 98, find all possible asymptotes (vertical, horizontal, and/or slant) of the given function. Sketch the graph of f.

97. $f(x) = \dfrac{x^2 - 16}{x - 4}$

98. $f(x) = \dfrac{x^3}{x^2 - 5}$

99. Average Cost The cost C (in dollars) of producing x charcoal grills is $C = 125{,}000 + 9.65x$. The average cost $\overline{C}$ per charcoal grill is

$$\overline{C} = \frac{C}{x} = \frac{125{,}000 + 9.65x}{x}, \quad x > 0.$$

(a) Sketch the graph of the average cost function.

(b) Find the average cost of producing 1000, 10,000, 100,000, and 1,000,000 charcoal grills. What can you conclude?

100. Average Recycling Cost The cost C (in dollars) of recycling a waste product is

$$C = 325{,}000 + 8.5x, \quad x > 0$$

where x is the number of pounds of waste. The average recycling cost $\overline{C}$ per pound is

$$\overline{C} = \frac{C}{x} = \frac{325{,}000 + 8.5x}{x}, \quad x > 0.$$

(a) Sketch the graph of $\overline{C}$.

(b) Find the average cost of recycling 1000, 10,000, 100,000, and 1,000,000 pounds of waste. What can you conclude?

101. Population of Fish The Wildlife Commission introduces 60,000 game fish into a large lake. The population N (in thousands) of the fish is

$$N = \frac{20(3 + 5t)}{1 + 0.06t}, \quad t \geq 0$$

where t is time (in years).

(a) Find the populations when $t = 5$, 10, and 25.

(b) What is the limiting number of fish in the lake as time progresses?

102. Human Memory Model Consider the learning curve modeled by

$$P = \frac{0.7 + 0.65(n - 1)}{1 + 0.65(n - 1)}, \quad n \geq 0$$

where P is the percent of correct responses (in decimal form) after n trials.

(a) Complete the table.

n	1	2	3	4	5	6	7	8	9	10
P										

(b) According to this model, what is the limiting percent of correct responses as n increases?

103. Smokestack Emissions The cost C (in dollars) of removing p percent of the air pollutants in the stack emissions of a utility company that burns coal to generate electricity is

$$C = \frac{105{,}000p}{100 - p}, \quad 0 \leq p < 100.$$

(a) Find the cost of removing 25% of the pollutants.

(b) Find the cost of removing 60% of the pollutants.

(c) Find the cost of removing 99% of the pollutants.

(d) According to the model, would it be possible to remove 100% of the pollutants? Explain.

Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

1. Sketch the graph of the quadratic function given by

$f(x) = -\frac{1}{2}(x - 1)^2 - 5.$

Identify the vertex and intercepts.

2. Describe the right-hand and left-hand behavior of the graph of f.

(a) $f(x) = 12x^3 - 5x^2 - 49x + 15$

(b) $f(x) = 5x^4 - 3x^3 + 2x^2 + 11x + 12$

3. Simplify $\dfrac{x^4 + 4x^3 - 19x^2 - 106x - 120}{x^2 - 3x - 10}$.

4. List all the possible rational zeros of

$f(x) = 4x^4 - 16x^3 + 3x^2 + 36x - 27.$

Use synthetic division to show that $x = -\frac{3}{2}$ and $x = \frac{3}{2}$ are zeros of f. Using these results, completely factor the polynomial.

5. The sales per share S (in dollars) for Cost Plus, Inc. for the years 1996 to 2005 are shown in the table at the left. *(Source: Cost Plus Inc.)*

Year	Sales per share, S
1996	11.79
1997	13.33
1998	15.81
1999	19.60
2000	23.50
2001	26.38
2002	32.12
2003	36.73
2004	41.62
2005	43.99

Table for 5

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a linear model, a quadratic model, and a cubic model for the data.

(c) Use a graphing utility to graph each model separately with the data in the same viewing window. How well does each model fit the data?

(d) Use each model to predict the year in which the sales per share will be about \$50. Then discuss the appropriateness of each model for predicting future values.

In Exercises 6–9, perform the indicated operation and write the result in standard form.

6. $(12 + 3i) + (4 - 6i)$

7. $(10 - 2i) - (3 + 7i)$

8. $\left(5 + \sqrt{-12}\right)\left(3 - \sqrt{-12}\right)$

9. $(4 + 3i)(2 - 5i)$

10. Write the quotient in standard form: $\dfrac{1 + i}{1 - i}$.

In Exercises 11 and 12, solve the quadratic equation.

11. $x^2 + 5x + 7 = 0$

12. $2x^2 - 5x + 11 = 0$

13. Find a polynomial with real coefficients that has 2, 5, $3i$, and $-3i$ as zeros.

14. Find all the zeros of $f(x) = x^3 + 2x^2 + 5x + 10$, given that $\sqrt{5}\,i$ is a zero.

15. Sketch the graph of $f(x) = \dfrac{3x}{x - 2}$. Label any intercepts and asymptotes. What is the domain of f?

Exponential and Logarithmic Functions

4



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Some scientists believe the duration of short-term memory is less than a minute. In contrast, the duration of long-term memory is theoretically unlimited. You can use logarithmic functions to model long-term memory to see how well humans retain information over time. (See Section 4.2, Example 10.)

Applications

Exponential and logarithmic functions are used to model and solve many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Population Growth, Exercises 65 and 66, page 352
- Bone Graft Procedures, Example 11, page 379
- Super Bowl Ad Revenue, Exercise 43, page 393

4.1 Exponential Functions
4.2 Logarithmic Functions
4.3 Properties of Logarithms
4.4 Solving Exponential and Logarithmic Equations
4.5 Exponential and Logarithmic Models

Section 4.1

Exponential Functions

- Evaluate an exponential expression.
- Sketch the graph of an exponential function.
- Evaluate and sketch the graph of the natural exponential function.
- Use the compound interest formulas.
- Use an exponential model to solve an application problem.

Exponential Functions

So far, this text has dealt only with **algebraic functions,** which include polynomial functions and rational functions. In this chapter, you will study two types of nonalgebraic functions—*exponential* functions and *logarithmic* functions. These functions are examples of **transcendental functions.**

Definition of Exponential Function

The **exponential function *f* with base *a*** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

The base $a = 1$ is excluded because it yields

$$f(x) = 1^x = 1.$$

This is a constant function, not an exponential function.

You already know how to evaluate a^x for integer and rational values of x. For example, you know that $4^3 = 64$ and $4^{1/2} = 2$. However, to evaluate 4^x for any real number x, you need to interpret forms with *irrational* exponents. For the purposes of this text, it is sufficient to think of

$$a^{\sqrt{2}} \qquad (\text{where } \sqrt{2} \approx 1.414214)$$

as that value having the successively closer approximations

$$a^{1.4},\ a^{1.41},\ a^{1.414},\ a^{1.4142},\ a^{1.41421},\ a^{1.414214},\ \ldots$$

Example 1 Evaluating an Exponential Expression

Scientific Calculator

Number	*Keystrokes*	*Display*
$2^{-\pi}$	2 y^2 π +/− =	0.113314732

Graphing Calculator

Number	*Keystrokes*	*Display*
$2^{-\pi}$	2 ^ (−) π ENTER	.1133147323

✓CHECKPOINT 1

Use a calculator to evaluate $(2.2)^{1.8}$. Round your result to three decimal places. ■

Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Examples 2, 3, and 4.

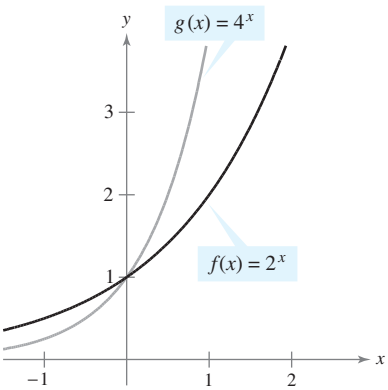


FIGURE 4.1

Example 2 Graphs of $y = a^x$

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$ **b.** $g(x) = 4^x$

SOLUTION The table below lists some values for each function, and Figure 4.1 shows their graphs. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x) = 4^x$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

✓CHECKPOINT 2

Sketch the graph of $f(x) = 5^x$. ■

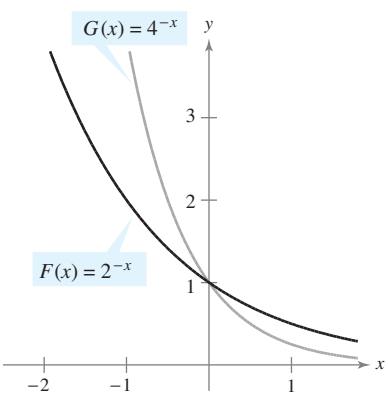


FIGURE 4.2

Example 3 Graphs of $y = a^{-x}$

In the same coordinate plane, sketch the graph of each function.

a. $F(x) = 2^{-x}$ **b.** $G(x) = 4^{-x}$

SOLUTION The table below lists some values for each function, and Figure 4.2 shows their graphs. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

x	-3	-2	-1	0	1	2
$F(x) = 2^{-x}$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$G(x) = 4^{-x}$	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$

✓CHECKPOINT 3

Sketch the graph of $F(x) = 5^{-x}$. ■

The tables in Examples 2 and 3 were evaluated by hand. You could, of course, use the *table* feature of a graphing utility to construct tables with even more values.

In Example 3, note that the functions given by $F(x) = 2^{-x}$ and $G(x) = 4^{-x}$ can be rewritten with positive exponents.

$$F(x) = 2^{-x} = \left(\tfrac{1}{2}\right)^x \quad \text{and} \quad G(x) = 4^{-x} = \left(\tfrac{1}{4}\right)^x$$

Comparing the functions in Examples 2 and 3, observe that

$$F(x) = 2^{-x} = f(-x) \quad \text{and} \quad G(x) = 4^{-x} = g(-x).$$

Consequently, the graph of F is a reflection (in the y-axis) of the graph of f. The graphs of G and g have the same relationship.

The graphs in Figures 4.1 and 4.2 are typical of the exponential functions $y = a^x$ and $y = a^{-x}$. They have one y-intercept and one horizontal asymptote (the x-axis), and they are continuous. The basic characteristics of these exponential functions are summarized in Figures 4.3 and 4.4.

Characteristics of Exponential Functions

Graph of $y = a^x$, $a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Increasing
- x-axis is a horizontal asymptote ($a^x \to 0$ as $x \to -\infty$)
- Continuous

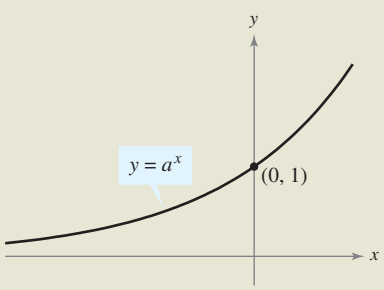


FIGURE 4.3

Graph of $y = a^{-x}$, $a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Decreasing
- x-axis is a horizontal asymptote ($a^{-x} \to 0$ as $x \to \infty$)
- Continuous
- Reflection of graph of $y = a^x$ about y-axis

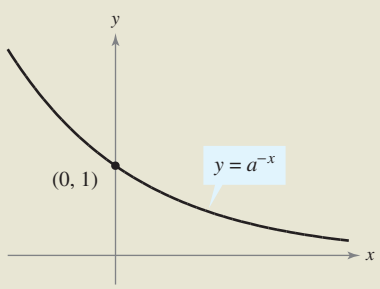


FIGURE 4.4

DISCOVERY

Use a graphing utility to graph

$$y = a^x$$

for $a = 3$, 5, and 7 in the same viewing window. (Use a viewing window in which $-2 \le x \le 1$ and $0 \le y \le 2$.) How do the graphs compare with each other? Which graph is on the top in the interval $(-\infty, 0)$? Which is on the bottom? Which graph is on the top in the interval $(0, \infty)$? Which is on the bottom?

Repeat this experiment with the graphs of $y = b^x$ for $b = \frac{1}{3}, \frac{1}{5}$, and $\frac{1}{7}$. (Use a viewing window in which $-1 \le x \le 2$ and $0 \le y \le 2$.) What can you conclude about the shape of the graph of $y = b^x$ and the value of b?

In the following example, notice how the graph of $y = a^x$ is used to sketch the graphs of functions of the form $f(x) = b \pm a^{x+c}$.

Example 4 Transformations of Graphs of Exponential Functions

Each of the following graphs is a transformation of the graph of $f(x) = 3^x$, as shown in Figure 4.5.

a. Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *left*.

b. Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f *downward* two units.

c. Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by *reflecting* the graph of f in the x-axis.

d. Because $j(x) = 3^{-x} = f(-x)$, the graph of j can be obtained by *reflecting* the graph of f in the y-axis.

STUDY TIP

Notice in Example 4(b) that shifting the graph downward two units also shifts the horizontal asymptote of $f(x) = 3^x$ from the x-axis $(y = 0)$ to the line $y = -2$.

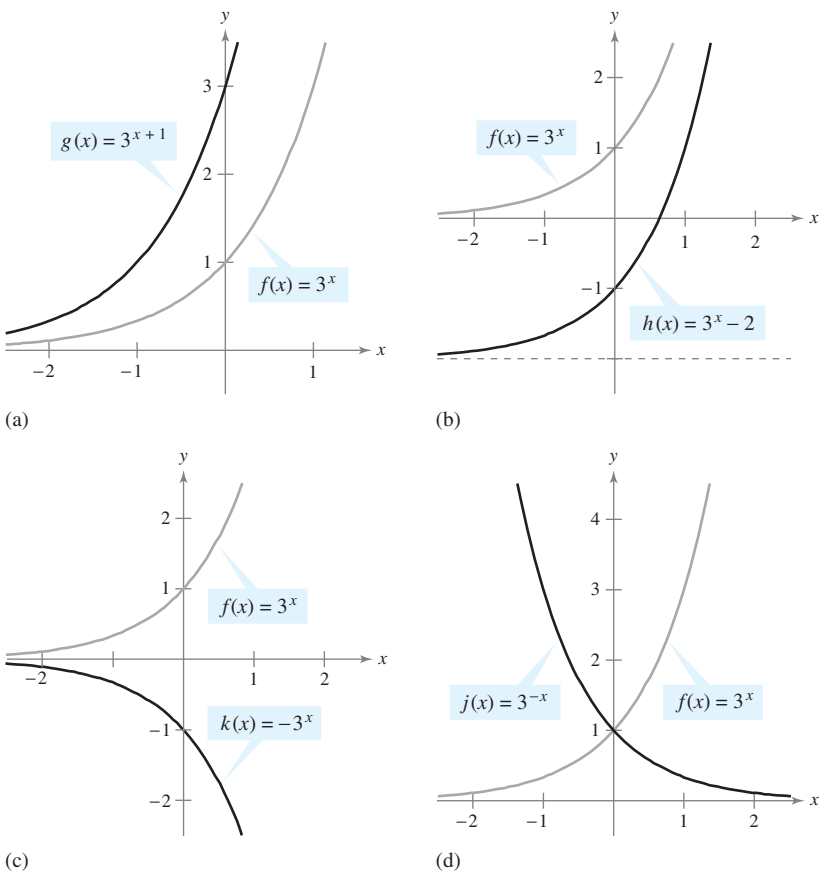


FIGURE 4.5

✓CHECKPOINT 4

Sketch the graph of $f(x) = 2^{x-1}$. ■

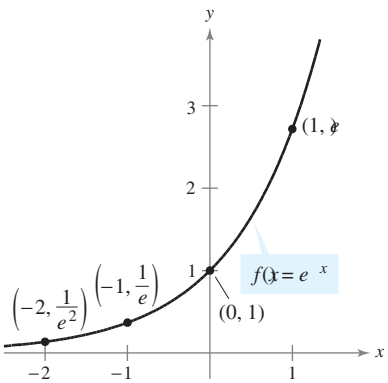
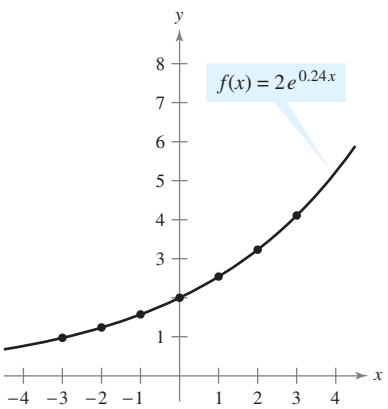
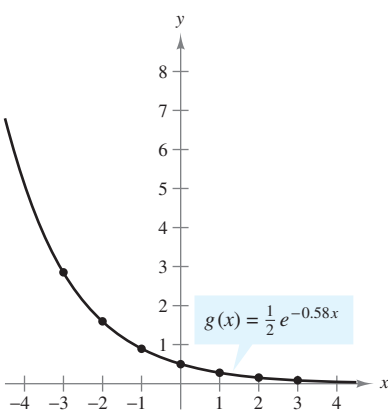


FIGURE 4.6



(a)



(b)

FIGURE 4.7

✓CHECKPOINT 6

Sketch the graph of $f(x) = e^{0.5x}$. ■

The Natural Base *e*

In many applications, the most convenient choice for a base is the irrational number

$$e = 2.718281828 \ldots$$

called the **natural base.** The function given by $f(x) = e^x$ is called the **natural exponential function.** Its graph is shown in Figure 4.6. The graph of the natural exponential function has the same basic characteristics as the graph of the exponential function given by $f(x) = a^x$ (see page 344). Be sure you see that for the exponential function given by $f(x) = e^x$, e is the constant 2.718281828 . . . , whereas x is the variable.

Example 5 Evaluating the Natural Exponential Function

Use a calculator to evaluate the function given by $f(x) = e^x$ when $x = 2$ and $x = -1$.

SOLUTION

Scientific Calculator

Number	*Keystrokes*	*Display*
e^2	2 [2nd] [e^x]	7.389056099
e^{-1}	1 [+/−] [2nd] [e^x]	0.367879441

Graphing Calculator

Number	*Keystrokes*	*Display*
e^2	[2nd] [e^x] 2 [)] [ENTER]	7.389056099
e^{-1}	[2nd] [e^x] [(−)] 1 [)] [ENTER]	.3678794412

✓CHECKPOINT 5

Use a calculator to evaluate $f(x) = e^x$ when $x = 6$. ■

Example 6 Graphing Natural Exponential Functions

Sketch the graph of each natural exponential function.

a. $f(x) = 2e^{0.24x}$ **b.** $g(x) = \frac{1}{2}e^{-0.58x}$

SOLUTION To sketch these two graphs, you can use a calculator to plot several points on each graph, as shown in the table. Then, connect the points with smooth curves, as shown in Figure 4.7. Note that the graph in part (a) is increasing, whereas the graph in part (b) is decreasing.

x	−3	−2	−1	0	1	2	3
$f(x) = 2e^{0.24x}$	0.974	1.238	1.573	2	2.542	3.232	4.109
$g(x) = \frac{1}{2}e^{-0.58x}$	2.849	1.595	0.893	0.5	0.280	0.157	0.088

Compound Interest

One of the most familiar examples of exponential growth is that of an investment earning **continuously compounded interest.** The formula for the balance in an account that is compounded n times per year is $A = P(1 + r/n)^{nt}$, where A is the balance in the account, P is the initial deposit, r is the annual interest rate (in decimal form), and t is the number of years. Using exponential functions, you will *develop* this formula and show how it leads to continuous compounding.

Suppose a principal P is invested at an annual interest rate r, compounded once a year. The principal at the end of the first year, P_1, is equal to the initial deposit P plus the interest earned, Pr. So,

$$P_1 = P + Pr.$$

This can be rewritten by factoring out P from each term as follows.

$$\begin{aligned} P_1 &= P + Pr \\ &= P(1 + r) \end{aligned}$$

DISCOVERY

Use a calculator and the formula $A = P(1 + r/n)^{nt}$ to calculate the amount in an account when $P = \$3000$, $r = 6\%$, t is 10 years, and the number of compoundings is (1) by the day, (2) by the hour, (3) by the minute, and (4) by the second. Use these results to present an argument that increasing the number of compoundings does not mean unlimited growth of the amount in the account.

This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown below.

Year	*Balance After Each Compounding*
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
3	$P_3 = P_2(1 + r) = P(1 + r)^2(1 + r) = P(1 + r)^3$
	$\vdots$
t	$P_t = P(1 + r)^t$

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings per year and let t be the number of years. Then the rate per compounding is r/n and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$ Amount (balance) with n compoundings per year

If you let the number of compoundings n increase without bound, the process approaches what is called **continuous compounding.** In the formula for n compoundings per year, let $m = n/r$. This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Amount with n compoundings per year

$$= P\left(1 + \frac{1}{m}\right)^{mrt}$$ Substitute mr for n and simplify.

$$= P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$ Property of exponents

As m increases without bound, it can be shown that $[1 + (1/m)]^m$ approaches e. From this, you can conclude that the formula for continuous compounding is $A = Pe^{rt}$.

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$

Be sure that the annual interest rate is written in decimal form. For instance, 6% should be written as 0.06 when using compound interest formulas.

Example 7 *MAKE A DECISION* **Compound Interest**

You invest \$12,000 at an annual rate of 3%. Find the balance after 5 years when the interest is compounded (a) quarterly, (b) monthly, and (c) continuously. Which type of compounding earns the most money?

SOLUTION

a. For quarterly compounding, you have $n = 4$. So, in 5 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Formula for compound interest

$$= 12{,}000\left(1 + \frac{0.03}{4}\right)^{4(5)}$$ Substitute for P, r, n, and t.

$$\approx \$13{,}934.21.$$ Use a calculator.

b. For monthly compounding, you have $n = 12$. So, in 5 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Formula for compound interest

$$= 12{,}000\left(1 + \frac{0.03}{12}\right)^{12(5)}$$ Substitute for P, r, n and t.

$$\approx \$13{,}939.40$$ Use a calculator.

c. For continuous compounding, the balance is

$$A = Pe^{rt}$$ Formula for continuous compounding

$$= 12{,}000e^{0.03(5)}$$ Substitute for P, r, and t.

$$\approx \$13{,}942.01$$ Use a calculator.

Note that continuous compounding yields more than quarterly and monthly compounding. This is typical of the two types of compounding. That is, for a given principal, interest rate, and time, continuous compounding will always yield a larger balance than compounding n times a year.

✓CHECKPOINT 7

You invest \$6000 at an annual rate of 4%. Find the balance after 7 years when the interest is compounded continuously. ■

Another Application

Example 8
MAKE A DECISION **Radioactive Decay**

In 1986, a nuclear reactor accident occurred in Chernobyl in what was then the Soviet Union. The explosion spread highly toxic radioactive chemicals, such as plutonium, over hundreds of square miles, and the government evacuated the city and the surrounding area. Consider the model

$$P = 10\left(\tfrac{1}{2}\right)^{t/24{,}100}$$

which represents the amount of plutonium P that remains (from an initial amount of 10 pounds) after t years. Sketch the graph of this function over the interval from $t = 0$ to $t = 100{,}000$, where $t = 0$ represents 1986. How much of the 10 pounds of plutonium will remain in the year 2010? How much of the 10 pounds will remain after 100,000 years? Why is this city uninhabited?

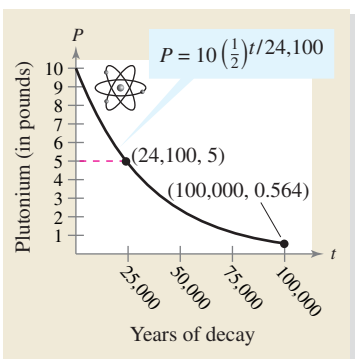


FIGURE 4.8

SOLUTION The graph of this function is shown in Figure 4.8. Note from this graph that plutonium has a *half-life* of about 24,100 years. That is, after 24,100 years, *half* of the original amount of plutonium will remain. After another 24,100 years, one-quarter of the original amount will remain, and so on. In the year 2010 ($t = 24$), there will still be

$$P = 10\left(\tfrac{1}{2}\right)^{24/24{,}100} \approx 10\left(\tfrac{1}{2}\right)^{0.0009959} \approx 9.993 \text{ pounds}$$

of the original amount of plutonium remaining. After 100,000 years, there will still be

$$P = 10\left(\tfrac{1}{2}\right)^{100{,}000/24{,}100} \approx 10\left(\tfrac{1}{2}\right)^{4.149} \approx 0.564 \text{ pound}$$

of the original amount of plutonium remaining. This city is uninhabited because much of the original amount of radioactive plutonium still remains in the city.

✓CHECKPOINT 8

In Example 8, how much of the initial 10 pounds of plutonium will remain in the year 2086? ■

CONCEPT CHECK

1. Is the value of 8^x when $x = 3$ equivalent to the value of 8^{-x} when $x = -3$? Explain.
2. What formula would you use to find the balance A of an account after t years with a principal of \$1000 earning an annual interest rate of 5% compounded continuously?
3. What is the range of the graph of $f(x) = 5^x - 1$?
4. Write a natural exponential function whose graph is the graph of $y = e^x$ shifted two units to the left and three units upward.

Skills Review 4.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 0.4.

In Exercises 1–12, use the properties of exponents to simplify the expression.

1. $5^{2x}(5^{-x})$ **2.** $3^{-x}(3^{3x})$ **3.** $\dfrac{4^{5x}}{4^{2x}}$

4. $\dfrac{10^{2x}}{10^{x}}$ **5.** $(4^{x})^{2}$ **6.** $(4^{2x})^{5}$

7. $\left(\dfrac{2^{x}}{3^{x}}\right)^{-1}$ **8.** $(4^{6x})^{1/2}$ **9.** $(2^{3x})^{-1/3}$

10. $\left(\dfrac{3^{4x}}{5^{4x}}\right)^{1/4}$ **11.** $(16^{x})^{1/4}$ **12.** $(27^{x})^{1/3}$

Exercises 4.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, use a calculator to evaluate the expression. Round your result to three decimal places.

1. $(2.6)^{1.3}$ **2.** $(1.07)^{50}$

3. $100(1.03)^{-1.4}$ **4.** $1500(2^{-5/2})$

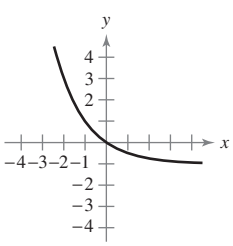
5. $6^{-\sqrt{2}}$ **6.** $1.3^{\sqrt{5}}$

7. e^{4} **8.** e^{-5}

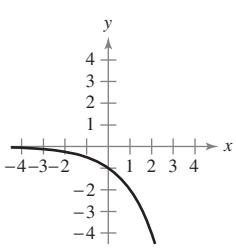
9. $e^{2/3}$ **10.** $e^{-2.7}$

In Exercises 11–18, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

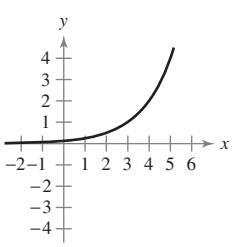
(a)



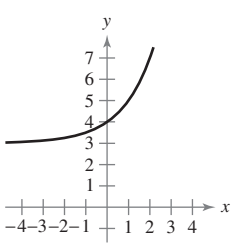
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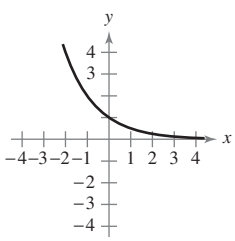
(c)



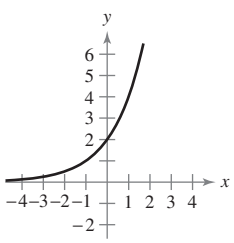
(d)



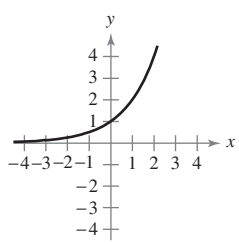
(e)



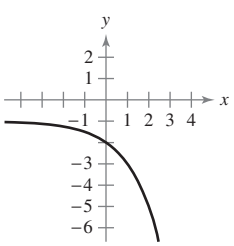
(f)



(g)



(h)



11. $f(x) = 2^{x}$ **12.** $f(x) = 2^{-x}$

13. $f(x) = -2^{x}$ **14.** $f(x) = -2^{x} - 1$

15. $f(x) = 2^{x} + 3$ **16.** $f(x) = 2^{-x} - 1$

17. $f(x) = 2^{x+1}$ **18.** $f(x) = 2^{x-3}$

In Exercises 19–36, sketch the graph of the function.

19. $g(x) = 4^{x}$ **20.** $f(x) = \left(\frac{3}{2}\right)^{x}$

21. $f(x) = 4^{-x}$ **22.** $h(x) = \left(\frac{3}{2}\right)^{-x}$

23. $h(x) = 4^{x-3}$ **24.** $g(x) = \left(\frac{3}{2}\right)^{x+2}$

25. $g(x) = 4^{-x} - 2$ **26.** $f(x) = \left(\frac{3}{2}\right)^{-x} + 2$

27. $y = 2^{-x^{2}}$ **28.** $y = 3^{-x^{2}}$

29. $y = e^{-0.1x}$ **30.** $y = e^{0.2x}$

31. $f(x) = 2e^{0.12x}$ **32.** $f(x) = 3e^{-0.2x}$

33. $f(x) = e^{2x}$ **34.** $h(x) = e^{x-2}$

35. $g(x) = 1 + e^{-x}$ **36.** $N(t) = 1000e^{-0.2t}$

Compound Interest In Exercises 37–40, complete the table to find the balance A for P dollars invested at rate r for t years, compounded n times per year.

n	1	2	4	12	365	Continuous
A						

37. $P = \$5000$, $r = 8\%$, $t = 5$ years

38. $P = \$1000$, $r = 10\%$, $t = 10$ years

39. $P = \$2500$, $r = 12\%$, $t = 20$ years

40. $P = \$1000$, $r = 10\%$, $t = 40$ years

Compound Interest In Exercises 41–44, complete the table to find the amount P that must be invested at rate r to obtain a balance of $A = \$100{,}000$ in t years.

t	1	10	20	30	40	50
P						

41. $r = 9\%$, compounded continuously

42. $r = 12\%$, compounded continuously

43. $r = 10\%$, compounded monthly

44. $r = 7\%$, compounded daily

45. Compound Interest A bank offers two types of interest-bearing accounts. The first account pays 5% interest compounded quarterly. The second account pays 3% interest compounded continuously. Which account earns more money? Why?

46. Compound Interest A bank offers two types of interest-bearing accounts. The first account pays 6% interest compounded monthly. The second account pays 5% interest compounded continuously. Which account earns more money? Why?

47. *MAKE A DECISION: CASH SETTLEMENT* You invest a cash settlement of $10,000 for 5 years. You have a choice between an account that pays 6.25% interest compounded monthly with a monthly online access fee of $5 and an account that pays 5.25% interest compounded continuously with free online access. Which account should you choose? Explain your reasoning.

48. *MAKE A DECISION: SALES COMMISSION* You invest a sales commission of $12,000 for 6 years. You have a choice between an account that pays 4.85% interest compounded monthly with a monthly online access fee of $3 and an account that pays 4.25% interest compounded continuously with free online access. Which account should you choose? Explain your reasoning.

Compound Interest On the day a child was born, a lump sum P was deposited in a trust fund paying 6.5% interest compounded continuously. In Exercises 49–52, use the balance A of the fund on the child's 25th birthday to find P.

49. $A = \$100{,}000$ **50.** $A = \$500{,}000$

51. $A = \$750{,}000$ **52.** $A = \$1{,}000{,}000$

Compound Interest On the day you were born, a lump sum P was deposited in a trust fund paying 7.5% interest compounded continuously. In Exercises 53–56, use the balance A of the fund, which is the balance on your 21st birthday, to find P.

53. $A = \$100{,}000$ **54.** $A = \$500{,}000$

55. $A = \$750{,}000$ **56.** $A = \$1{,}000{,}000$

57. Demand Function The demand function for a limited edition comic book is given by

$$p = 3000\left(1 - \frac{5}{5 + e^{-0.015x}}\right).$$

(a) Find the price p for a demand of $x = 75$ units.

(b) Find the price p for a demand of $x = 200$ units.

(c) Use a graphing utility to graph the demand function.

(d) Use the graph from part (c) to approximate the demand when the price is $100.

58. Demand Function The demand function for a home theater sound system is given by

$$p = 7500\left(1 - \frac{7}{7 + e^{-0.003x}}\right).$$

(a) Find the price p for a demand of $x = 200$ units.

(b) Find the price p for a demand of $x = 900$ units.

(c) Use a graphing utility to graph the demand function.

(d) Use the graph from part (c) to approximate the demand when the price is $400.

59. Bacteria Growth The number of a certain type of bacteria increases according to the model

$$P(t) = 100e^{0.01896t}$$

where t is time (in hours).

(a) Find $P(0)$. (b) Find $P(5)$.

(c) Find $P(10)$. (d) Find $P(24)$.

60. Bacteria Growth As a result of a medical treatment, the number of a certain type of bacteria decreases according to the model

$$P(t) = 100e^{-0.685t}$$

where t is time (in hours).

(a) Find $P(0)$. (b) Find $P(5)$.

(c) Find $P(10)$. (d) Find $P(24)$.

Present Value The present value of money is the principal P you need to invest today so that it will grow to an amount A at the end of a specified time. The present value formula

$$P = A\left(1 + \frac{r}{n}\right)^{-nt}$$

is obtained by solving the compound interest formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

for P. Recall that t is the number of years, r is the interest rate per year, and n is the number of compoundings per year. In Exercises 61–64, find the present value of amount A invested at rate r for t years, compounded n times per year.

61. $A = \$10{,}000$, $r = 6\%$, $t = 5$ years, $n = 4$

62. $A = \$50{,}000$, $r = 7\%$, $t = 10$ years, $n = 12$

63. $A = \$20{,}000$, $r = 8\%$, $t = 6$ years, $n = 4$

64. $A = \$1{,}000{,}000$, $r = 8\%$, $t = 20$ years, $n = 2$

65. Population Growth The population P of a town increases according to the model

$$P(t) = 4500e^{0.0272t}$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to predict the population in each year.

(a) 2010 (b) 2012

(c) 2015 (d) 2020

66. Population Growth The population P of a small city increases according to the model

$$P(t) = 36{,}000e^{0.0156t}$$

where t represents the year, with $t = 0$ corresponding to 2000. Use the model to predict the population in each year.

(a) 2009 (b) 2011

(c) 2015 (d) 2018

67. Radioactive Decay Strontium-90 has a half-life of 29.1 years. The amount S of 100 kilograms of strontium-90 present after t years is given by

$$S = 100e^{-0.0238t}.$$

How much of the 100 kilograms will remain after 50 years?

68. Radioactive Decay Neptunium-237 has a half-life of 2.1 million years. The amount N of 200 kilograms of neptunium-237 present after t years is given by

$$N = 200e^{-0.00000033007t}.$$

How much of the 200 kilograms will remain after 20,000 years?

69. Radioactive Decay Five pounds of the element plutonium (^{230}Pu) is released in a nuclear accident. The amount of plutonium P that is present after t months is given by $P = 5e^{-0.1507t}$.

(a) Use a graphing utility to graph this function over the interval from $t = 0$ to $t = 10$.

(b) How much of the 5 pounds of plutonium will remain after 10 months?

(c) Use the graph to estimate the half-life of ^{230}Pu. Explain your reasoning.

70. Radioactive Decay One hundred grams of radium (^{226}Ra) is stored in a container. The amount of radium R present after t years is given by $R = 100e^{-0.0004335t}$.

(a) Use a graphing utility to graph this function over the interval from $t = 0$ to $t = 10{,}000$.

(b) How much of the 100 grams of radium will remain after 10,000 years?

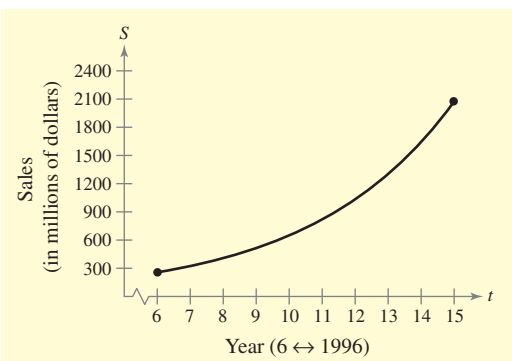
(c) Use the graph to estimate the half-life of ^{226}Ra. Explain your reasoning.

71. Guitar Sales The sales S (in millions of dollars) for Guitar Center, Inc. from 1996 to 2005 can be modeled by

$$S = 63.7e^{0.2322t}, \quad 6 \le t \le 15$$

where t represents the year, with $t = 6$ corresponding to 1996. *(Source: Guitar Center, Inc.)*

(a) Use the graph to estimate *graphically* the sales for Guitar Center, Inc. in 1998, 2000, and 2005.



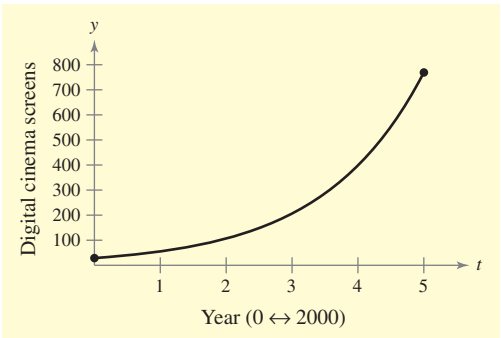
(b) Use the model to confirm *algebraically* the estimates obtained in part (a).

72. Digital Cinema Screens The numbers y of digital cinema screens in the world from 2000 to 2005 can be modeled by

$$y = 28.7e^{0.6577t}, \quad 0 \le t \le 5$$

where t represents the year, with $t = 0$ corresponding to 2000. *(Source: Screen Digest)*

(a) Use the graph to estimate *graphically* the numbers of digital cinema screens in 2001 and 2004.



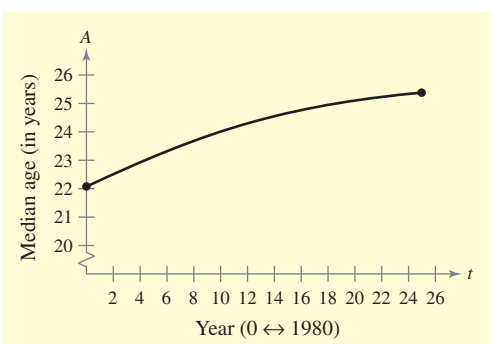
(b) Use the model to confirm *algebraically* the estimates obtained in part (a).

73. Age at First Marriage From 1980 to 2005, the median age A of an American woman at her first marriage can be approximated by the model

$$A = 17.91 + \frac{7.88}{1 + e^{-0.1117t - 0.1138}}, \quad 0 \le t \le 25$$

where t represents the year, with $t = 0$ corresponding to 1980. *(Source: U.S. Census Bureau)*

(a) Use the graph to estimate *graphically* the median age of an American woman at her first marriage in 1980, 1990, 2000, and 2005.



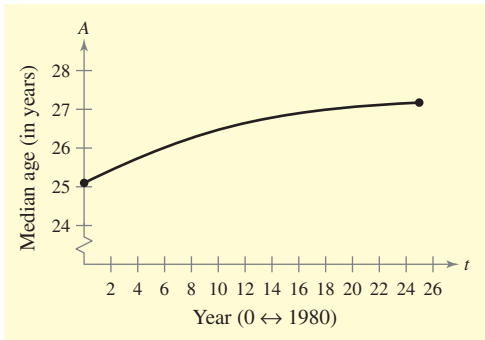
(b) Use the model to confirm *algebraically* the estimates obtained in part (a).

74. Age at First Marriage From 1980 to 2005, the median age A of an American man at his first marriage can be approximated by the model

$$A = 22.55 + \frac{4.74}{1 + e^{-0.1412t - 0.1513}}, \quad 0 \le t \le 25$$

where t represents the year, with $t = 0$ corresponding to 1980. *(Source: U.S. Census Bureau)*

(a) Use the graph to estimate *graphically* the median age of an American man at his first marriage in 1980, 1990, 2000, and 2005.



(b) Use the model to confirm *algebraically* the estimates obtained in part (a).

75. Compare Compare the results of Exercises 73 and 74. What can you conclude about the differences in men's and women's ages at first marriage?

76. Hospital Employment The numbers of people E (in thousands) employed in hospitals from 1999 to 2005 can be modeled by

$$E = 3331(1.0182)^t, \quad 9 \le t \le 15$$

where t represents the year, with $t = 9$ corresponding to 1999. *(Source: U.S. Bureau of Labor Statistics)*

(a) Use a graphing utility to graph E for the years 1999 to 2005.

(b) Use the graph from part (a) to estimate the numbers of hospital employees in 2000, 2002, and 2005.

77. Prescriptions The numbers of prescriptions P (in millions) filled in the United States from 1998 to 2005 can be modeled by

$$P = -11{,}415 + \frac{15{,}044}{1 + e^{-0.2166t - 0.7667}}, \quad 8 \le t \le 15$$

where t represents the year, with $t = 8$ corresponding to 1998. *(Source: National Association of Chain Drug Stores)*

(a) Use a graphing utility to graph P for the years 1998 to 2005.

(b) Use the graph from part (a) to estimate the numbers of prescriptions filled in 1999, 2002, and 2005.

78. Writing Determine whether $e = \frac{271{,}801}{99{,}990}$. Justify your answer.

79. Extended Application To work an extended application involving the healing rate of a wound, visit this text's website at *college.hmco.com/info/larsonapplied*.

Section 4.2

Logarithmic Functions

- Recognize and evaluate a logarithmic function with base a.
- Sketch the graph of a logarithmic function.
- Recognize and evaluate the natural logarithmic function.
- Use a logarithmic model to solve an application problem.

Logarithmic Functions

In Section 2.8, you studied inverse functions. There, you learned that if a function has the property that no horizontal line intersects the graph of the function more than once, the function must have an inverse function. By looking back at the graphs of the exponential functions introduced in Section 4.1, you will see that every function of the form $f(x) = a^x$ (where $a > 0$ and $a \neq 1$) passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base *a*.**

Definition of a Logarithmic Function

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

The function given by

$$f(x) = \log_a x$$

is called the **logarithmic function with base *a*.**

STUDY TIP

By the definition of a logarithmic function,

$$3^4 = 81$$

(a = 3, y = 4, x = 81)

can be written as

$$\log_3 81 = 4.$$

(a = 3, x = 81, y = 4)

The equations $y = \log_a x$ and $x = a^y$ are equivalent. The first equation is in logarithmic form and the second is in exponential form.

When evaluating logarithms, remember that *a logarithm is an exponent.* This means that $\log_a x$ is the exponent to which a must be raised to obtain x. For instance, $\log_2 8 = 3$ because 2 must be raised to the third power to obtain 8.

Example 1 Evaluating Logarithmic Expressions

a. $\log_2 32 = 5$ because $2^5 = 32$.

b. $\log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.

c. $\log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

d. $\log_3 1 = 0$ because $3^0 = 1$.

✓CHECKPOINT 1

Evaluate the expression $\log_7 \frac{1}{49}$. ■

The logarithmic function with base 10 is called the **common logarithmic function.** On most calculators, this function is denoted by LOG.

STUDY TIP

Because $\log_a x$ is the inverse function of $y = a^x$, it follows that the domain of $y = \log_a x$ is the range of $y = a^x$, $(0, \infty)$. In other words, $y = \log_a x$ is defined only if x is positive.

Example 2 Evaluating Logarithmic Expressions on a Calculator

Scientific Calculator

	Number	*Keystrokes*	*Display*
a.	$\log_{10} 10$	10 LOG	1
b.	$2 \log_{10} 2.5$	2.5 LOG × 2 =	0.795880017
c.	$\log_{10}(-2)$	2 +/− LOG	ERROR

Graphing Calculator

	Number	*Keystrokes*	*Display*
a.	$\log_{10} 10$	LOG 10) ENTER	1
b.	$2 \log_{10} 2.5$	2 LOG 2.5) ENTER	.7958800173
c.	$\log_{10}(-2)$	LOG (−) 2) ENTER	ERROR

Many calculators display an error message (or a complex number) when you try to evaluate $\log_{10}(-2)$. This is because the domain of every logarithmic function is the set of *positive real numbers.* In other words, there is no real number power to which 10 can be raised to obtain -2.

✓CHECKPOINT 2

Use a calculator to evaluate the expression $\log_{10} 200$. Round your result to three decimal places. ■

The following properties follow directly from the definition of the logarithmic function with base a.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$ Inverse Properties
4. If $\log_a x = \log_a y$, then $x = y$. One-to-One Property

Example 3 Using Properties of Logarithms

a. Solve the equation $\log_2 x = \log_2 3$ for x.

b. Solve the equation $\log_5 x = 1$ for x.

SOLUTION

a. Using the One-to-One Property (Property 4), you can conclude that $x = 3$.

b. Using Property 2, you can conclude that $x = 5$.

✓CHECKPOINT 3

Solve the equation $\log_4 1 = x$ for x. ■

Graphs of Logarithmic Functions

To sketch the graph of $y = \log_a x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

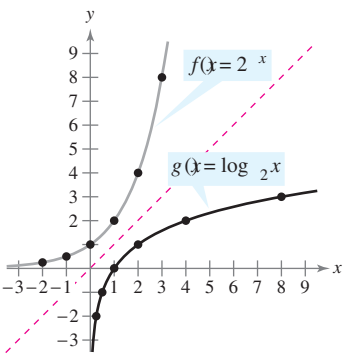


FIGURE 4.9 Inverse Functions

Example 4 Graphs of Exponential and Logarithmic Functions

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$

b. $g(x) = \log_2 x$

SOLUTION

a. For $f(x) = 2^x$, construct a table of values, as follows.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

By plotting these points and connecting them with a smooth curve, you obtain the graph shown in Figure 4.9.

b. Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve. The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 4.9.

Before you can confirm the result of Example 4 with a graphing utility, you need to know how to enter $\log_2 x$. You will learn how to do this using the *change-of-base formula* discussed in Section 4.3.

✓CHECKPOINT 4

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 4^x$

b. $g(x) = \log_4 x$ ■

Example 5 Sketching the Graph of a Logarithmic Function

Sketch the graph of the common logarithmic function given by $f(x) = \log_{10} x$.

SOLUTION Begin by constructing a table of values. Note that some of the values can be obtained without a calculator by using the Inverse Property of logarithms. Others require a calculator. Next, plot the points and connect them with a smooth curve, as shown in Figure 4.10.

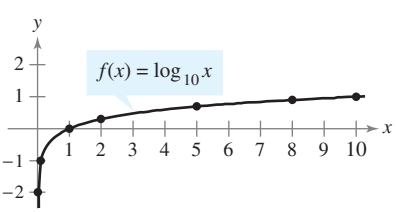


FIGURE 4.10

	Without Calculator				With Calculator		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$f(x) = \log_{10} x$	-2	-1	0	1	0.301	0.699	0.903

✓CHECKPOINT 5

Sketch the graph of the function given by $f(x) = 2 \log_{10} x$. ■

The nature of the graph in Figure 4.10 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x-intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$. The basic characteristics of logarithmic graphs are summarized in Figure 4.11. Note that the vertical asymptote occurs at $x = 0$, where $\log_a x$ is *undefined.*

Characteristics of Logarithmic Functions

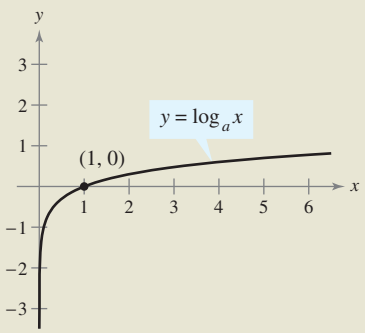


FIGURE 4.11

Graph of $y = \log_a x$, $a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Intercept: $(1, 0)$
- Increasing
- One-to-one; therefore has an inverse function
- y-axis is a vertical asymptote ($\log_a x \to -\infty$ as $x \to 0^+$)
- Continuous
- Reflection of graph of $y = a^x$ about the line $y = x$

Example 6 Sketching the Graphs of Logarithmic Functions

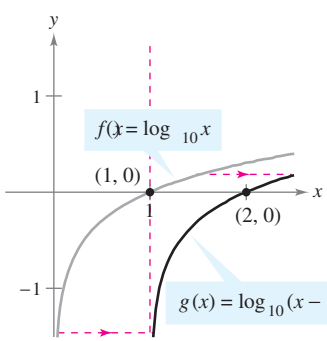
The graph of each function below is similar to the graph of $f(x) = \log_{10} x$.

a. Because $g(x) = \log_{10}(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *right.* See Figure 4.12(a).

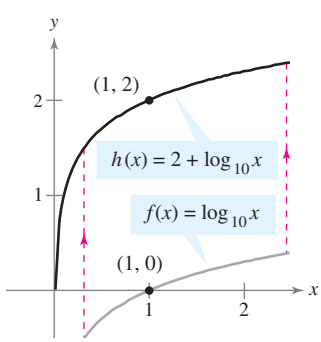
b. Because $h(x) = 2 + \log_{10} x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units *upward.* See Figure 4.12(b).

STUDY TIP

Notice in Example 6(a) that shifting the graph of $f(x)$ one unit to the right also shifts the vertical asymptote from the y-axis ($x = 0$) to the line $x = 1$.



(a) Right shift of one unit



(b) Upward shift of two units

FIGURE 4.12

✓CHECKPOINT 6

Sketch the graph of $f(x) = \log_{10}(x + 3)$. ■

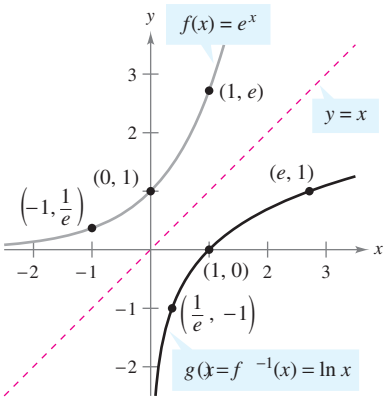


FIGURE 4.13

The Natural Logarithmic Function

By looking back at the graph of the natural exponential function introduced in Section 4.1, you will see that $f(x) = e^x$ is one-to-one and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol $\ln x$, read as "el en of x".

The Natural Logarithmic Function

The function defined by

$$f(x) = \log_e x = \ln x, \quad x > 0$$

is called the **natural logarithmic function.**

Because the functions given by $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions of each other, their graphs are reflections of each other in the line $y = x$. This reflective property is illustrated in Figure 4.13. The four properties of logarithms listed on page 355 are also valid for natural logarithms.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.
2. $\ln e = 1$ because $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$ Inverse Properties
4. If $\ln x = \ln y$, then $x = y$. One-to-One Property

Example 7 Using Properties of Natural Logarithms

Evaluate each logarithmic expression.

a. $\ln \frac{1}{e}$ **b.** $e^{\ln 5}$ **c.** $\frac{\ln 1}{3}$ **d.** $2 \ln e$

SOLUTION

a. $\ln \frac{1}{e} = \ln e^{-1} = -1$ Inverse Property

b. $e^{\ln 5} = 5$ Inverse Property

c. $\frac{\ln 1}{3} = \frac{0}{3} = 0$ Property 1

d. $2 \ln e = 2(1) = 2$ Property 2

✓CHECKPOINT 7

Evaluate each logarithmic expression.

a. $\ln e^7$ **b.** $5 \ln 1$ ■

On most calculators, the natural logarithm is denoted by LN, as illustrated in Example 8.

Example 8 Evaluating the Natural Logarithmic Function

Use a calculator to evaluate each expression.

a. $\ln 2$ **b.** $\ln 0.3$ **c.** $\ln e^2$ **d.** $\ln(-1)$

SOLUTION

Scientific Calculator

	Number	*Keystrokes*	*Display*
a.	$\ln 2$	2 LN	0.69314718
b.	$\ln 0.3$	.3 LN	−1.203972804
c.	$\ln e^2$	2 2nd [e^x] LN	2
d.	$\ln(-1)$	1 +/− LN	ERROR

Graphing Calculator

	Number	*Keystrokes*	*Display*
a.	$\ln 2$	LN 2) ENTER	.6931471806
b.	$\ln 0.3$	LN .3) ENTER	−1.203972804
c.	$\ln e^2$	LN 2nd [e^x] 2)) ENTER	2
d.	$\ln(-1)$	LN (−) 1) ENTER	ERROR

In Example 8, note that $\ln(-1)$ gives an error message on most calculators. This occurs because the domain of $\ln x$ is the set of positive real numbers (see Figure 4.13). So, $\ln(-1)$ is undefined.

Example 9 Finding the Domains of Logarithmic Functions

Find the domain of each function.

a. $f(x) = \ln(x - 2)$ **b.** $g(x) = \ln(2 - x)$ **c.** $h(x) = \ln x^2$

SOLUTION

a. Because $\ln(x - 2)$ is defined only if $x - 2 > 0$, it follows that the domain of f is $(2, \infty)$.

b. Because $\ln(2 - x)$ is defined only if $2 - x > 0$, it follows that the domain of g is $(-\infty, 2)$. The graph of g is shown in Figure 4.14.

c. Because $\ln x^2$ is defined only if $x^2 > 0$, it follows that the domain of h is all real numbers except $x = 0$.

✓CHECKPOINT 8

Use a calculator to evaluate the expression $\ln 0.1$. Round your result to three decimal places. ■

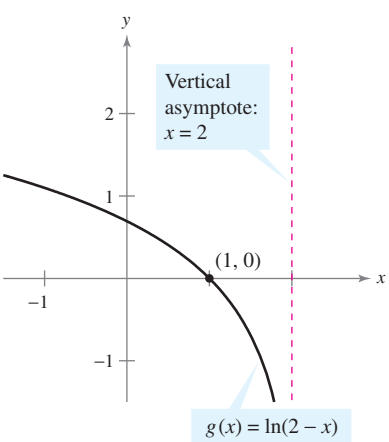


FIGURE 4.14

✓CHECKPOINT 9

Find the domain of the function given by $f(x) = \ln(x + 5)$. ■

Application

Example 10 **Human Memory Model**

A group of students participating in a psychological experiment attended several lectures on a subject. Every month for a year after that, the group of students were tested to see how much of the material they remembered. The average scores for the group are given by the *human memory model*

$$f(t) = 75 - 6\ln(t + 1), \quad 0 \le t \le 12$$

where t is the time (in months). Based on the results of the experiment, how many months can a student wait before retaking the exam and still expect to score 60 or better? (Do not count portions of months.)

SOLUTION To determine how many months a student can wait before retaking the exam and still expect to score 60 or better, use the model to create a table of values showing the scores for several months.

Month, t	0	1	2	3	4	5	6
Score, $f(t)$	75	70.84	68.41	66.68	65.34	64.25	63.32

Month, t	7	8	9	10	11	12
Score, $f(t)$	62.52	61.82	61.18	60.61	60.09	59.61

From the table, you can see that a student would need to retake the exam by the 11th month in order to score 60 or better. The graph of f is shown in Figure 4.15.

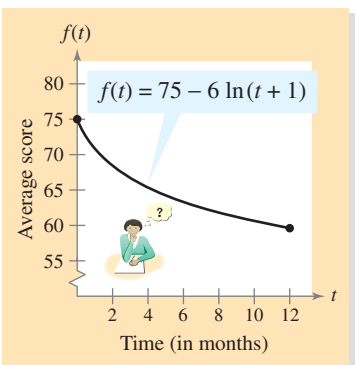


FIGURE 4.15

✓CHECKPOINT 10

Biologists have found that an alligator's length l (in inches) can be approximated by the model

$$l = 27.1 \ln w - 32.8$$

where w is the weight (in pounds) of the alligator. Find the lengths of alligators for which $w = 150$, 225, 380, 450, and 625 pounds. Round your results to the nearest tenth of an inch. ■

CONCEPT CHECK

1. Is $\log_c b = a$ equivalent to $a^b = c$ when a, b, and c are greater than 0, $a \neq 1$, and $c \neq 1$? Explain.
2. What property would you use to solve $\log_x 7 = 1$ for x?
3. Explain how you can use the graph of an exponential function to sketch the graph of $f(x) = \log_5 x$.
4. How is the graph of $g(x) = -\ln x$ related to the graph of $f(x) = \ln x$?

Skills Review 4.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.6 and 4.1.

In Exercises 1–4, solve for x.

1. $2^x = 8$ **2.** $4^x = 1$ **3.** $10^x = 0.1$ **4.** $e^x = e$

In Exercises 5 and 6, evaluate the expression. (Round the result to three decimal places.)

5. e^2 **6.** e^{-1}

In Exercises 7–10, describe how the graph of g is related to the graph of f.

7. $g(x) = f(x + 2)$ **8.** $g(x) = -f(x)$ **9.** $g(x) = -1 + f(x)$ **10.** $g(x) = f(-x)$

Exercises 4.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the logarithmic equation with its exponential form. [The exponential forms are labeled (a), (b), (c), (d), (e), and (f).]

1. $\log_4 16 = 2$ (a) $4^{1/2} = 2$

2. $\log_2 16 = 4$ (b) $2^{-4} = \frac{1}{16}$

3. $\log_2 \frac{1}{16} = -4$ (c) $4^2 = 16$

4. $\log_4 \frac{1}{16} = -2$ (d) $4^{-2} = \frac{1}{16}$

5. $\log_4 2 = \frac{1}{2}$ (e) $16^{1/2} = 4$

6. $\log_{16} 4 = \frac{1}{2}$ (f) $2^4 = 16$

In Exercises 7–16, use the definition of a logarithm to write the equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.

7. $4^4 = 256$ **8.** $7^3 = 343$

9. $81^{1/4} = 3$ **10.** $9^{3/2} = 27$

11. $6^{-2} = \frac{1}{36}$ **12.** $10^{-3} = 0.001$

13. $e^1 = e$ **14.** $e^4 = 54.5981\ldots$

15. $e^x = 4$ **16.** $e^{-x} = 2$

In Exercises 17–26, use the definition of a logarithm to write the equation in exponential form. For example, the exponential form of $\log_5 125 = 3$ is $5^3 = 125$.

17. $\log_4 16 = 2$ **18.** $\log_{10} 1000 = 3$

19. $\log_2 \frac{1}{2} = -1$ **20.** $\log_3 \frac{1}{9} = -2$

21. $\ln e = 1$ **22.** $\ln \frac{1}{e} = -1$

23. $\log_5 0.2 = -1$ **24.** $\log_{10} 0.1 = -1$

25. $\log_{27} 3 = \frac{1}{3}$ **26.** $\log_8 2 = \frac{1}{3}$

In Exercises 27–42, evaluate the expression without using a calculator.

27. $\log_3 9$ **28.** $\log_5 125$

29. $\log_2 \frac{1}{16}$ **30.** $\log_6 \frac{1}{36}$

31. $\log_8 2$ **32.** $\log_{64} 4$

33. $\log_7 7$ **34.** $\log_{12} 1$

35. $\log_{10} 0.0001$ **36.** $\log_{10} 100$

37. $\ln e$ **38.** $\ln e^{10}$

39. $\ln e^{-4}$ **40.** $\ln \frac{1}{e^3}$

41. $\log_a a^5$ **42.** $\log_a 1$

In Exercises 43–54, use a calculator to evaluate the logarithm. Round your result to three decimal places.

43. $\log_{10} 345$ **44.** $\log_{10} 163$

45. $\log_{10} \frac{4}{5}$ **46.** $\log_{10} \frac{3}{4}$

47. $\log_{10} \sqrt{8}$ **48.** $\log_{10} \sqrt{3}$

49. $\ln 7$ **50.** $2 \ln 9$

51. $\ln 18.42$ **52.** $\ln 36.7$

53. $\ln \sqrt{6}$ **54.** $\ln \sqrt{10}$

In Exercises 55–58, sketch the graphs of f and g in the same coordinate plane.

55. $f(x) = 3^x$, $g(x) = \log_3 x$

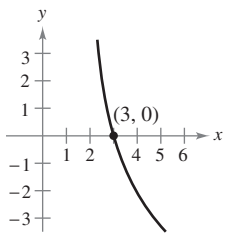
56. $f(x) = 5^x$, $g(x) = \log_5 x$

57. $f(x) = e^x$, $g(x) = \ln x$

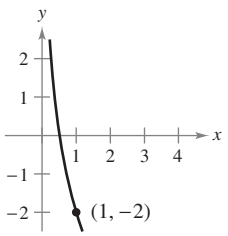
58. $f(x) = 10^x$, $g(x) = \log_{10} x$

In Exercises 59–64, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

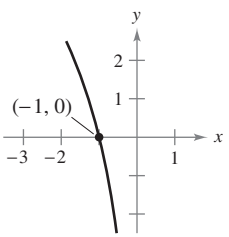
(a)



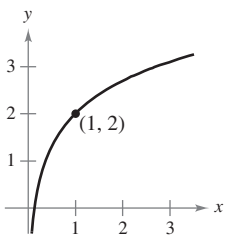
(b)



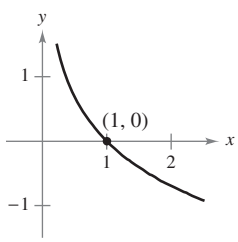
(c)



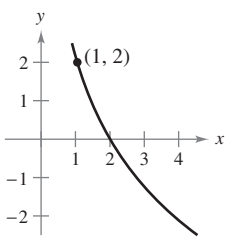
(d)



(e)



(f)



59. $f(x) = \ln x + 2$

60. $f(x) = -\ln x$

61. $f(x) = -3\ln(x - 2)$

62. $f(x) = 4\ln(-x)$

63. $f(x) = -3\ln x + 2$

64. $f(x) = -3\ln x - 2$

In Exercises 65–74, find the domain, vertical asymptote, and x-intercept of the logarithmic function. Then sketch its graph.

65. $f(x) = \log_2 x$

66. $g(x) = \log_4 x$

67. $h(x) = \log_2(x + 4)$

68. $f(x) = \log_4(x - 3)$

69. $f(x) = -\log_2 x$

70. $h(x) = -\log_4(x - 1)$

71. $g(x) = \ln(-x)$

72. $f(x) = \ln(3 - x)$

73. $h(x) = \ln(x + 1)$

74. $f(x) = 3 + \ln x$

Ⓣ In Exercises 75–80, use a graphing utility to graph the function. Be sure to use an appropriate viewing window.

75. $f(x) = \log(x + 1)$

76. $f(x) = \log(x - 1)$

77. $f(x) = \ln(x - 1)$

78. $f(x) = \ln(x + 2)$

79. $f(x) = \ln x + 1$

80. $f(x) = 3\ln x - 1$

81. Population Growth The population of a town will double in

$$t = \frac{8 \ln 3}{\ln 63 - \ln 45}$$

years. Find t.

82. Work The work W (in foot-pounds) done in compressing a volume of 9 cubic feet at a pressure of 15 pounds per square inch to a volume of 3 cubic feet is $W = 19{,}440(\ln 9 - \ln 3)$. Find W.

83. Human Memory Model Students in a mathematics class were given an exam and then retested monthly with an equivalent exam. The average score g for the class can be approximated by the human memory model

$$g(t) = 78 - 14\log_{10}(t + 1), \quad 0 \le t \le 12$$

where t is the time (in months).

(a) What was the average score on the original exam?

(b) What was the average score after 4 months?

(c) When did the average score drop below 70?

84. Human Memory Model Students in a seventh-grade class were given an exam. During the next 2 years, the same students were retested several times. The average score g can be approximated by the model

$$g(t) = 87 - 16\log_{10}(t + 1), \quad 0 \le t \le 24$$

where t is the time (in months).

(a) What was the average score on the original exam?

(b) What was the average score after 6 months?

(c) When did the average score drop below 70?

85. Investment Time A principal P, invested at 5.25% interest and compounded continuously, increases to an amount that is K times the principal after t years, where t is given by

$$t = \frac{\ln K}{0.0525}.$$

(a) Complete the table.

K	1	2	4	6	8	10	12
t							

(b) Use the table in part (a) to graph the function.

Ⓣ **86. Investment Time** A principal P, invested at 4.85% interest and compounded continuously, increases to an amount that is K times the principal after t years, where t is given by

$$t = \frac{\ln K}{0.0485}.$$

Use a graphing utility to graph this function.

Skill Retention Model In Exercises 87 and 88, participants in an industrial psychology study were taught a simple mechanical task and tested monthly on this mechanical task for a period of 1 year. The average scores for the participants are given by the model

$f(t) = 98 - 14 \log_{10}(t + 1), \quad 0 \le t \le 12$

where t is the time (in months).

87. Use a graphing utility to graph the function. Use the graph to discuss the domain and range of the function.

88. Think About It Based on the graph of f, do you think the study's participants practiced the simple mechanical task very often? Cite the behavior of the graph to justify your answer.

Productivity In Exercises 89 and 90, the productivity of a new employee (in units produced per day) is given by the model

$g(t) = 2 + 12 \ln t, \quad 1 \le t \le 15$

where t is the time (in work days).

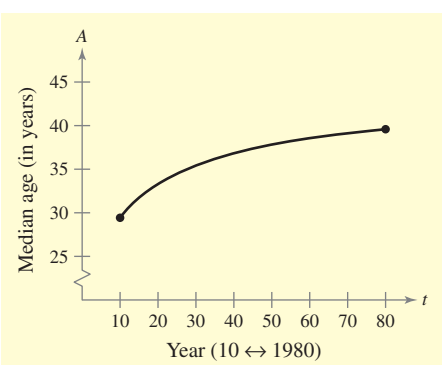
89. Use a graphing utility to graph the function. Use the graph to discuss the domain and range of the function.

90. Think About It Based on the graph of g, do you think the new employee will reach a benchmark of 40 units produced per day by the end of three work weeks? Explain.

91. Median Age of U.S. Population The model

$A = 15.68 - 0.037t + 6.131 \ln t, \quad 10 \le t \le 80$

approximates the median age A of the United States population from 1980 to 2050. In the model, t represents the year, with $t = 10$ corresponding to 1980 (see figure). *(Source: U.S. Census Bureau)*



(a) Use the model to approximate the median age in the United States in 1980.

(b) Use the model to approximate the median age in the United States in 1990.

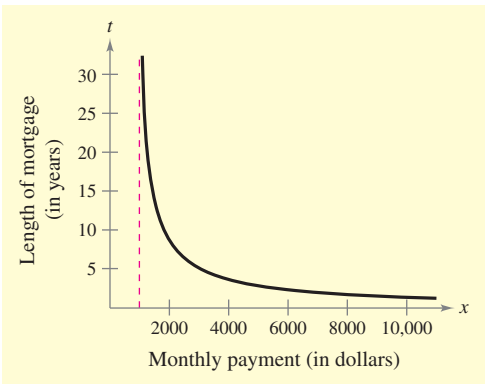
(c) Use the model to approximate the change in median age in the United States from 1980 to 2000.

(d) Use the model to project the change in median age in the United States from 1980 to 2050.

92. Monthly Payment The model

$t = 12.542 \ln\left(\frac{x}{x - 1000}\right), \quad x > 1000$

approximates the length of a home mortgage of \$150,000 at 8% interest in terms of the monthly payment. In the model, t is the length of the mortgage (in years) and x is the monthly payment (in dollars) (see figure).



(a) Use the model to approximate the length of a \$150,000 mortgage at 8% interest when the monthly payment is \$1100.65 and when the monthly payment is \$1254.68.

(b) Approximate the total amount paid over the term of the mortgage with a monthly payment of \$1100.65 and with a monthly payment of \$1254.68.

(c) Approximate the total interest charge for a monthly payment of \$1100.65 and for a monthly payment of \$1254.68.

(d) What is the vertical asymptote of the model? Interpret its meaning in the context of the problem.

93. Think About It The table of values was obtained by evaluating a function. Determine which of the statements may be true and which must be false.

x	1	2	8
y	0	1	3

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

Section 4.3

Properties of Logarithms

- Evaluate a logarithm using the change-of-base formula.
- Use properties of logarithms to evaluate or rewrite a logarithmic expression.
- Use properties of logarithms to expand or condense a logarithmic expression.
- Use logarithmic functions to model and solve real-life applications.

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e). Although common logs and natural logs are the most frequently used, you may occasionally need to evaluate logarithms with other bases. To do this, you can use the following **change-of-base formula.**

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows.

Base b

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Base 10

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}$$

Base e

$$\log_a x = \frac{\ln x}{\ln a}$$

One way to look at the change-of-base formula is that logarithms to base a are simply *constant multiples* of logarithms to base b. The constant multiplier is $1/(\log_b a)$.

Example 1 Changing Bases Using Common Logarithms

a. $\log_4 30 = \dfrac{\log_{10} 30}{\log_{10} 4} \approx \dfrac{1.47712}{0.60206} \approx 2.4534$

b. $\log_2 14 = \dfrac{\log_{10} 14}{\log_{10} 2} \approx \dfrac{1.14613}{0.30103} \approx 3.8074$

✓CHECKPOINT 1

Evaluate $\log_8 56$ using common logarithms. Round your result to three decimal places. ■

Example 2 Changing Bases Using Natural Logarithms

a. $\log_4 30 = \dfrac{\ln 30}{\ln 4} \approx \dfrac{3.40120}{1.386294} \approx 2.4534$

b. $\log_2 14 = \dfrac{\ln 14}{\ln 2} \approx \dfrac{2.63906}{0.693147} \approx 3.8074$

✓CHECKPOINT 2

Evaluate $\log_8 56$ using natural logarithms. Round your result to three decimal places. ■

Notice in Examples 1 and 2 that the result is the same whether common logarithms or natural logarithms are used in the change-of-base formula.

Properties of Logarithms

You know from the preceding section that the logarithmic function with base a is the *inverse function* of the exponential function with base a. So, it makes sense that the properties of exponents should have corresponding properties involving logarithms. For instance, the exponential property $a^0 = 1$ has the corresponding logarithmic property $\log_a 1 = 0$.

Properties of Logarithms

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

Logarithm with Base a	*Natural Logarithm*	
$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$	Product Rule
$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$	Quotient Rule
$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$	Power Rule

STUDY TIP

There is no general property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(x + y)$ is *not* equal to $\log_a x + \log_a y$.

Example 3 Using Properties of Logarithms

Write each logarithm in terms of ln 2 and ln 3.

a. $\ln 6$ **b.** $\ln \frac{2}{27}$

SOLUTION

a. $\ln 6 = \ln(2 \cdot 3)$ Rewrite 6 as $2 \cdot 3$.

$= \ln 2 + \ln 3$ Product Rule

b. $\ln \frac{2}{27} = \ln 2 - \ln 27$ Quotient Rule

$= \ln 2 - \ln 3^3$ Rewrite 27 as 3^3.

$= \ln 2 - 3 \ln 3$ Power Rule

✓CHECKPOINT 3

Write $\log_{10} \frac{25}{3}$ in terms of $\log_{10} 3$ and $\log_{10} 5$. ■

Example 4 Using Properties of Logarithms

Use the properties of logarithms to verify that $-\log_{10} \frac{1}{100} = \log_{10} 100$.

SOLUTION

$$-\log_{10} \frac{1}{100} = -\log_{10}(100^{-1}) = -(-1)\log_{10} 100 = \log_{10} 100$$

Try checking this result on your calculator.

✓CHECKPOINT 4

Use the properties of logarithms to verify that $-\ln \frac{2}{e} = 1 - \ln 2$. ■

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because these properties convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

DISCOVERY

Use a calculator to approximate $\ln \sqrt[3]{e^2}$. Now find the exact value by rewriting $\ln \sqrt[3]{e^2}$ with a rational exponent using the properties of logarithms. How do the two values compare?

Example 5 Expanding Logarithmic Expressions

Expand each logarithmic expression.

a. $\log_4 5x^3y$ **b.** $\ln \dfrac{\sqrt{3x-5}}{7}$

SOLUTION

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$ Product Rule

$= \log_4 5 + 3\log_4 x + \log_4 y$ Power Rule

b. $\ln \dfrac{\sqrt{3x-5}}{7} = \ln \dfrac{(3x-5)^{1/2}}{7}$ Rewrite using rational exponent.

$= \ln(3x-5)^{1/2} - \ln 7$ Quotient Rule

$= \dfrac{1}{2}\ln(3x-5) - \ln 7$ Power Rule

✓CHECKPOINT 5

Expand the expression $\ln 2mn^2$. ■

In Example 5, the properties of logarithms were used to *expand* logarithmic expressions. In Example 6, this procedure is reversed and the properties of logarithms are used to *condense* logarithmic expressions.

STUDY TIP

When applying the properties of logarithms to a logarithmic function, you should be careful to check the domain of the function. For example, the domain of $f(x) = \ln x^2$ is all real $x \neq 0$, whereas the domain of $g(x) = 2\ln x$ is all real $x > 0$.

Example 6 Condensing Logarithmic Expressions

Condense each logarithmic expression.

a. $\frac{1}{2}\log_{10} x + 3\log_{10}(x+1)$ **b.** $2\ln(x+2) - \ln x$

c. $\frac{1}{3}[\log_2 x + \log_2(x+1)]$

SOLUTION

a. $\frac{1}{2}\log_{10} x + 3\log_{10}(x+1) = \log_{10} x^{1/2} + \log_{10}(x+1)^3$ Power Rule

$= \log_{10}\left[\sqrt{x}(x+1)^3\right]$ Product Rule

b. $2\ln(x+2) - \ln x = \ln(x+2)^2 - \ln x$ Power Rule

$= \ln \dfrac{(x+2)^2}{x}$ Quotient Rule

c. $\frac{1}{3}[\log_2 x + \log_2(x+1)] = \frac{1}{3}\{\log_2[x(x+1)]\}$ Product Rule

$= \log_2[x(x+1)]^{1/3}$ Power Rule

$= \log_2 \sqrt[3]{x(x+1)}$ Rewrite with a radical.

✓CHECKPOINT 6

Condense the expression $2\log_{10}(x+1) - 3\log_{10}(x-1)$. ■

Applications

One method of determining how the x- and y-values of a set of nonlinear data are related begins by taking the natural logarithm of each of the x- and y-values. If you graph the points $(\ln x, \ln y)$ and they fall in a straight line, then you can determine that the x- and y-values are related by the equation

$$\ln y = m \ln x$$

where m is the slope of the straight line.

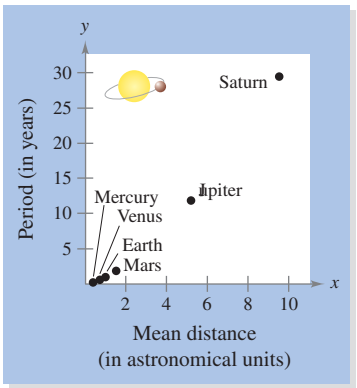


FIGURE 4.16

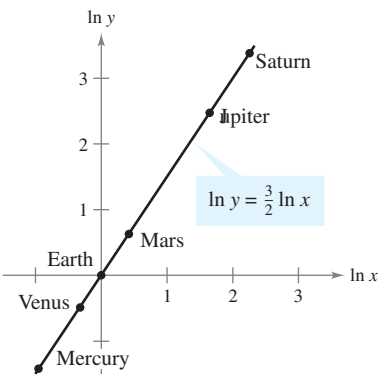


FIGURE 4.17

Example 7 Finding a Mathematical Model

The table shows the mean distance from the sun x and the period (the time it takes a planet to orbit the sun) y for each of the six planets that are closest to the sun. In the table, the mean distance is given in astronomical units (where Earth's mean distance is defined as 1.0), and the period is given in years. Find an equation that relates y and x.

Planet	Mean distance, x	Period, y
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.863
Saturn	9.537	29.447

SOLUTION The points in the table are plotted in Figure 4.16. From this figure it is not clear how to find an equation that relates y and x. To solve this problem, take the natural logarithm of each of the x- and y-values in the table. This produces the following results.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
$\ln x$	-0.949	-0.324	0.000	0.421	1.649	2.255
$\ln y$	-1.423	-0.486	0.000	0.632	2.473	3.383

Now, by plotting the points in the second table, you can see that all six of the points appear to lie in a line (see Figure 4.17). Choose any two points to determine the slope of the line. Using the two points $(0.421, 0.632)$ and $(0, 0)$, you can determine that the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}.$$

By the point-slope form, the equation of the line is $Y = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$. You can therefore conclude that $\ln y = \frac{3}{2} \ln x$.

✓CHECKPOINT 7

Find a logarithmic equation that relates y and x.

x	1	2	3	4
y	1	1.414	1.732	2

Example 8 Sound Intensity



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The level of sound L (in decibels) with an intensity of I (in watts per square meter) is given by

$$L = 10 \log_{10} \frac{I}{I_0}$$

where I_0 represents the faintest sound that can be heard by the human ear, and is approximately equal to 10^{-12} watt per square meter. You and your roommate are playing your stereos at the same time and at the same intensity. How much louder is the music when both stereos are playing compared with when just one stereo is playing?

SOLUTION Let L_1 represent the level of sound when one stereo is playing and let L_2 represent the level of sound when both stereos are playing. Using the formula for level of sound, you can express L_1 as

$$L_1 = 10 \log_{10} \frac{I}{10^{-12}}.$$

For L_2, multiply I by 2 as shown below

$$L_2 = 10 \log_{10} \frac{2I}{10^{-12}}$$

because L_2 represents the level of sound when *two* stereos are playing at the same intensity I. To determine the increase in loudness, subtract L_1 from L_2 as follows.

$$\begin{aligned} L_2 - L_1 &= 10 \log_{10} \frac{2I}{10^{-12}} - 10 \log_{10} \frac{I}{10^{-12}} \\ &= 10\left(\log_{10} \frac{2I}{10^{-12}} - \log_{10} \frac{I}{10^{-12}}\right) \\ &= 10\left(\log_{10} 2 + \log_{10} \frac{I}{10^{-12}} - \log_{10} \frac{I}{10^{-12}}\right) \\ &= 10 \log_{10} 2 \approx 3 \end{aligned}$$

So, the music is about 3 decibels louder. Notice that the variable I drops out of the equation when it is simplified. This means that the loudness increases by 3 decibels when both stereos are played at the same intensity, regardless of the individual intensities of the stereos.

✓CHECKPOINT 8

Two sounds have intensities of $I_1 = 10^{-6}$ watt per square meter and $I_2 = 10^{-9}$ watt per square meter. Use the formula for the level of sound in Example 8 to find the difference in loudness between the two sounds. ■

CONCEPT CHECK

In Exercises 1–4, let x and y be positive real numbers. Determine whether the statement is true or false. Explain your reasoning.

1. The expression $\log_{10} 25$ can be rewritten as $2 \log_{10} 5$.
2. The expression $\ln(x + y)$ can be rewritten as $\ln x + \ln y$.
3. The expression $2 \ln(x + 1) - \ln y$ can be rewritten as $\ln \frac{(x + 1)^2}{y}$.
4. The expressions $\frac{\log_{10} x}{\log_{10} y}$ and $\frac{\ln x}{\ln y}$ are equivalent.

Skills Review 4.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3, 0.4, and 4.2.

In Exercises 1–4, evaluate the expression without using a calculator.

1. $\log_7 49$ **2.** $\log_2 \frac{1}{32}$ **3.** $\ln \frac{1}{e^2}$ **4.** $\log_{10} 0.001$

In Exercises 5–8, simplify the expression.

5. e^2e^3 **6.** $\frac{e^2}{e^3}$ **7.** $(e^2)^3$ **8.** $(e^2)^0$

In Exercises 9–12, rewrite the equation in exponential form.

9. $y = \frac{1}{x^2}$ **10.** $y = \sqrt{x}$ **11.** $\log_4 64 = 3$ **12.** $\log_{16} 4 = \frac{1}{2}$

Exercises 4.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, write the logarithm in terms of common logarithms.

1. $\log_5 8$ **2.** $\log_7 12$
3. $\ln 30$ **4.** $\ln 20$
5. $\log_3 n$ **6.** $\log_4 m$
7. $\log_{1/5} x$ **8.** $\log_{1/3} x$
9. $\log_x \frac{3}{10}$ **10.** $\log_x \frac{3}{4}$
11. $\log_{2.6} x$ **12.** $\log_{7.1} x$

In Exercises 13–24, write the logarithm in terms of natural logarithms.

13. $\log_5 8$ **14.** $\log_7 12$
15. $\log_{10} 5$ **16.** $\log_{10} 20$
17. $\log_3 n$ **18.** $\log_2 m$
19. $\log_{1/5} x$ **20.** $\log_{1/3} x$
21. $\log_x \frac{3}{10}$ **22.** $\log_x \frac{3}{4}$
23. $\log_{2.6} x$ **24.** $\log_{7.1} x$

In Exercises 25–36, evaluate the logarithm. Round your result to three decimal places.

25. $\log_2 6$ **26.** $\log_8 3$
27. $\log_{27} 35$ **28.** $\log_{19} 42$
29. $\log_{15} 1250$ **30.** $\log_{20} 1575$
31. $\log_5 \frac{1}{3}$ **32.** $\log_9 \frac{3}{5}$
33. $\log_{1/4} 10$ **34.** $\log_{1/3} 5$
35. $\log_{1/2} 0.2$ **36.** $\log_{1/6} 0.025$

In Exercises 37–50, approximate the logarithm using the properties of logarithms, given $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.

37. $\log_b 10$ **38.** $\log_b 15$
39. $\log_b \frac{2}{3}$ **40.** $\log_b \frac{3}{5}$
41. $\log_b 8$ **42.** $\log_b 81$
43. $\log_b \sqrt{2}$ **44.** $\log_b \sqrt{5}$
45. $\log_b 40$ **46.** $\log_b 45$
47. $\log_b (2b)^{-2}$ **48.** $\log_b (3b^2)$
49. $\log_b \sqrt[3]{4b}$ **50.** $\log_b \sqrt[3]{3b}$

In Exercises 51–56, find the *exact* value of the logarithmic expression without using a calculator.

51. $\log_4 \sqrt[3]{4}$ **52.** $\log_8 \sqrt[4]{8}$
53. $\ln \frac{1}{\sqrt{e}}$ **54.** $\ln \sqrt[4]{e^3}$
55. $\log_5 \frac{1}{125}$ **56.** $\log_7 \frac{49}{343}$

In Exercises 57–64, use the properties of logarithms to simplify the given logarithmic expression.

57. $\log_9 \frac{1}{18}$ **58.** $\log_5 \frac{1}{15}$
59. $\log_7 \sqrt{70}$ **60.** $\log_5 \sqrt{75}$
61. $\log_5 \frac{1}{250}$ **62.** $\log_{10} \frac{9}{300}$
63. $\ln(5e^6)$ **64.** $\ln \frac{6}{e^2}$

In Exercises 65–86, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms. (Assume all variables are positive.)

65. $\log_2(4^3 \cdot 3^5)$

66. $\log_3(3^2 \cdot 4^2)$

67. $\log_3 4n$

68. $\log_6 6x$

69. $\log_5 \frac{x}{25}$

70. $\log_{10} \frac{y}{2}$

71. $\log_2 x^4$

72. $\log_2 z^{-3}$

73. $\ln \sqrt{z}$

74. $\ln \sqrt[3]{t}$

75. $\ln xyz$

76. $\ln \frac{xy}{z}$

77. $\ln \sqrt{a-1}, \quad a > 1$

78. $\ln \sqrt[3]{y-2}, \quad y > 2$

79. $\ln \left[\frac{(z-1)^2}{z}\right]$

80. $\ln \left(\frac{x}{\sqrt{x^2+1}}\right)$

81. $\ln \frac{z}{\sqrt[3]{z+3}}$

82. $\log_9 \frac{\sqrt{y}}{z^2}$

83. $\ln \sqrt[3]{\frac{x}{y}}$

84. $\ln \sqrt{\frac{x^2}{y^3}}$

85. $\ln \sqrt[4]{x^3(x^2+3)}$

86. $\ln \sqrt{x^2(x+2)}$

In Exercises 87–102, condense the expression to the logarithm of a single quantity.

87. $\log_3 x + \log_3 5$

88. $\log_5 y + \log_5 x$

89. $\log_4 8 - \log_4 x$

90. $\log_{10} 4 - \log_{10} z$

91. $2 \log_{10}(x+4)$

92. $-4 \log_{10} 2x$

93. $-\ln x - 3 \ln 6$

94. $2 \ln 8 + 5 \ln z$

95. $\frac{1}{3} \ln 5x - \ln(x+1)$

96. $\frac{3}{2} \ln(z-2) + \ln z$

97. $\log_8(x-2) - \log_8(x+2)$

98. $3 \log_7 x + 2 \log_7 y - 4 \log_7 z$

99. $2 \ln 3 - \frac{1}{2} \ln(x^2+1)$

100. $\frac{3}{2} \ln t^6 - \frac{3}{4} \ln t^4$

101. $\ln x - \ln(x+2) - \ln(x-2)$

102. $\ln(x+1) + 2 \ln(x-1) + 3 \ln x$

Curve Fitting In Exercises 103–106, find a logarithmic equation that relates y and x. Explain the steps used to find the equation.

103.

x	1	2	3	4	5	6
y	1	1.189	1.316	1.414	1.495	1.565

104.

x	1	2	3	4	5	6
y	1	1.587	2.080	2.520	2.924	3.302

105.

x	1	2	3	4	5	6
y	2.5	2.102	1.9	1.768	1.672	1.597

106.

x	1	2	3	4	5	6
y	0.5	2.828	7.794	16	27.951	44.091

107. Nail Length The approximate lengths and diameters (in inches) of common nails are shown in the table. Find a logarithmic equation that relates the diameter y of a common nail to its length x.

Length, x	Diameter, y	Length, x	Diameter, y
1	0.070	4	0.176
2	0.111	5	0.204
3	0.146	6	0.231

108. Galloping Speeds of Animals Four-legged animals run with two different types of motion: trotting and galloping. An animal that is trotting has at least one foot on the ground at all times, whereas an animal that is galloping has all four feet off the ground at some point in its stride. The number of strides per minute at which an animal breaks from a trot to a gallop depends on the weight of the animal. Use the table to find a logarithmic equation that relates an animal's lowest galloping speed y (in strides per minute) to its weight x (in pounds).

Weight, x	Galloping speed, y	Weight, x	Galloping speed, y
25	191.5	75	164.2
35	182.7	500	125.9
50	173.8	1000	114.2

109. Sound Intensity Use the equation for the level of sound in Example 8 to find the difference in loudness between an average office and a broadcast studio with the intensities given below.

Office: 1.26×10^{-7} watt per square meter

Broadcast studio: 3.16×10^{-10} watt per square meter

110. Sound Intensity Use the equation for the level of sound in Example 8 to find the difference in loudness between a bird singing and rustling leaves with the intensities given below.

Bird singing: 10^{-8} watt per square meter

Rustling leaves: 10^{-10} watt per square meter

111. Graphical Analysis Use a graphing utility to graph

$$f(x) = \ln 5x \quad \text{and} \quad g(x) = \ln 5 + \ln x$$

in the same viewing window. What do you observe about the two graphs? What property of logarithms is being demonstrated graphically?

112. Graphical Analysis You are helping another student learn the properties of logarithms. How would you use a graphing utility to demonstrate to this student the logarithmic property

$$\log_a u^v = v \log_a u$$

(u is a positive number, v is a real number, and a is a positive number such that $a \neq 1$)? What two functions could you use? Briefly describe your explanation of this property using these functions and their graphs.

113. Reasoning An algebra student claims that the following is true:

$$\log_a \frac{x}{y} = \frac{\log_a x}{\log_a y} = \log_a x - \log_a y.$$

Discuss how you would use a graphing utility to demonstrate that this claim is not true. Describe how to demonstrate the actual property of logarithms that is hidden in this faulty claim.

114. Reasoning A classmate claims that the following is true:

$$\ln(x + y) = \ln x + \ln y = \ln xy.$$

Discuss how you would use a graphing utility to demonstrate that this claim is not true. Describe how to demonstrate the actual property of logarithms that is hidden in this faulty claim.

115. Complete the proof of the logarithmic property

$$\log_a uv = \log_a u + \log_a v.$$

Let $\log_a u = x$ and $\log_a v = y$.

$a^x =$ ☐ and $a^y =$ ☐	Rewrite in exponential form.
$u \cdot v =$ ☐ $\cdot$ ☐ $= a$☐	Multiply and substitute for u and v.
☐ $= x + y$	Rewrite in logarithmic form.
$\log_a uv =$ ☐ $+$ ☐	Substitute for x and y.

116. Complete the proof of the logarithmic property

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

Let $\log_a u = x$ and $\log_a v = y$.

$a^x =$ ☐ and $a^y =$ ☐	Rewrite in exponential form.
$\frac{u}{v} = \frac{\square}{\square} = a$☐	Divide and substitute for u and v.
☐ $= x - y$	Rewrite in logarithmic form.
$\log_a \frac{u}{v} =$ ☐ $-$ ☐	Substitute for x and y.

Business Capsule



Photo courtesy of Maggie's Place

Co-founded by five recent college graduates, Maggie's Place is a community of homes that provides hospitality for pregnant women who are alone or living on the streets. Maggie's Place provides for immediate needs such as shelter, clothing, food, and community support. Expectant mothers are connected to community resources such as prenatal care, education programs, and low-cost housing. Maggie's Place opened its first home, the Magdalene House, on May 13, 2000 in Phoenix, Arizona and has since expanded to four homes. Maggie's Place is working with other local and national groups to develop homes in other communities.

117. Research Project Use your campus library, the Internet, or some other reference source to find information about a nonprofit group or company whose growth can be modeled by a logarithmic function. Write a brief report about the growth of the group or company.

Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

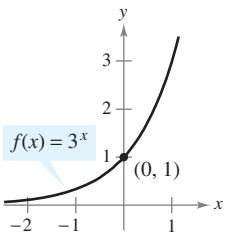


Figure for 1–4

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, use the graph of $f(x) = 3^x$ to sketch the graph of the function.

1. $g(x) = 3^x - 2$

2. $h(x) = 3^{-x}$

3. $k(x) = \log_3 x$

4. $j(x) = \log_3 (x - 1)$

5. For $P = \$10{,}000$, $r = 5.15\%$, and $t = 8$ years, find the balance in an account when interest is compounded (a) monthly and (b) continuously.

6. The numbers of children C (in millions) participating in the Federal School Breakfast Program from 1997 to 2005 can be approximated by the model

$$C = 5.26(1.04)^t, \quad 7 \le t \le 15$$

where t represents the year, with $t = 7$ corresponding to 1997. *(Source: U.S. Department of Agriculture)*

(a) Use the model to estimate the numbers of children participating in the Federal School Breakfast Program in 2000 and in 2005.

(b) Use the model to predict the numbers of children that will participate in the Federal School Breakfast Program in 2009 and in 2010.

7. The size of a bacteria population is modeled by

$$P(t) = 100e^{0.2154t}$$

where t is the time in hours. Find (a) $P(0)$, (b) $P(6)$, and (c) $P(12)$.

8. Use the demand function

$$p = 4000\left(1 - \frac{8}{8 + e^{-0.003x}}\right)$$

to find the price for a demand of $x = 500$ MP3 players.

In Exercises 9–12, evaluate the expression without using a calculator.

9. $\log_{10} 100$

10. $\ln e^4$

11. $\log_4 \frac{1}{16}$

12. $\ln 1$

13. Sketch the graphs of $f(x) = 4^x$ and $g(x) = \log_4 x$ in the same coordinate plane. Identify the domains of f and g. Discuss the special relationship between f and g that is demonstrated by their graphs.

x	y
1	1
2	1.260
3	1.442
4	1.587
6	1.817
8	2.000

Table for 20

In Exercises 14 and 15, find the *exact* value of the logarithm.

14. $\log_7 \sqrt{343}$

15. $\ln \sqrt[5]{e^6}$

In Exercises 16 and 17, expand the logarithmic expression.

16. $\log_{10} \sqrt[3]{\frac{xy}{z}}$

17. $\ln\left(\frac{x^2 + 3}{x^3}\right)$

In Exercises 18 and 19, condense the logarithmic expression.

18. $\ln x + \ln y - \ln 3$

19. $-3 \log_{10} 4 - 3 \log_{10} x$

20. Use the values in the table at the left to find a logarithmic equation that relates y and x.

Section 4.4

Solving Exponential and Logarithmic Equations

- Solve an exponential equation.
- Solve a logarithmic equation.
- Use an exponential or a logarithmic model to solve an application problem.

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for *solving equations* involving these exponential and logarithmic functions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

One-to-One Properties	*Inverse Properties*
$a^x = a^y$ if and only if $x = y$.	$a^{\log_a x} = x$
$\log_a x = \log_a y$ if and only if $x = y$.	$\log_a a^x = x$

Example 1 Solving Simple Equations

	Original Equation	*Rewritten Equation*	*Solution*	*Property*
a.	$2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b.	$\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
c.	$e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
d.	$\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
e.	$\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse

✓CHECKPOINT 1

Solve each equation for x.

a. $3^x = 81$ **b.** $\log_6 x = 3$ ■

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Property of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

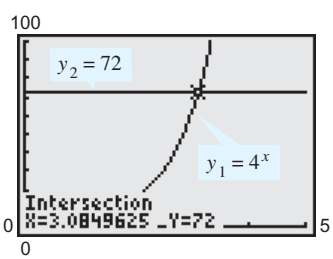
Solving Exponential Equations

TECHNOLOGY

When solving an exponential or logarithmic equation, you can check your solution graphically by "graphing the left and right sides separately" and using the *intersect* feature of your graphing utility to determine the point of intersection. For instance, to check the solution of the equation in Example 2(a), you can graph

$$y_1 = 4^x \quad \text{and} \quad y_2 = 72$$

in the same viewing window, as shown below. Using the *intersect* feature of your graphing utility, you can determine that the graphs intersect when $x \approx 3.085$, which confirms the solution found in Example 2(a).



For instructions on how to use the *intersect* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

✓CHECKPOINT 3

Solve $62 - 10^x = 24$ and approximate the result to three decimal places. ■

Example 2 Solving Exponential Equations

Solve each equation and approximate the result to three decimal places.

a. $4^x = 72$ **b.** $3(2^x) = 42$

SOLUTION

a.
$$4^x = 72$$ Write original equation.
$$\log_4 4^x = \log_4 72$$ Take log (base 4) of each side.
$$x = \log_4 72$$ Inverse Property
$$x = \frac{\ln 72}{\ln 4} \approx 3.085$$ Change-of-base formula

The solution is $x = \log_4 72 \approx 3.085$. Check this in the original equation.

b.
$$3(2^x) = 42$$ Write original equation.
$$2^x = 14$$ Divide each side by 3 to isolate the exponential expression.
$$\log_2 2^x = \log_2 14$$ Take log (base 2) of each side.
$$x = \log_2 14$$ Inverse Property
$$x = \frac{\ln 14}{\ln 2} \approx 3.807$$ Change-of-base formula

The solution is $x = \log_2 14 \approx 3.807$. Check this in the original equation.

✓CHECKPOINT 2

Solve $6^x = 84$ and approximate the result to three decimal places. ■

In Example 2(a), the exact solution is $x = \log_4 72$ and the approximate solution is $x \approx 3.085$. An exact answer is preferred when the solution is an intermediate step in a larger problem. For a final answer, an approximate solution in decimal form is easier to comprehend.

Example 3 Solving an Exponential Equation

Solve $e^x + 5 = 60$ and approximate the result to three decimal places.

SOLUTION

$$e^x + 5 = 60$$ Write original equation.
$$e^x = 55$$ Subtract 5 from each side to isolate the exponential expression.
$$\ln e^x = \ln 55$$ Take natural log of each side.
$$x = \ln 55$$ Inverse Property
$$x \approx 4.007$$ Use a calculator.

The solution is $x = \ln 55 \approx 4.007$. Check this in the original equation.

Example 4 Solving an Exponential Equation

Solve $2(3^{2t-5}) - 4 = 11$ and approximate the result to three decimal places.

SOLUTION

$$2(3^{2t-5}) - 4 = 11$$ Write original equation.

$$2(3^{2t-5}) = 15$$ Add 4 to each side.

$$3^{2t-5} = \frac{15}{2}$$ Divide each side by 2.

$$\log_3 3^{2t-5} = \log_3 \frac{15}{2}$$ Take log (base 3) of each side.

$$2t - 5 = \log_3 \frac{15}{2}$$ Inverse Property

$$2t = 5 + \log_3 7.5$$ Add 5 to each side.

$$t = \frac{5}{2} + \frac{1}{2}\log_3 7.5$$ Divide each side by 2.

$$t \approx 3.417$$ Use a calculator.

The solution is $t = \frac{5}{2} + \frac{1}{2}\log_3 7.5 \approx 3.417$. Check this in the original equation.

✓CHECKPOINT 4

Solve $4(4^{2t-7}) + 14 = 110$ and approximate the result to three decimal places. ■

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in Examples 2, 3, and 4. However, the algebra is a bit more complicated.

Example 5 Solving an Exponential Equation of Quadratic Type

Solve $e^{2x} - 3e^x + 2 = 0$.

SOLUTION

$$e^{2x} - 3e^x + 2 = 0$$ Write original equation.

$$(e^x)^2 - 3e^x + 2 = 0$$ Write in quadratic form.

$$(e^x - 2)(e^x - 1) = 0$$ Factor.

$$e^x - 2 = 0 \Rightarrow x = \ln 2$$ Set 1st factor equal to 0.

$$e^x - 1 = 0 \Rightarrow x = 0$$ Set 2nd factor equal to 0.

The solutions are $x = \ln 2$ and $x = 0$. Check these in the original equation. Or, check by graphing $y = e^{2x} - 3e^x + 2$ using a graphing utility. The graph should have two x-intercepts: $x = \ln 2$ and $x = 0$, as shown in Figure 4.18.

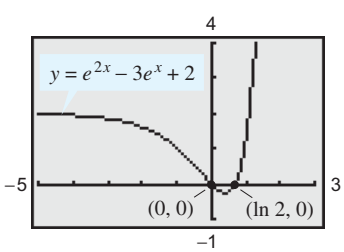


FIGURE 4.18

✓CHECKPOINT 5

Solve $e^{2x} - 7e^x + 12 = 0$. ■

Solving Logarithmic Equations

To solve a logarithmic equation such as

$\ln x = 3$ Logarithmic form

write the equation in exponential form as follows.

$e^{\ln x} = e^3$ Exponentiate each side.

$x = e^3$ Exponential form

This procedure is called **exponentiating** each side of an equation.

Example 6 Solving Logarithmic Equations

a. Solve $\ln x = 2$.

b. Solve $2 \log_5 3x = 4$.

SOLUTION

a. $\ln x = 2$ Write original equation.

$e^{\ln x} = e^2$ Exponentiate each side.

$x = e^2$ Inverse Property

The solution is $x = e^2$. Check this in the original equation.

b. $2 \log_5 3x = 4$ Write original equation.

$\log_5 3x = 2$ Divide each side by 2.

$5^{\log_5 3x} = 5^2$ Exponentiate each side (base 5).

$3x = 25$ Inverse Property

$x = \frac{25}{3}$ Divide each side by 3.

The solution is $x = \frac{25}{3}$. Check this in the original equation.

✓CHECKPOINT 6

Solve $\log_3 2x = 4$. ■

Example 7 Solving a Logarithmic Equation

Solve $\log_3(5x - 1) = \log_3(x + 7)$.

SOLUTION

$\log_3(5x - 1) = \log_3(x + 7)$ Write original equation.

$5x - 1 = x + 7$ One-to-One Property

$4x = 8$ Add $-x$ and 1 to each side.

$x = 2$ Divide each side by 4.

The solution is $x = 2$. Check this in the original equation.

✓CHECKPOINT 7

Solve $\ln(3x + 2) = \ln(x + 8)$. ■

TECHNOLOGY

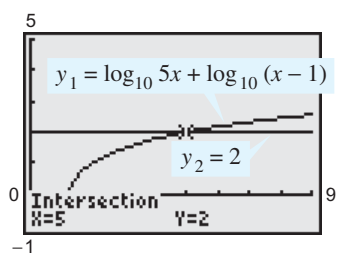
You can use a graphing utility to verify that the equation in Example 9 has $x = 5$ as its only solution. Graph

$$y_1 = \log_{10} 5x + \log_{10}(x - 1)$$

and

$$y_2 = 2$$

in the same viewing window. From the graph shown below, it appears that the graphs of the two equations intersect at one point. Use the *intersect* feature or the *zoom* and *trace* features to determine that $x = 5$ is the solution. You can verify this algebraically by substituting $x = 5$ into the original equation.



For instructions on how to use the *zoom* and *trace* features, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

✓CHECKPOINT 9

Solve $\log_6 x + \log_6(x + 5) = 2$. ■

Example 8 Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$ and approximate the result to three decimal places.

SOLUTION

$5 + 2 \ln x = 4$	Write original equation.
$2 \ln x = -1$	Subtract 5 from each side.
$\ln x = -\frac{1}{2}$	Divide each side by 2.
$e^{\ln x} = e^{-1/2}$	Exponentiate each side.
$x = e^{-1/2}$	Inverse Property
$x \approx 0.607$	Use a calculator.

The solution is $x = e^{-1/2} \approx 0.607$. Check this in the original equation.

✓CHECKPOINT 8

Solve $4 + 3 \ln x = 16$ and approximate the result to three decimal places. ■

Because the domain of a logarithmic function generally does not include all real numbers, be sure to check for extraneous solutions of logarithmic equations.

Example 9 Checking for Extraneous Solutions

Solve $\log_{10} 5x + \log_{10}(x - 1) = 2$.

SOLUTION

$\log_{10} 5x + \log_{10}(x - 1) = 2$	Write original equation.
$\log_{10}[5x(x - 1)] = 2$	Product Rule of logarithms
$10^{\log_{10}(5x^2 - 5x)} = 10^2$	Exponentiate each side (base 10).
$5x^2 - 5x = 100$	Inverse Property
$x^2 - x - 20 = 0$	Write in general form.
$(x - 5)(x + 4) = 0$	Factor.
$x - 5 = 0$	Set 1st factor equal to 0.
$x = 5$	Solution
$x + 4 = 0$	Set 2nd factor equal to 0.
$x = -4$	Solution

The solutions appear to be $x = 5$ and $x = -4$. However, when you check these in the original equation, you can see that $x = 5$ is the only solution.

In Example 9, the domain of $\log_{10} 5x$ is $x > 0$ and the domain of $\log_{10}(x - 1)$ is $x > 1$, so the domain of the original equation is $x > 1$. Because the domain is all real numbers greater than 1, the solution $x = -4$ is extraneous.

Applications

Example 10 Doubling and Tripling an Investment

You deposit $500 in an account that pays 6.75% interest, compounded continuously.

a. How long will it take your money to double?

b. How long will it take your money to triple?

SOLUTION Using the formula for compound interest with continuous compounding, you can find that the balance in the account is given by

$$A = Pe^{rt}$$
$$= 500e^{0.0675t}.$$

a. To find the time required for the balance to double, let $A = 1000$ and solve the resulting equation for t.

$$500e^{0.0675t} = 1000 \quad \text{Write original equation.}$$
$$e^{0.0675t} = 2 \quad \text{Divide each side by 500.}$$
$$\ln e^{0.0675t} = \ln 2 \quad \text{Take natural log of each side.}$$
$$0.0675t = \ln 2 \quad \text{Inverse Property}$$
$$t = \frac{1}{0.0675} \ln 2 \quad \text{Divide each side by 0.0675.}$$
$$t \approx 10.27 \quad \text{Use a calculator.}$$

The balance in the account will double after approximately 10.27 years.

b. To find the time required for the balance to triple, let $A = 1500$ and solve the resulting equation for t.

$$500e^{0.0675t} = 1500 \quad \text{Write original equation.}$$
$$e^{0.0675t} = 3 \quad \text{Divide each side by 500.}$$
$$\ln e^{0.0675t} = \ln 3 \quad \text{Take natural log of each side.}$$
$$0.0675t = \ln 3 \quad \text{Inverse Property}$$
$$t = \frac{1}{0.0675} \ln 3 \quad \text{Divide each side by 0.0675.}$$
$$t \approx 16.28 \quad \text{Use a calculator.}$$

The balance in the account will triple after approximately 16.28 years.

Notice that it took 10.27 years to earn the first $500 and only 6.01 years to earn the second $500. This result is graphically demonstrated in Figure 4.19.

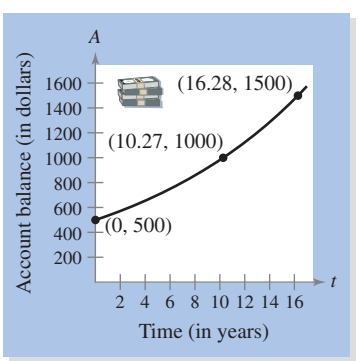


FIGURE 4.19

✓CHECKPOINT 10

In Example 10, how long will it take for the account balance to reach $600? ■

Example 11 Bone Graft Procedures

From 1998 to 2005, the numbers of bone graft procedures y (in thousands) performed in the United States can be approximated by

$$y = 144.32e^{0.164t}$$

where t represents the year, with $t = 8$ corresponding to 1998 (see Figure 4.20). Use the model to estimate the year in which the number of bone graft procedures reached about 880,000. *(Source: U.S. Department of Health and Human Services)*

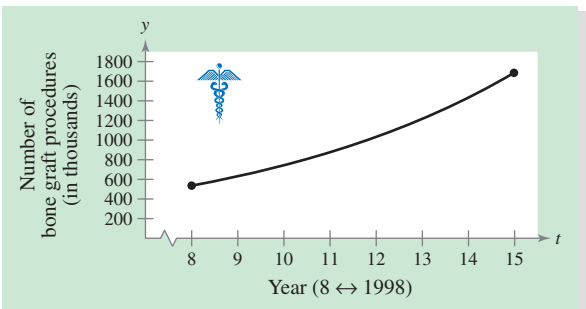


FIGURE 4.20

SOLUTION

$144.32e^{0.164t} = y$	Write original model.
$144.32e^{0.164t} = 880$	Substitute 880 for y.
$e^{0.164t} \approx 6.098$	Divide each side by 144.32.
$\ln e^{0.164t} \approx \ln 6.098$	Take natural log of each side.
$0.164t \approx 1.808$	Inverse Property.
$t \approx 11$	Divide each side by 0.164.

The solution is $t \approx 11$. Because $t = 8$ represents 1998, it follows that there were about 880,000 bone graft procedures performed in 2001.

✓CHECKPOINT 11

Use the model in Example 11 to estimate the year in which the number of bone graft procedures performed in the United States reached about 1,440,000. ■

CONCEPT CHECK

1. What property would you use to solve the equation $3^x = 81$? Explain.
2. What strategy would you use for solving an equation of the form $\log_a x = b$?
3. Describe and correct the error in solving the equation.

$$\log_8 x = 10$$
$$8^{\log_8 x} = 10^8$$
$$x = 100{,}000{,}000$$

4. Discuss the steps you would take to solve the equation $\ln(x + 1) - \ln x = 100$.

Skills Review 4.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.4, 3.5, and 4.2.

In Exercises 1–6, solve for x.

1. $x \ln 2 = \ln 3$
2. $(x - 1) \ln 4 = 2$
3. $2xe^2 = e^3$
4. $4xe^{-1} = 8$
5. $x^2 - 4x + 5 = 0$
6. $2x^2 - 3x + 1 = 0$

In Exercises 7–10, simplify the expression.

7. $\log_{10} 10^x$
8. $\log_{10} 10^{2x}$
9. $\ln e^{2x}$
10. $\ln e^{-x^2}$

Exercises 4.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, solve for x.

1. $5^x = 125$
2. $2^x = 64$
3. $7^x = \frac{1}{49}$
4. $4^x = \frac{1}{256}$
5. $4^{2x-1} = 64$
6. $3^{x-1} = 27$
7. $\log_4 x = 3$
8. $\log_5 5x = 2$
9. $\log_{10} x = -1$
10. $\ln(2x - 1) = 0$

In Exercises 11–22, apply the Inverse Property of logarithmic or exponential functions to simplify the expression.

11. $\ln e^{x^2}$
12. $\ln e^{2x-1}$
13. $\log_{10} 10^{x^2} + 1$
14. $\log_{10} 10^{2x+3}$
15. $\log_5 5^{x^3} - 7$
16. $\log_8 8^{x^5} + 1$
17. $-8 + e^{\ln x^3}$
18. $-1 + \ln e^{2x}$
19. $10^{\log_{10}(x+5)}$
20. $10^{\log_{10}(x^2+7x+10)}$
21. $2^{\log_2 x^2}$
22. $9^{\log_9(3x+7)}$

In Exercises 23–60, solve the exponential equation algebraically. Approximate the result to three decimal places.

23. $3e^x = 9$
24. $5e^x = 20$
25. $2(3^x) = 16$
26. $3(4^x) = 81$
27. $e^x - 9 = 19$
28. $6^x + 10 = 47$
29. $3^{2x} = 80$
30. $6^{5x} = 3000$
31. $5^{-t/2} = 0.20$
32. $4^{-t/3} = 0.15$
33. $3^{x-1} = 28$
34. $2^{x-3} = 31$
35. $2^{3-x} = 565$
36. $8^{-2-x} = 431$
37. $8(10^{3x}) = 12$
38. $5(10^{x-6}) = 7$
39. $3(5^{x-1}) = 21$
40. $8(3^{6-x}) = 40$
41. $e^{3x} = 12$
42. $e^{2x} = 50$
43. $500e^{-x} = 300$
44. $1000e^{-4x} = 75$
45. $7 - 2e^x = 6$
46. $-14 + 3e^x = 11$
47. $6(2^{3x-1}) - 7 = 9$
48. $8(4^{6-2x}) + 13 = 41$
49. $e^{2x} - 8e^x + 12 = 0$
50. $e^{2x} - 5e^x + 6 = 0$
51. $e^{2x} - 3e^x - 4 = 0$
52. $e^{2x} - 9e^x - 36 = 0$
53. $\dfrac{500}{100 - e^{x/2}} = 20$
54. $\dfrac{400}{1 + e^{-x}} = 350$
55. $\dfrac{3000}{2 + e^{2x}} = 2$
56. $\dfrac{119}{e^{6x} - 14} = 7$
57. $\left(1 + \dfrac{0.065}{365}\right)^{365t} = 4$
58. $\left(1 + \dfrac{0.075}{4}\right)^{4t} = 5$
59. $\left(1 + \dfrac{0.10}{12}\right)^{12t} = 2$
60. $\left(1 + \dfrac{0.0825}{26}\right)^{26t} = 9$

In Exercises 61–90, solve the logarithmic equation algebraically. Approximate the result to three decimal places.

61. $\log_{10} x = 4$
62. $\ln x = 5$
63. $\ln x = -3$
64. $\log_{10} x = -5$
65. $\ln 2x = 2.4$
66. $\ln 4x = 1$
67. $\log_{10} 2x = 7$
68. $\log_{10} 3z = 2$
69. $5 \log_3(x + 1) = 12$
70. $5 \log_{10}(x - 2) = 11$
71. $3 \ln 5x = 10$
72. $2 \ln x = 7$
73. $\ln \sqrt{x + 2} = 1$
74. $\ln \sqrt{x - 8} = 5$
75. $7 + 3 \ln x = 5$
76. $2 - 6 \ln x = 10$
77. $\ln x - \ln(x + 1) = 2$
78. $\ln x - \ln(x + 2) = 3$
79. $\ln x + \ln(x - 2) = 1$
80. $\ln x + \ln(x + 3) = 1$
81. $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$

82. $\ln(x + 1) - \ln(x - 2) = \ln x$

83. $\log_2(2x - 3) = \log_2(x + 4)$

84. $\log_3(x + 8) = \log_3(3x + 2)$

85. $\log_{10}(x + 4) - \log_{10} x = \log_{10}(x + 2)$

86. $\log_{10} x + \log_{10}(x + 1) = \log_{10}(x + 3)$

87. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$

88. $\log_3 x + \log_3(x - 8) = 2$

89. $\log_{10} 8x - \log_{10}(1 + \sqrt{x}) = 2$

90. $\log_{10} 4x - \log_{10}(12 + \sqrt{x}) = 2$

In Exercises 91–94, solve for y in terms of x.

91. $\ln y = \ln(2x + 1) + \ln 1$

92. $\ln y = 2 \ln x + \ln(x - 3)$

93. $\log_{10} y = 2 \log_{10}(x - 1) - \log_{10}(x + 2)$

94. $\log_{10}(y - 4) + \log_{10} x = 3 \log_{10} x$

(T) In Exercises 95–98, use a graphing utility to solve the equation. Approximate the result to three decimal places. Verify your result algebraically.

95. $2^x - 7 = 0$

96. $500 - 1500e^{-x/2} = 0$

97. $3 - \ln x = 0$

98. $10 - 4 \ln(x - 2) = 0$

Compound Interest In Exercises 99 and 100, find the time required for a \$1000 investment to double at interest rate r, compounded continuously.

99. $r = 0.0625$

100. $r = 0.085$

Compound Interest In Exercises 101 and 102, find the time required for a \$1000 investment to triple at interest rate r, compounded continuously.

101. $r = 0.0725$

102. $r = 0.0875$

103. Suburban Wildlife The number V of varieties of suburban nondomesticated wildlife in a community is approximated by the model

$$V = 15 \cdot 10^{0.02x}, \quad 0 \le x \le 36$$

where x is the number of months since the development of the community was completed. Use this model to approximate the number of months since the development was completed when $V = 50$.

104. Native Prairie Grasses The number A of varieties of native prairie grasses per acre within a farming region is approximated by the model

$$A = 10.5 \cdot 10^{0.04x}, \quad 0 \le x \le 24$$

where x is the number of months since the farming region was plowed. Use this model to approximate the number of months since the region was plowed using a test acre for which $A = 70$.

105. Demand Function The demand function for a special limited edition coin set is given by

$$p = 1000\left(1 - \frac{5}{5 + e^{-0.001x}}\right).$$

(a) Find the demand x for a price of $p = \$139.50$.

(b) Find the demand x for a price of $p = \$99.99$.

(T) (c) Use a graphing utility to confirm graphically the results found in parts (a) and (b).

106. Demand Function The demand function for a hot tub spa is given by

$$p = 105{,}000\left(1 - \frac{3}{3 + e^{-0.002x}}\right).$$

(a) Find the demand x for a price of $p = \$25{,}000$.

(b) Find the demand x for a price of $p = \$21{,}000$.

(T) (c) Use a graphing utility to confirm graphically the results found in parts (a) and (b).

(T) **107. Forest Yield** The yield V (in millions of cubic feet per acre) for a forest at age t years is given by

$$V = 6.7e^{-48.1/t}, \quad t > 0.$$

(a) Use a graphing utility to find the time necessary to obtain a yield of 1.3 million cubic feet per acre.

(b) Use a graphing utility to find the time necessary to obtain a yield of 2 million cubic feet per acre.

108. Human Memory Model In a group project on learning theory, a mathematical model for the percent P (in decimal form) of correct responses after n trials was found to be

$$P = \frac{0.98}{1 + e^{-0.3n}}, \quad n \ge 0.$$

(a) After how many trials will 80% of the responses be correct? (That is, for what value of n will $P = 0.8$?)

(T) (b) Use a graphing utility to graph the memory model and confirm the result found in part (a).

(c) Write a paragraph describing the memory model.

(T) **109. U.S. Currency** The value y (in billions of dollars) of U.S. currency in circulation (outside the U.S. Treasury and not held by banks) from 1996 to 2005 can be approximated by the model

$$y = -302 + 374 \ln t, \quad 6 \le t \le 15$$

where t represents the year, with $t = 6$ corresponding to 1996. *(Source: Board of Governors of the Federal Reserve System)*

(a) Use a graphing utility to graph the model.

(b) Use a graphing utility to estimate the year when the value of U.S. currency in circulation exceeded \$600 billion.

(c) Verify your answer to part (b) algebraically.

110. Retail Trade The average monthly sales y (in billions of dollars) in retail trade in the United States from 1996 to 2005 can be approximated by the model

$$y = -22 + 117 \ln t, \quad 6 \le t \le 15$$

where t represents the year, with $t = 6$ corresponding to 1996. *(Source: U.S. Council of Economic Advisors)*

(a) Use a graphing utility to graph the model.

(b) Use a graphing utility to estimate the year in which the average monthly sales first exceeded \$270 billion.

(c) Verify your answer to part (b) algebraically.

111. *MAKE A DECISION: AUTOMOBILES* Automobiles are designed with crumple zones that help protect their occupants in crashes. The crumple zones allow the occupants to move short distances when the automobiles come to abrupt stops. The greater the distance moved, the fewer g's the crash victims experience. (One g is equal to the acceleration due to gravity. For very short periods of time, humans have withstood as much as 40 g's.) In crash tests with vehicles moving at 90 kilometers per hour, analysts measured the numbers of g's experienced during deceleration by crash dummies that were permitted to move x meters during impact. The data are shown in the table.

x	0.2	0.4	0.6	0.8	1.0
g's	158	80	53	40	32

A model for these data is given by

$$y = -3.00 + 11.88 \ln x + \frac{36.94}{x}$$

where y is the number of g's.

(a) Complete the table using the model.

x	0.2	0.4	0.6	0.8	1.0
y					

(b) Use a graphing utility to graph the data points and the model in the same viewing window. How do they compare?

(c) Use the model to estimate the least distance traveled during impact for which the passenger does not experience more than 30 g's.

(d) Do you think it is practical to lower the number of g's experienced during impact to fewer than 23? Explain your reasoning.

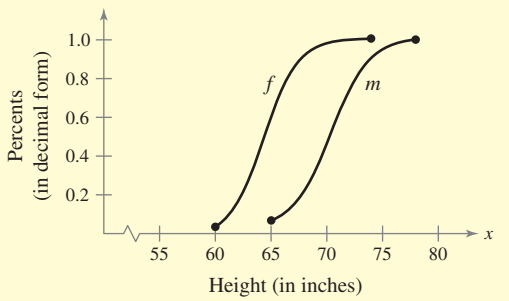
112. Average Heights The percent m of American males (between 20 and 29 years old) who are less than x inches tall is approximated by

$$m = 0.027 + \frac{0.986}{1 + e^{-0.5857(x - 70.38)}}, \quad 65 \le x \le 78$$

and the percent f of American females (between 20 and 29 years old) who are less than x inches tall is approximated by

$$f = -0.023 + \frac{1.031}{1 + e^{-0.6500(x - 64.34)}}, \quad 60 \le x \le 74$$

where m and f are the percents (in decimal form) and x is the height (in inches) (see figure). *(Source: U.S. National Center for Health Statistics)*



(a) What is the median height for each sex between 20 and 29 years old? (In other words, for what values of x are m and f equal to 0.5?)

(b) Write a paragraph describing each height model.

In Exercises 113–116, rewrite each verbal statement as an equation. Then decide whether the statement is true or false. Justify your answer.

113. The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

114. The logarithm of the sum of two numbers is equal to the product of the logarithms of the numbers.

115. The logarithm of the difference of two numbers is equal to the difference of the logarithms of the numbers.

116. The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

117. Think About It Is it possible for a logarithmic equation to have more than one extraneous solution? Explain.

118. Think About It Are the times required for the investments in Exercises 99 and 100 to quadruple twice as long as the times for them to double? Give a reason for your answer and verify your answer algebraically.

Section 4.5

Exponential and Logarithmic Models

- Construct and use a model for exponential growth or exponential decay.
- Use a Gaussian model to solve an application problem.
- Use a logistic growth model to solve an application problem.
- Use a logarithmic model to solve an application problem.
- Choose an appropriate model involving exponential or logarithmic functions for a real-life situation.

Introduction

The five most common types of mathematical models involving exponential functions and logarithmic functions are as follows.

1. Exponential growth model: $y = ae^{bx}, \quad b > 0$

2. Exponential decay model: $y = ae^{-bx}, \quad b > 0$

3. Gaussian model: $y = ae^{-(x-b)^2/c}$

4. Logistic growth model: $y = \dfrac{a}{1 + be^{-rx}}$

5. Logarithmic models: $y = a + b \ln x, \quad y = a + b \log_{10} x$

The basic shape of each of these graphs is shown in Figure 4.21.

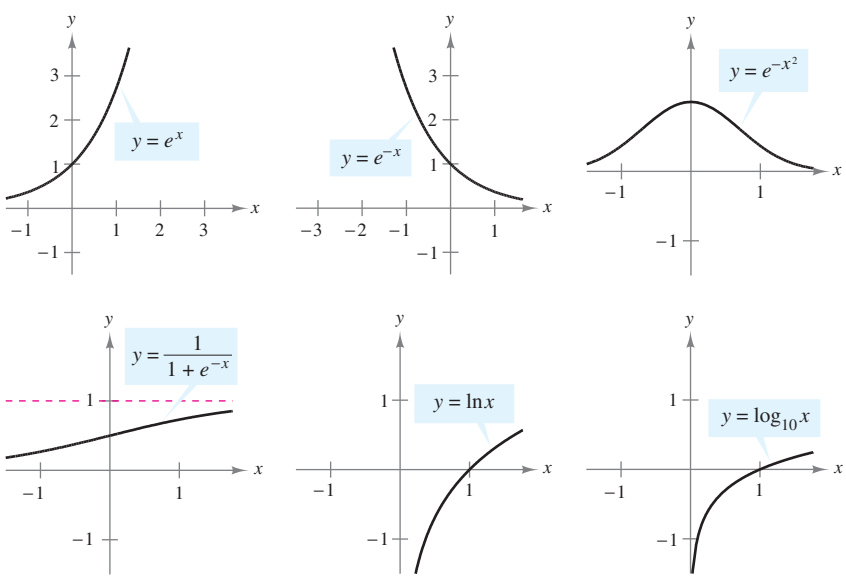


FIGURE 4.21

You can often gain quite a bit of insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the function's asymptotes. Use the graphs in Figure 4.21 to identify the asymptote(s) of the graph of each function.

Exponential Growth and Decay

Example 1 Population Increase

The world populations (in millions) for each year from 1996 through 2005 are shown in the table. *(Source: U.S. Census Bureau)*

Year	1996	1997	1998	1999	2000
Population	5763	5842	5920	5997	6073

Year	2001	2002	2003	2004	2005
Population	6149	6224	6299	6375	6451

An exponential growth model that approximates these data is

$$P = 5356e^{0.012469t}, \quad 6 \le t \le 15$$

where P is the population (in millions) and $t = 6$ represents 1996. Compare the estimates given by the model with the values given by the U.S. Census Bureau. Use the model to predict the year in which the world population reaches 6.7 billion.

SOLUTION The following table compares the two sets of population figures. The graph of the model and the original data values are shown in Figure 4.22.

Year	1996	1997	1998	1999	2000	2001
Population	5763	5842	5920	5997	6073	6149
Model	5772	5845	5918	5992	6067	6143

Year	2002	2003	2004	2005
Population	6224	6299	6375	6451
Model	6221	6299	6378	6458

To find the year in which the world population reaches 6.7 billion, let $P = 6700$ in the model and solve for t.

$$5356e^{0.012469t} = P \qquad \text{Write original model.}$$
$$5356e^{0.012469t} = 6700 \qquad \text{Let } P = 6700.$$
$$e^{0.012469t} \approx 1.25093 \qquad \text{Divide each side by 5356.}$$
$$\ln e^{0.012469t} \approx \ln 1.25093 \qquad \text{Take natural log of each side.}$$
$$0.012469t \approx 0.223890 \qquad \text{Inverse Property}$$
$$t \approx 18.0 \qquad \text{Divide each side by 0.012469.}$$

According to the model, the world population reaches 6.7 billion in 2008.

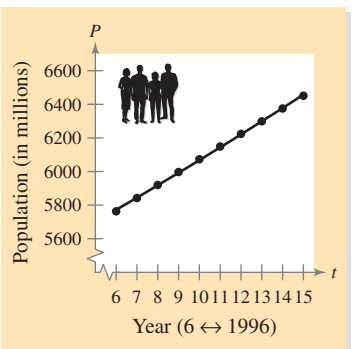


FIGURE 4.22

TECHNOLOGY

Some graphing utilities have an *exponential regression* feature that can be used to find an exponential model that represents data. If you have such a graphing utility, try using it to find a model for the data given in Example 1. How does your model compare with the model given in Example 1? For instructions on how to use the *regression* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

✓CHECKPOINT 1

Use the model in Example 1 to predict the year in which the world population will reach 7.3 billion. ■

The exponential model in Example 1 increases (or decreases) by the same percent each year. What is the annual percent increase for this exponential model?

In Example 1, you were given the exponential growth model. But suppose this model were not given; how could you find such a model? If you are given a set of data, as in Example 1, but you are not given the exponential growth model that fits the data, you can choose any two of the points and substitute them in the general exponential growth model $y = ae^{bx}$. This technique is demonstrated in Example 2.

Example 2 Finding an Exponential Growth Model

Find an exponential growth model whose graph passes through the points $(0, 4453)$ and $(7, 5024)$, as shown in Figure 4.23(a).

SOLUTION The general form of the model is

$$y = ae^{bx}.$$

From the fact that the graph passes through the point $(0, 4453)$, you know that $y = 4453$ when $x = 0$. By substituting these values into the general model, you have

$$4453 = ae^0 \quad \Rightarrow \quad a = 4453.$$

In a similar way, from the fact that the graph passes through the point $(7, 5024)$, you know that $y = 5024$ when $x = 7$. By substituting these values into the model, you obtain

$$5024 = 4453e^{7b} \quad \Rightarrow \quad b = \frac{1}{7}\ln\frac{5024}{4453} \approx 0.01724.$$

So, the exponential growth model is

$$y = 4453e^{0.01724x}.$$

The graph of the model is shown in Figure 4.23(b).

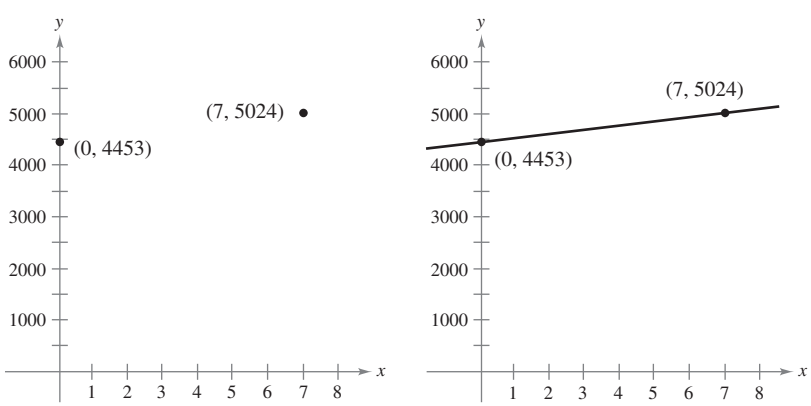


FIGURE 4.23

✓CHECKPOINT 2

Find an exponential growth model whose graph passes through the points $(0, 3)$ and $(5, 8)$. ■

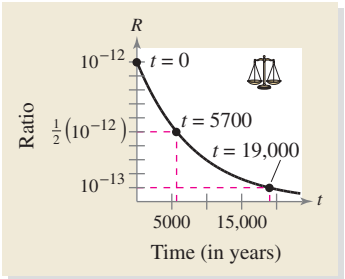


FIGURE 4.24

In living organic material, the ratio of the number of radioactive carbon isotopes (carbon 14) to the number of nonradioactive carbon isotopes (carbon 12) is about 1 to 10^{12}. When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of about 5700 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time t (in years).

$$R = \frac{1}{10^{12}} e^{-t/8223}$$ Carbon dating model

In Figure 4.24, note that R decreases as the time t increases. Any material that is composed of carbon, such as wood, bone, hair, pottery, paper, and water, can be dated.



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In 1960, Willard Libby of the University of Chicago won the Nobel Prize for Chemistry for the carbon 14 method, a valuable tool for estimating the ages of ancient materials.

✓CHECKPOINT 3

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{9^{13}}.$$

Estimate the age of the fossil. ■

Example 3 Carbon Dating ®

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{10^{13}}.$$

Estimate the age of the fossil.

SOLUTION In the carbon dating model, substitute the given value of R to obtain the following.

$$\frac{1}{10^{12}} e^{-t/8223} = R$$ Write original model.

$$\frac{e^{-t/8223}}{10^{12}} = \frac{1}{10^{13}}$$ Substitute $\frac{1}{10^{13}}$ for R.

$$e^{-t/8223} = \frac{1}{10}$$ Multiply each side by 10^{12}.

$$\ln e^{-t/8223} = \ln \frac{1}{10}$$ Take natural log of each side.

$$-\frac{t}{8223} \approx -2.3026$$ Inverse Property

$$t \approx 18{,}934$$ Multiply each side by -8223.

So, you can estimate the age of the fossil to be about 19,000 years.

An exponential model can be used to determine the *decay* of radioactive isotopes. For instance, to find how much of an initial 10 grams of radioactive radium (^{226}Ra) with a half-life of 1599 years is left after 500 years, you would use the exponential decay model, as follows.

$$y = ae^{-bt} \Rightarrow \frac{1}{2}(10) = 10e^{-b(1599)} \Rightarrow \ln\frac{1}{2} = -1599b \Rightarrow -\frac{\ln\frac{1}{2}}{1599} = b$$

Using the value of b found above, $a = 10$, and $t = 500$, the amount left is

$$y = 10e^{-[-\ln(1/2)/1599](500)} \approx 8.05 \text{ grams}$$

Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed.** For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The graph of a Gaussian model is called a **bell-shaped curve.** Try to sketch the standard normal distribution curve with a graphing utility. Can you see why it is called a bell-shaped curve?

The **average value** of a population can be found from the bell-shaped curve by observing where the maximum y-value of the function occurs. The x-value corresponding to the maximum y-value of the function represents the average value of the independent variable, x.

Example 4 SAT Scores

In 2006, the SAT (Scholastic Aptitude Test) mathematics scores for college-bound seniors in the United States roughly followed a normal distribution given by

$$y = 0.0035e^{-(x-518)^2/26{,}450}, \quad 200 \le x \le 800$$

where x is the SAT score for mathematics. Sketch the graph of this function. From the graph, estimate the average SAT score. *(Source: College Board)*

SOLUTION The graph of the function is shown in Figure 4.25. On this bell-shaped curve, the x-value corresponding to the maximum value of the curve represents the average score. From the graph, you can estimate that the average mathematics score for college-bound seniors in 2006 was 518.

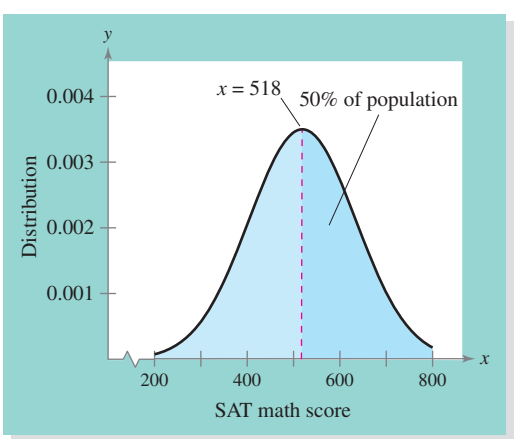


FIGURE 4.25

✓CHECKPOINT 4

In 2006, the SAT reading scores for college-bound seniors in the United States roughly followed a normal distribution given by

$$y = 0.0035e^{-(x-503)^2/25{,}538}, \quad 200 \le x \le 800$$

where x is the SAT score for reading. Sketch the graph of this function. From the graph, estimate the average SAT score. *(Source: College Board)* ■

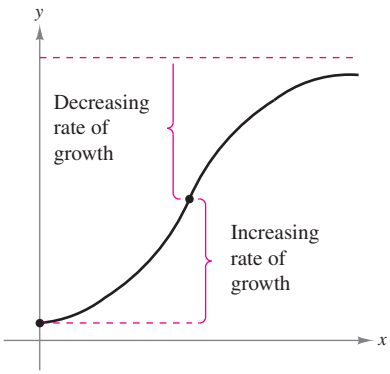


FIGURE 4.26 Logistic Curve

Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as shown by the graph in Figure 4.26. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where y is the population size and x is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions, followed by less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**

Example 5 Spread of a Virus

On a college campus of 5000 students, one student returned from vacation with a contagious flu virus. The spread of the virus through the student population is given by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

where y is the total number of students infected after t days. The college will cancel classes when 40% or more of the students become infected.

a. How many students are infected after 5 days?

b. After how many days will the college cancel classes?

SOLUTION

a. After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}} = \frac{5000}{1 + 4999e^{-4}} \approx 54.$$

b. Classes are cancelled when the number infected is $(0.40)(5000) = 2000$. So, solve for t in the following equation.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} = \frac{1.5}{4999}$$

$$\ln e^{-0.8t} = \ln \frac{1.5}{4999}$$

$$-0.8t = \ln \frac{1.5}{4999}$$

$$t \approx 10.1$$

So, after 10 days, at least 40% of the students will become infected, and the college will cancel classes. The graph of the function is shown in Figure 4.27.

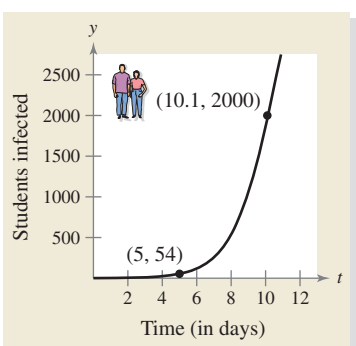


FIGURE 4.27

✓CHECKPOINT 5

In Example 5, how many days does it take for 25% of the students to become infected? ■



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The severity of the destruction caused by an earthquake depends on its magnitude and duration. Earthquakes can destroy buildings, and can cause landslides and tsunamis.

Logarithmic Models

Example 6 Magnitudes of Earthquakes

On the Richter scale, the magnitude R of an earthquake of intensity I per unit of area is given by

$$R = \log_{10}\left(\frac{I}{I_0}\right)$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensity per unit of area for each earthquake. (Intensity is a measure of the wave energy of an earthquake.)

a. Prince William Sound, Alaska, in 1964; $R = 9.2$

b. Off the coast of Northern California in 2005; $R = 7.2$

SOLUTION

a. Because $I_0 = 1$ and $R = 9.2$,

$$9.2 = \log_{10} I$$
$$I = 10^{9.2} \approx 1{,}584{,}893{,}000.$$

b. For $R = 7.2$,

$$7.2 = \log_{10} I$$
$$I = 10^{7.2} \approx 15{,}849{,}000$$

Note that an increase of 2.0 units on the Richter scale (from 7.2 to 9.2) represents an intensity change by a factor of

$$\frac{1{,}584{,}893{,}000}{15{,}849{,}000} \approx 100.$$

In other words, the Prince William Sound earthquake in 1964 had a magnitude about 100 times greater than that of the earthquake off the coast of Northern California in 2005.

✓CHECKPOINT 6

Find the intensity I per unit of area of an earthquake measuring $R = 6.4$ on the Richter scale. (Let $I_0 = 1$.) ■

Example 7 pH Levels

Acidity, or pH level, is a measure of the hydrogen ion concentration $[H^+]$ (measured in moles of hydrogen per liter) of a solution. Use the model given by

$$\text{pH} = -\log_{10}[H^+]$$

to determine the hydrogen ion concentration of milk of magnesia, which has a pH of 10.5.

SOLUTION

$$\text{pH} = -\log_{10}[H^+] \quad \text{Write original model.}$$
$$10.5 = -\log_{10}[H^+] \quad \text{Substitute 10.5 for pH.}$$
$$-10.5 = \log_{10}[H^+] \quad \text{Multiply each side by } -1.$$
$$10^{-10.5} = 10^{\log_{10}[H^+]} \quad \text{Exponentiate each side (base 10).}$$
$$3.16 \times 10^{-11} = [H^+] \quad \text{Simplify.}$$

So, the hydrogen ion concentration of milk of magnesia is 3.16×10^{-11} mole of hydrogen per liter.

✓CHECKPOINT 7

Use the model in Example 7 to determine the hydrogen ion concentration of coffee, which has a pH of 5.0 ■

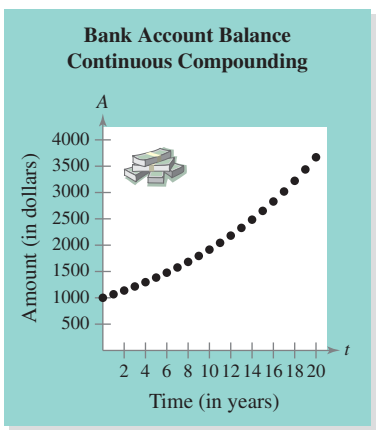
Comparing Models

So far you have been given the type of model to use for a data set. Now you will use the general trends of the graphs of the five models presented in this section to choose appropriate models for real-life situations.

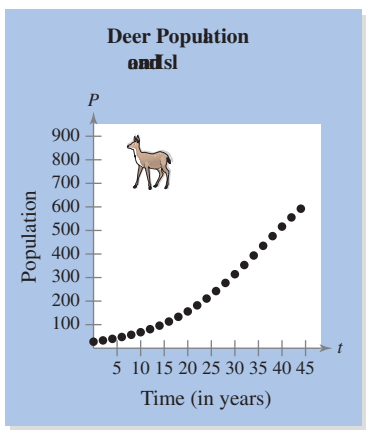
Example 8
MAKE A DECISION
Choosing an Appropriate Model

Decide whether to use an exponential growth model or a logistic growth model to represent each data set.

a.



b.



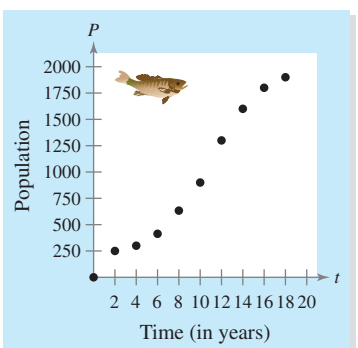
SOLUTION

a. As long as withdrawals and deposits are not made and the interest rate remains constant, the bank account balance will grow exponentially. So, an exponential growth model is an appropriate model.

b. The growth of the deer population will slow as the population approaches the carrying capacity of the island. So, a logistic growth model is an appropriate model.

✓CHECKPOINT 8

Decide whether to use an exponential growth model or a logistic growth model to represent the data set for the fish population of a lake in the figure below.



CONCEPT CHECK

1. What type of model is the function $y = 8e^{-0.5x^2}$?
2. Does a Gaussian model generally represent population growth well? Explain your reasoning.
3. Explain why the growth of a population of bacteria in a petri dish can be modeled by a logistic growth model.
4. Is it possible for the graph of an exponential decay model to pass through the points (0, 220) and (−4, 736)? Explain.

Skills Review 4.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 4.1, 4.2, and 4.4.

In Exercises 1–6, sketch the graph of the equation.

1. $y = e^{0.1x}$ **2.** $y = e^{-0.25x}$ **3.** $y = e^{-x^2/5}$

4. $y = \dfrac{2}{1 + e^{-x}}$ **5.** $y = \log_{10} 2x$ **6.** $y = \ln 4x$

In Exercises 7 and 8, solve the equation algebraically.

7. $3e^{2x} = 7$ **8.** $4 \ln 5x = 14$

In Exercises 9 and 10, solve the equation graphically.

9. $2e^{-0.2x} = 0.002$ **10.** $6 \ln 2x = 12$

Exercises 4.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Compound Interest In Exercises 1–10, complete the table for a savings account in which interest is compounded continuously.

	Initial Investment	*Annual % Rate*	*Time to Double*	*Amount After 10 Years*
1.	\$5000	7%		
2.	\$1000	$9\frac{1}{4}$%		
3.	\$500		10 yr	
4.	\$10,000		5 yr	
5.	\$1000			\$2281.88
6.	\$2000			\$3000
7.		11%		\$19,205
8.		8%		\$20,000
9.	\$5000			\$11,127.70
10.	\$250			\$600

Radioactive Decay In Exercises 11–16, complete the table for the radioactive isotope.

	Isotope	*Half-Life (Years)*	*Initial Quantity*	*Amount After 1000 Years*
11.	^{226}Ra	1599	4 g	
12.	^{226}Ra	1599		0.15 g
13.	^{14}C	5715		3.5 g
14.	^{14}C	5715	8 g	
15.	^{239}Pu	24,100		1.6 g
16.	^{239}Pu	24,100		0.38 g

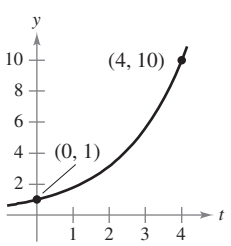
In Exercises 17–20, classify the model as an exponential growth model or an exponential decay model.

17. $y = 3e^{0.5t}$ **18.** $y = 2e^{-0.6t}$

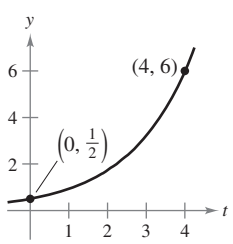
19. $y = 20e^{-1.5t}$ **20.** $y = 4e^{0.07t}$

In Exercises 21–24, find the constants C and k such that the exponential function $y = Ce^{kt}$ passes through the points on the graph.

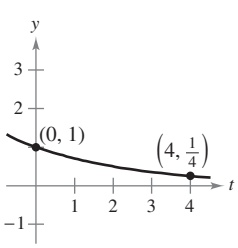
21.



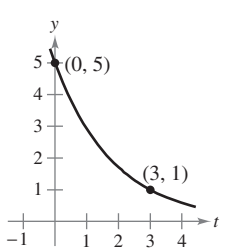
22.



23.



24.



25. Population The population P of a city is given by

$$P = 120{,}000e^{0.016t}$$

where t represents the year, with $t = 0$ corresponding to 2000. Sketch the graph of this equation. Use the model to predict the year in which the population of the city will reach about 180,000.

26. Population The population P of a city is given by

$$P = 25{,}000e^{kt}$$

where t represents the year, with $t = 0$ corresponding to 2000. In 1980, the population was 15,000. Find the value of k and use this result to predict the population in the year 2010.

27. Bacteria Growth The number N of bacteria in a culture is given by the model $N = 100e^{kt}$, where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 100$. When $t = 6$, there are 140 bacteria. How long does it take the bacteria population to double in size? To triple in size?

28. Bacteria Growth The number N of bacteria in a culture is given by the model $N = 250e^{kt}$, where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 250$. When $t = 10$, there are 320 bacteria. How long does it take the bacteria population to double in size? To triple in size?

29. Carbon Dating The ratio of carbon 14 to carbon 12 in a piece of wood discovered in a cave is $R = 1/8^{14}$. Estimate the age of the piece of wood.

30. Carbon Dating The ratio of carbon 14 to carbon 12 in a piece of paper buried in a tomb is $R = 1/13^{11}$. Estimate the age of the piece of paper.

31. Radioactive Decay What percent of a present amount of radioactive cesium (^{137}Cs) will remain after 100 years? Use the fact that radioactive cesium has a half-life of 30 years.

32. Radioactive Decay Find the half-life of radioactive iodine (^{131}I) if, after 20 days, 0.53 kilogram of an initial 3 kilograms remains.

33. Learning Curve The management at a factory has found that the maximum number of units a worker can produce in a day is 40. The learning curve for the number of units N produced per day after a new employee has worked t days is given by

$$N = 40(1 - e^{kt}).$$

After 20 days on the job, a particular worker produced 25 units in 1 day.

(a) Find the learning curve for this worker (first find the value of k).

(b) How many days should pass before this worker is producing 35 units per day?

34. Learning Curve The management at a customer service center has found that the maximum number of customer calls an employee can process effectively in a day is 90. The learning curve for the number N of calls processed per day after a new employee has worked t days is given by

$$N = 90(1 - e^{kt}).$$

After 15 days on the job, a particular employee processed 60 calls in 1 day.

(a) Find the learning curve for this worker (first find the value of k).

(b) How many days should pass before this employee will process 80 calls per day?

35. Motorola The sales per share S (in dollars) for Motorola from 1992 to 2005 can be approximated by the function

$$S = \begin{cases} 2.33 - 0.909t + 10.394 \ln t, & 2 \le t \le 10 \\ 0.6157t^2 - 15.597t + 110.25, & 11 \le t \le 15 \end{cases}$$

where t represents the year, with $t = 2$ corresponding to 1992. *(Source: Motorola)*

(a) Use a graphing utility to graph the function.

(b) Describe the change in sales per share that occurred in 2001.

36. Intel The sales per share S (in dollars) for Intel from 1992 to 2005 can be approximated by the function

$$S = \begin{cases} -1.48 + 2.65 \ln t, & 2 \le t \le 10 \\ 0.1586t^2 - 3.465t + 22.87, & 11 \le t \le 15 \end{cases}$$

where t represents the year, with $t = 2$ corresponding to 1992. *(Source: Intel)*

(a) Use a graphing utility to graph the function.

(b) Describe the change in sales per share that occurred in 2001.

37. Women's Heights The distribution of heights of American women (between 30 and 39 years of age) can be approximated by the function

$$p = 0.163e^{-(x - 64.9)^2/12.03}, \quad 60 \le x \le 74$$

where x is the height (in inches) and p is the percent (in decimal form). Use a graphing utility to graph the function. Then determine the average height of women in this age bracket. *(Source: U.S. National Center for Health Statistics)*

38. Men's Heights The distribution of heights of American men (between 30 and 39 years of age) can be approximated by the function

$$p = 0.131e^{-(x - 69.9)^2/18.66}, \quad 63 \le x \le 77$$

where x is the height (in inches) and p is the percent (in decimal form). Use a graphing utility to graph the function. Then determine the average height of men in this age bracket. *(Source: U.S. National Center for Health Statistics)*

39. Stocking a Lake with Fish A lake is stocked with 500 fish, and the fish population P increases according to the logistic curve

$$P = \frac{10{,}000}{1 + 19e^{-t/5}}, \quad t \ge 0$$

where t is the time (in months).

(a) Use a graphing utility to graph the logistic curve.

(b) Find the fish population after 5 months.

(c) After how many months will the fish population reach 2000?

40. Endangered Species A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1000 animals and that the growth of the herd will be modeled by the logistic curve

$$p = \frac{1000}{1 + 9e^{-kt}}, \quad t \geq 0$$

where p is the number of animals and t is the time (in years). The herd size is 134 after 2 years. Find k. Then find the population after 5 years.

41. Aged Population The table shows the projected U.S. populations P (in thousands) of people who are 85 years old or older for several years from 2010 to 2050. *(Source: U.S. Census Bureau)*

Year	85 years and older
2010	6123
2015	6822
2020	7269
2025	8011
2030	9603
2035	12,430
2040	15,409
2045	18,498
2050	20,861

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 10$ corresponding to 2010.

(b) Use the *regression* feature of a graphing utility to find an exponential model for the data. Use the Inverse Property

$$b = e^{\ln b}$$

to rewrite the model as an exponential model in base e.

(c) Use a graphing utility to graph the exponential model in base e.

(d) Use the exponential model in base e to estimate the populations of people who are 85 years old or older in 2022 and in 2042.

42. Super Bowl Ad Cost The table shows the costs C (in millions of dollars) of a 30-second TV ad during the Super Bowl for several years from 1987 to 2006. *(Source: TNS Media Intelligence)*

Year	Cost
1987	0.6
1992	0.9
1997	1.2
2002	2.2
2006	2.5

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 1987.

(b) Use the *regression* feature of a graphing utility to find an exponential model for the data. Use the Inverse Property $b = e^{\ln b}$ to rewrite the model as an exponential model in base e.

(c) Use a graphing utility to graph the exponential model in base e.

(d) Use the exponential model in base e to predict the costs of a 30-second ad during the Super Bowl in 2009 and in 2010.

43. Super Bowl Ad Revenue The table shows Super Bowl TV ad revenues R (in millions of dollars) for several years from 1987 to 2006. *(Source: TNS Media Intelligence)*

Year	Revenue
1987	31.5
1992	48.2
1997	72.2
2002	134.2
2006	162.5

(a) Use a spreadsheet software program to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 1987.

(b) Use the *regression* feature of a spreadsheet software program to find an exponential model for the data. Use the Inverse Property $b = e^{\ln b}$ to rewrite the model as an exponential model in base e.

(c) Use a spreadsheet software program to graph the exponential model in base e.

(d) Use the exponential model in base e to predict the Super Bowl ad revenues in 2009 and in 2010.

44. Domestic Demand The domestic demands D (in thousands of barrels) for refined oil products in the United States from 1995 to 2005 are shown in the table. *(Source: U.S. Energy Information Administration)*

Year	Demand
1995	6,469,625
1996	6,701,094
1997	6,796,300
1998	6,904,705
1999	7,124,435
2000	7,210,566

Year	Demand
2001	7,171,885
2002	7,212,765
2003	7,312,410
2004	7,587,546
2005	7,539,440

(a) Use a spreadsheet software program to create a scatter plot of the data. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use the *regression* feature of a spreadsheet software program to find an exponential model for the data. Use the Inverse Property $b = e^{\ln b}$ to rewrite the model as an exponential model in base e.

(c) Use the *regression* feature of a spreadsheet software program to find a logarithmic model $(y = a + b \ln x)$ for the data.

(d) Use a spreadsheet software program to graph the exponential model in base e and the logarithmic model with the scatter plot.

(e) Use both models to predict domestic demands in 2008, 2009, and 2010. Do both models give reasonable predictions? Explain.

45. Population The populations P of the United States (in thousands) from 1990 to 2005 are shown in the table. *(Source: U.S. Census Bureau)*

Year	Population
1990	250,132
1991	253,493
1992	256,894
1993	260,255
1994	263,436
1995	266,557
1996	269,667
1997	272,912

Year	Population
1998	276,115
1999	279,295
2000	282,403
2001	285,335
2002	288,216
2003	291,089
2004	293,908
2005	296,639

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 1990.

(b) Use the *regression* feature of a graphing utility to find an exponential model for the data. Use the Inverse Property $b = e^{\ln b}$ to rewrite the model as an exponential model in base e.

(c) Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data.

(d) Use a graphing utility to graph the exponential model in base e and the models in part (c) with the scatter plot.

(e) Use each model to predict the populations in 2008, 2009, and 2010. Do all models give reasonable predictions? Explain.

46. Population The population P of the United States officially reached 300 million at about 7:46 A.M. E.S.T. on Tuesday, October 17, 2006. The table shows the U.S. populations (in millions) since 1900. *(Source: U.S. Census Bureau)*

Year	Population
1900	76
1910	92
1920	106
1930	123
1940	132
1950	151

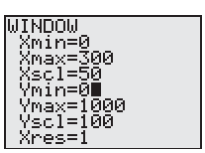
Year	Population
1960	179
1970	203
1980	227
1990	250
2000	282
2006	300

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 1900.

(b) Use the *regression* feature of a graphing utility to find an exponential model for the data. Use the Inverse Property $b = e^{\ln b}$ to rewrite the model as an exponential model in base e.

(c) Graph the exponential model in base e with the scatter plot of the data. What appears to be happening to the relationship between the data points and the regression curve at $t = 100$ and $t = 106$?

(d) Use the *regression* feature of a graphing utility to find a logistic growth model for the data. Graph each model using the window settings shown below. Which model do you think will give more accurate predictions of the population well beyond 2006?



(e) The U.S. Census Bureau predicts that the U.S. population will be about 420 million in 2050. Use each model to predict the population in 2050. Which model gives an estimate closer to the prediction of 420 million?

Earthquake Magnitudes In Exercises 47 and 48, use the Richter scale (see Example 6) for measuring the magnitudes of earthquakes.

47. Find the magnitude R (on the Richter scale) of an earthquake of intensity I. (Let $I_0 = 1$.)

(a) $I = 80{,}500{,}000$ (b) $I = 48{,}275{,}000$

48. Find the intensity I of an earthquake measuring R on the Richter scale. (Let $I_0 = 1$.)

(a) Vanuatu Islands in 2002, $R = 7.3$

(b) Near coast of Peru in 2001, $R = 8.4$

Intensity of Sound In Exercises 49 and 50, find the level of sound using the following information for determining sound intensity. The level of sound L (in decibels) of a sound with an intensity of I is given by

$$L = 10 \log_{10} \frac{I}{I_0}$$

where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear.

49. (a) $I = 10^{-10}$ watt per square meter (quiet room)

(b) $I = 10^{-5}$ watt per square meter (busy street corner)

50. (a) $I = 10^{-3}$ watt per square meter (loud car horn)

(b) $I \approx 10^{0}$ watt per square meter (threshold of pain)

pH Levels In Exercises 51–54, use the acidity model given in Example 7.

51. Compute $[H^+]$ for a solution for which pH $= 5.8$.

52. Compute $[H^+]$ for a solution for which pH $= 7.3$.

53. A grape has a pH of 3.5, and baking soda has a pH of 8.0. The hydrogen ion concentration of the grape is how many times that of the baking soda?

54. The pH of a solution is decreased by one unit. The hydrogen ion concentration is increased by what factor?

55. Estimating the Time of Death At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 A.M. the temperature was 82.8°F. From these two temperature readings, the coroner was able to determine that the time elapsed since death and the body temperature are related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where t is the time (in hours) elapsed since the person died and T is the temperature (in degrees Fahrenheit) of the person's body. The coroner assumed that the person had a normal body temperature of 98.6°F at death, and that the room temperature was a constant 70°F. Use this formula to estimate the time of death of the person.

56. Thawing a Package of Steaks You take a three-pound package of steaks out of the freezer at 11 A.M. and place it in the refrigerator. Will the steaks be thawed in time to be grilled at 6 P.M.? Assume that the refrigerator temperature is 40°F and that the freezer temperature is 0°F. Use the formula for Newton's Law of Cooling

$$t = -5.05 \ln \frac{T - 40}{0 - 40}$$

where t is the time in hours (with $t = 0$ corresponding to 11 A.M.) and T is the temperature of the package of steaks (in degrees Fahrenheit).

57. ***MAKE A DECISION: WORKER'S PRODUCTIVITY*** The numbers n of units per day that a new worker can produce after t days on the job are listed in the table. Use a graphing utility to create a scatter plot of the data. Do the data fit an exponential model or a logarithmic model? Use the *regression* feature of the graphing utility to find the model. Graph the model with the original data. Is the model a good fit? Can you think of a better model to use for these data? Explain.

Days, t	5	10	15	20	25
Units, n	6	13	22	34	56

58. Chemical Reaction The table shows the yield y (in milligrams) of a chemical reaction after x minutes. Use a graphing utility to create a scatter plot of the data. Do the data fit an exponential model or a logarithmic model? Use the [redacted] feature of the graphing utility to find the model. Graph the model with the original data. Is this model a good fit for the data?

Minutes, x	1	2	3	4	5
Yield, y	1.5	7.4	10.2	13.4	15.8

Minutes, x	6	7	8
Yield, y	16.3	18.2	18.3

Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 398.
Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 4.1

Review Exercises

■ Sketch the graph of an exponential function. *1–4, 9–16*

Characteristics of Exponential Functions

Graph of $y = a^x$, $a > 1$	Graph of $y = a^{-x}$, $a > 1$
•Domain: $(-\infty, \infty)$	•Domain: $(-\infty, \infty)$
•Range: $(0, \infty)$	•Range: $(0, \infty)$
•Intercept: $(0, 1)$	•Intercept: $(0, 1)$
•Increasing	•Decreasing
• x-axis is a horizontal asymptote: $(a^x \to 0 \text{ as } x \to -\infty)$	• x-axis is a horizontal asymptote: $(a^{-x} \to 0 \text{ as } x \to \infty)$
•Continuous	•Continuous
	•Reflection of graph of $y = a^x$ about y-axis

■ Use the compound interest formulas. *17–20*

For n compoundings per year: $A = P(1 + r/n)^{nt}$

For continuous compounding: $A = Pe^{rt}$

■ Use an exponential model to solve an application problem. *21, 22*

Section 4.2

■ Recognize and evaluate a logarithmic function. *23–36*

$y = \log_a x$ if and only if $x = a^y$

$y = \log_e x = \ln x$

■ Sketch the graph of a logarithmic function. *5–8, 37–42*

Characteristics of Logarithmic Functions

Graph of $y = \log_a x$, $a > 1$

•Domain: $(0, \infty)$
•Range: $(-\infty, \infty)$
•Intercept: $(1, 0)$
•Increasing
•One-to-one;therefore has an inverse function
• y-axis is a vertical asymptote $(\log_a x \to -\infty \text{ as } x \to 0^+)$
•Continuous
•Reflection of graph of $y = a^x$about the line $y = x$

■ Use a logarithmic model to solve an application problem. *43–46*

*Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

Section 4.3

Review Exercises

- Evaluate a logarithm using the change-of-base formula. *47–50*

$$\log_a x = \frac{\log_b x}{\log_b a}, \quad \log_a x = \frac{\ln x}{\ln a}$$

- Use properties of logarithms to evaluate or rewrite a logarithmic expression. *51–58*

$$\log_a(uv) = \log_a u + \log_a v \qquad \ln(uv) = \ln u + \ln v$$

$$\log_a \frac{u}{v} = \log_a u - \log_a v \qquad \ln \frac{u}{v} = \ln u - \ln v$$

$$\log_a u^n = n \log_a u \qquad \ln u^n = n \ln u$$

- Use properties of logarithms to expand or condense a logarithmic expression. *59–70*
- Use logarithmic functions to model and solve real-life applications. *71, 72*

Section 4.4

- Solve an exponential equation. *73–78*
- Solve a logarithmic equation. *79–86*
- Use an exponential or a logarithmic model to solve an application problem. *87, 88*

Section 4.5

- Construct and use a model for exponential growth or exponential decay. *89–95*

$$y = ae^{bx}, \quad b > 0 \qquad y = ae^{-bx}, \quad b > 0$$

- Use a Gaussian model to solve an application problem. *96*

$$y = ae^{-(x-b)^2/c}$$

- Use a logistic growth model to solve an application problem. *97*

$$y = \frac{a}{1 + be^{-rx}}$$

- Use a logarithmic model to solve an application problem. *98, 99*

$$y = a + b \ln x, \quad y = a + b \log_{10} x$$

- Choose an appropriate model involving exponential or logarithmic functions for a real-life situation. *100*

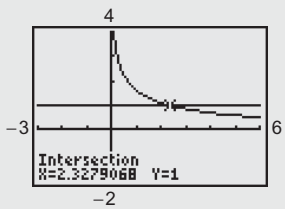
Study Strategies

- **Solve Problems Analytically or Graphically** When solving an exponential or logarithmic equation, you could use a variety of problem-solving strategies. For instance, if you were asked to solve the logarithmic equation

$$\ln(x + 4) - \ln x = 1$$

you could solve the equation *analytically*. That is, you could use the properties of logarithms to rewrite the equation, exponentiate each side, use the Inverse Property, and solve the resulting equation to determine that $x \approx 2.328$.

You could also solve the equation *graphically*. That is, you could use a graphing utility to graph $y_1 = \ln(x + 4) - \ln x$ and $y_2 = 1$ in the same viewing window. Then you could use the *intersect* feature or the *zoom* and *trace* features to determine that the solution of the original equation is $x \approx 2.328$.

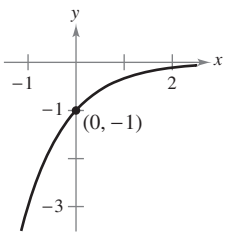


Review Exercises

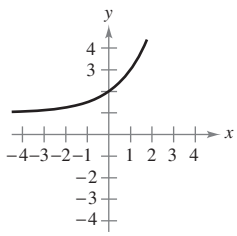
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

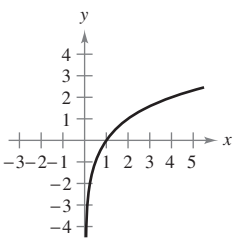
(a)



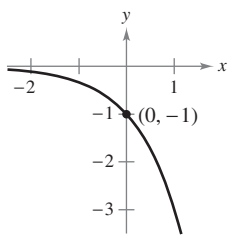
(b)



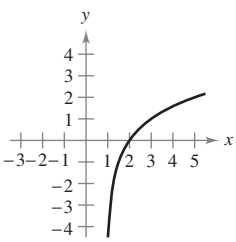
(c)



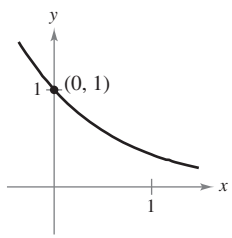
(d)



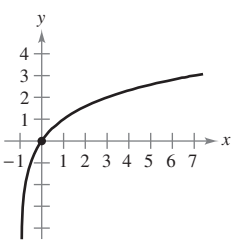
(e)



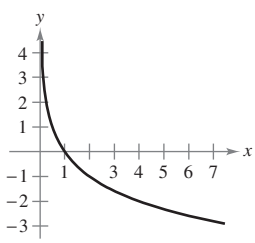
(f)



(g)



(h)



1. $f(x) = -3^x$

2. $f(x) = 3^{-x}$

3. $f(x) = -3^{-x}$

4. $f(x) = 2^x + 1$

5. $f(x) = \log_2 x$

6. $f(x) = \log_2(x - 1)$

7. $f(x) = -\log_2 x$

8. $f(x) = \log_2(x + 1)$

In Exercises 9–16, sketch the graph of the function.

9. $f(x) = 4^x$

10. $f(x) = 4^{x-1}$

11. $f(x) = \left(\frac{1}{2}\right)^x$

12. $f(x) = \left(\frac{1}{2}\right)^{x+1}$

13. $f(x) = 3e^{0.2x}$

14. $f(x) = 10e^{-0.1x}$

15. $f(x) = 3^{-x^2}$

16. $f(x) = 2^{1-x^2}$

Compound Interest In Exercises 17 and 18, complete the table to find the balance A for P dollars invested at a rate r for t years, compounded n times per year.

n	1	2	4	12	365	Continuous
A						

17. $P = \$000$, $r = 8.75\%$ $t = 12$ years

18. $P = \$000$, $r = 6.5\%$ $t = 25$ years

Compound Interest In Exercises 19 and 20, complete the table to determine the amount P that should be invested at a rate r to produce a final balance of $A = \$200{,}000$ in t years.

t	1	10	20	30	40	50
P						

19. $r = 7.5\%$ compounded continuously

20. $r = 9.5\%$ compounded quarterly

21. Investment Plan You deposit $000 in a fund that yields 5.75% interest, compounded continuously. How much money will be in the fund after 6 years?

22. Population The population P of a town increases according to the model

$$P(t) = 15{,}000e^{0.025t}$$

where t is the time in years, with $t = 8$ corresponding to 2008. Use the model to approximate the population in 2009 and 2011.

In Exercises 23–26, use the definition of a logarithm to write the equation in logarithmic form.

23. $4^3 = 64$

24. $25^{3/2} = 125$

25. $e^2 = 7.3890\ldots$

26. $e^x = 8$

In Exercises 27–30, use the definition of a logarithm to write the equation in exponential form.

27. $\log_3 81 = 4$

28. $\log_5 0.2 = -1$

29. $\ln 1 = 0$

30. $\ln 4 = 1.3862\ldots$

In Exercises 31–36, evaluate the expression without using a calculator.

31. $\log_2 32$

32. $\log_9 3$

33. $\ln e^7$

34. $\log_4 \frac{1}{4}$

35. $\ln e^{-1/2}$

36. $\ln 1$

In Exercises 37 and 38, use the fact that f and g are inverse functions of each other to sketch their graphs in the same coordinate plane.

37. $f(x) = 10^x,\ g(x) = \log_{10} x$

38. $f(x) = e^x,\ g(x) = \ln x$

In Exercises 39–42, find the domain, vertical asymptote, and x-intercept of the logarithmic function. Then sketch its graph.

39. $f(x) = \log_2(x - 3)$

40. $f(x) = 5 - 2\log_{10} x$

41. $g(x) = 2\ln x$

42. $g(x) = \ln(4 - x)$

43. Human Memory Model Students in a sociology class were given an exam and then retested monthly for 6 months with an equivalent exam. The average score for the class is given by the human memory model

$$f(t) = 82 - 16\log_{10}(t + 1), \quad 0 \le t \le 6$$

where t is the time (in months). How did the average score change over the six-month period?

44. Investment Time A principal P, invested at 5.85% interest and compounded continuously, increases to an amount that is K times the principal after t years, where t is given by

$$t = \frac{\ln K}{0.0585}.$$

Complete the table and describe the result.

K	1	2	3	4	6	8	10
t							

45. Antler Spread The antler spread a (in inches) and shoulder height h (in inches) of an adult American elk are related by the model

$$h = 116\log_{10}(a + 40) - 176.$$

(a) Approximate the shoulder height of an elk with an antler spread of 55 inches.

(T) (b) Use a graphing utility to graph the model.

46. Snow Removal The number of miles s of roads cleared of snow is approximated by the model

$$s = 25 - \frac{13\ln(h/12)}{\ln 3}, \quad 2 \le h \le 15$$

where h is the depth of the snow in inches.

(a) Use the model to find s when $h = 10$ inches.

(T) (b) Use a graphing utility to graph the model.

In Exercises 47–50, evaluate the logarithm using the change-of-base formula. Do each problem twice, once with common logarithms and once with natural logarithms. (Round your answer to three decimal places.)

47. $\log_3 10$

48. $\log_{1/4} 7$

49. $\log_{12} 200$

50. $\log_3 0.28$

In Exercises 51–54, approximate the logarithm using the properties of logarithms, given $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.

51. $\log_b 6$

52. $\log_b \frac{4}{25}$

53. $\log_b \sqrt{3}$

54. $\log_b 30$

In Exercises 55–58, find the *exact* value of the logarithm.

55. $\log_7 49$

56. $\log_6 \frac{1}{36}$

57. $\ln e^{3.2}$

58. $\ln \sqrt[5]{e^3}$

In Exercises 59–64, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms. (Assume that all variables are positive.)

59. $\log_{10} \frac{x}{y}$

60. $\log_{10} \frac{xy^3}{z^2}$

61. $\ln\left(x\sqrt{x - 3}\right)$

62. $\ln \sqrt[3]{\frac{x^3}{y^2}}$

63. $\log_5(y - 3)^4$

64. $\log_2 2xy^2 z$

In Exercises 65–70, condense the expression to the logarithm of a single quantity.

65. $\log_4 2 + \log_4 3$

66. $\ln y + 2\ln z$

67. $\frac{1}{2}\ln x$

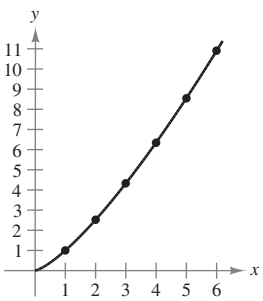
68. $4\log_3 x + \log_3 y - 2\log_3 z$

69. $\ln x - \ln(x - 3) - \ln(x + 1)$

70. $\log_{10}(x + 2) + 2\log_{10} x - 3\log_{10}(x + 4)$

71. Curve Fitting Find a logarithmic equation that relates y and x (see figure).

x	1	2	3	4	5	6
y	1	2.520	4.327	6.350	8.550	10.903



72. Human Memory Model Students in a learning theory study were given an exam and then retested monthly for 6 months with an equivalent exam. The average scores for the class are shown in the table, with $t = 1$ representing 1 month after the initial exam. Use the table to find a logarithmic equation that relates s and t.

Month, t	Score, s
1	87.9
2	79.7
3	74.8
4	71.3
5	68.6
6	66.5

In Exercises 73–86, solve the equation. Approximate the result to three decimal places.

73. $e^x = 8$

74. $2e^{x+1} = 7$

75. $3^{4x-1} - 4 = 23$

76. $2^{3x+1} + 5 = 133$

77. $e^{2x} - 3e^x - 4 = 0$

78. $e^{2x} - 8e^x + 12 = 0$

79. $\ln 3x = 8.2$

80. $2 \log_3 4x = 15$

81. $-2 + \ln 5x = 0$

82. $\ln 4x^2 = 21$

83. $\ln x - \ln 3 = 2$

84. $\log_3 x - \log_3 4 = 5$

85. $\log_2 \sqrt[3]{x + 1} = 1$

86. $\ln \sqrt{x + 1} = 2$

87. Demand Function The demand function for a desk is given by

$$p = 6000\left(1 - \frac{5}{5 + e^{-0.004x}}\right).$$

Find the demand x for each price p.

(a) $p =$ $500

(b) $p =$ $400.

88. Demand Function The demand function for a bicycle is given by

$$p = 4000\left(1 - \frac{3}{3 + e^{-0.004x}}\right).$$

Find the demand x for each price p.

(a) $p =$ $500

(b) $p =$ $400.

Radioactive Decay In Exercises 89 and 90, complete the table for the radioactive isotope.

	Isotope	*Half-Life (Years)*	*Initial Quantity*	*Amount After 1000 Years*
89.	^{14}C	5715	12 g	
90.	^{239}Pu	24,100		3.1 g

91. Population The population P of a city is given by

$$P = 185{,}000e^{0.018t}$$

where t represents the year, with $t = 8$ corresponding to 2008.

(a) Use a graphing utility to graph this equation.

(b) Use the model to predict the year in which the population of the city will reach 250,000.

92. Population The population P of a city is given by

$$P = 50{,}000e^{kt}$$

where t represents the year, with $t = 0$ corresponding to 2000. In 1990, the population was 34,500.

(a) Find the value of k and use this result to predict the population in the year 2030.

(b) Use a graphing utility to confirm the result of part (a).

93. Bacteria Growth The number of bacteria N in a culture is given by the model

$$N = 250e^{kt}$$

where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 250$. When $t = 6$, there are 380 bacteria. How long does it take the bacteria population to double in size? To triple in size?

94. Bacteria Growth The number of bacteria N in a culture is given by the model

$N = 200e^{kt}$

where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 200$. When $t = 5$, there are 325 bacteria. How long does it take for the bacteria population to double in size? To triple in size?

95. Learning Curve The management at a factory has found that the maximum number of units a worker can produce in a day is 50. The learning curve for the number of units N produced per day after a new employee has worked t days is given by

$N = 50(1 - e^{kt})$.

After 20 days on the job, a particular worker produced 31 units in 1 day.

(a) Find the learning curve for this worker.

(b) How many days should pass before this worker is producing 45 units per day?

96. Test Scores The scores on a general aptitude test roughly follow a normal distribution given by

$y = 0.0040e^{-[(x-300)^2]/20{,}000}, \quad 100 \le x \le 500.$

Sketch the graph of this function. Estimate the average score on this test.

97. Wildlife Management A state parks and wildlife department releases 100 deer into a wilderness area. The department believes that the carrying capacity of the area is 400 deer and that the growth of the herd will be modeled by the logistic curve

$$P = \frac{400}{1 + 3e^{-kt}}, \quad t \ge 0$$

where P is the number of deer and t is the time (in years).

(a) The herd size is 135 after 2 years. Find k.

(b) Find the populations after 5 years, after 10 years, and after 20 years.

98. Earthquake Magnitudes On the Richter scale, the magnitude R of an earthquake of intensity I is given by

$$R = \log_{10} \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensity per unit of area for each value of R.

(a) $R = 8.4$ (b) $R = 6.85$ (c) $R = 9.1$

99. Thawing a Package of Steaks You take a package of steaks out of a freezer at 10 A.M. and place it in the refrigerator. Will the steaks be thawed in time to be grilled at 6 P.M.? Assume that the refrigerator temperature is 40°F and the freezer temperature is 0°F. Use the formula

$$t = -3.95 \ln \frac{T - 40}{0 - 40}$$

where t is the time in hours (with $t = 0$ corresponding to 10 A.M.) and T is the temperature of the package of steaks (in degrees Fahrenheit).

100. MAKE A DECISION: COSTCO REVENUES The annual revenues R (in millions of dollars) for the Costco Wholesale Corporation from 1996 to 2005 are shown in the table. *(Source: Costco Wholesale Corporation)*

Year	Revenue, R
1996	19,566
1997	21,874
1998	24,270
1999	27,456
2000	32,164
2001	34,797
2002	38,762
2003	42,546
2004	48,107
2005	52,935

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find an exponential model for the data. Use the Inverse Property $b = e^{\ln b}$ to rewrite the model as an exponential model in base e.

(c) Use the *regression* feature of a graphing utility to find a logarithmic model for the data.

(d) Use the exponential model in base e and the logarithmic model to predict revenues in 2006 and in 2007. It is projected that revenues in 2006 and in 2007 will be $59,050 million and $64,500 million. Do the predictions from the two models agree with these projections? Explain.

(e) Use the exponential model in base e and the logarithmic model to predict revenues in 2009, 2010, and 2011. It is projected that revenue will reach $81,000 million during the period from 2009 to 2011. Does the prediction from each model agree with this projection? Explain.

Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, sketch the graph of the function.

1. $y = 2^x$

2. $y = e^{-2x}$

3. $y = \ln x$

4. $y = \log_3(x - 1)$

In Exercises 5 and 6, students in a psychology class were given an exam and then retested monthly with an equivalent exam. The average score for the class is given by the human memory model

$f(t) = 87 - 15 \log_{10}(t + 1), \quad 0 \le t \le 4$

where t is the time (in months).

5. What was the average score on the original exam? After 2 months? After 4 months?

6. The students in this psychology class participated in a study that required that they continue taking an equivalent exam every 6 months for 2 years. Use the model to predict the average score after 12 months and after 18 months. What could this indicate about human memory?

In Exercises 7–10, expand the logarithmic expression.

7. $\ln \dfrac{x^2y^3}{z}$

8. $\log_{10} 3xyz^2$

9. $\log_2\left(x\sqrt[3]{x - 2}\right)$

10. $\log_8 \sqrt[5]{x^2 + 1}$

In Exercises 11 and 12, condense the logarithmic expression.

11. $2 \ln x + 3 \ln y - \ln z$

12. $\frac{2}{3}(\log_{10} x + \log_{10} y)$

In Exercises 13–16, solve the equation. Approximate the result to three decimal places.

13. $2^{4x} = 21$

14. $e^{2x} - 8e^x + 12 = 0$

15. $\log_2(x + 1) - 7 = 0$

16. $\ln \sqrt{x + 2} = 3$

17. You deposit $10,000 in a fund that pays 6.75% interest, compounded continuously. When will the balance be greater than $20,000?

18. The population P of a city is given by

$P = 85{,}000e^{0.025t}$

where t represents the year, with $t = 8$ corresponding to 2008. When will the city have a population of 125,000? Explain.

19. The number of bacteria N in a culture is given by

$N = 100e^{kt}$

where t is the time (in hours), with $t = 0$ corresponding to the time when $N = 100$. When $t = 8$, $N = 175$. How long does it take the bacteria population to double?

20. Carbon 14 has a half-life of 5715 years. You have an initial quantity of 10 grams. How many grams will remain after 10,000 years? After 20,000 years?

21. If you are given the annual bear population on a small Alaskan island for the past decade, would you expect the bear population to grow exponentially or logistically? Explain your reasoning.

Cumulative Test: Chapters 2–4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–6, use the functions given by $f(x) = x^2 + 1$ and $g(x) = 3x - 5$ to find the indicated function.

1. $(f + g)(x)$ **2.** $(f - g)(x)$ **3.** $(fg)(x)$

4. $\left(\frac{f}{g}\right)(x)$ **5.** $(f \circ g)(x)$ **6.** $(g \circ f)(x)$

In Exercises 7–11, sketch the graph of the function. Describe the domain and range of the function.

7. $f(x) = (x - 2)^2 + 3$ **8.** $g(x) = \dfrac{2}{x - 3}$ **9.** $h(x) = 2^{-x}$

10. $f(x) = \log_3(x + 1)$ **11.** $g(x) = \begin{cases} x + 5, & x < 0 \\ 5, & x = 0 \\ x^2 + 5, & x > 0 \end{cases}$

12. The profit P (in dollars) for a software company is given by

$$P = -0.001x^2 + 150x - 175{,}000$$

where x is the number of units produced. What production level will yield a maximum profit?

In Exercises 13–15, perform the indicated operation and write the result in standard form.

13. $(5 + 3i)(6 - 5i)$ **14.** $(4 + 5i)^2$

15. Write the quotient in standard form: $\dfrac{1 - 3i}{5 + 2i}$.

16. Use the Quadratic Formula to solve $3x^2 - 5x + 7 = 0$.

17. Find all the zeros of

$$f(x) = x^4 + 17x^2 + 16$$

given that $4i$ is a zero. Explain your reasoning.

18. Use long division or synthetic division to divide.

(a) $(6x^3 - 4x^2) \div (2x^2 + 1)$

(b) $(2x^4 + 3x^3 - 6x + 5) \div (x + 2)$

In Exercises 19 and 20, solve the equation. Approximate the result to three decimal places.

19. $e^{2x} - 3e^x - 18 = 0$

20. $\frac{1}{3}\ln(x - 3) = 4$

21. The IQ scores for adults roughly follow the normal distribution given by

$$y = 0.0266e^{-(x-100)^2/450}, \quad 70 \le x \le 130$$

where x is the IQ score. Use a graphing utility to graph the function. From the graph, estimate the average IQ score.

5 Systems of Equations and Inequalities



Paul Grebliunas/Getty Images

Ancient Greeks and Romans used naturally occurring substances to control insects and protect crops. Today, farmers use chemicals to protect crops from insects. You can use a system of equations to find the amounts of chemicals needed to obtain a desired mixture. (See Section 5.3, Exercise 57.)

5.1 Solving Systems Using Substitution
5.2 Solving Systems Using Elimination
5.3 Linear Systems in Three or More Variables
5.4 Systems of Inequalities
5.5 Linear Programming

Applications

Systems of equations and inequalities are used to model and solve many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Atmosphere, Exercise 62, page 426
- Peregrine Falcons, Exercise 64, page 450
- Investments, Exercise 47, page 460

Section 5.1

Solving Systems Using Substitution

- Solve a system of equations by the method of substitution.
- Solve a system of equations graphically.
- Construct and use a system of equations to solve an application problem.

The Method of Substitution

Up to this point in the text, most problems have involved either a function of one variable or a single equation in two variables. However, many problems in science, business, and engineering involve two or more equations in two or more variables. To solve such a problem, you need to find the solutions of a **system of equations.** Here is an example of a system of two equations in x and y.

$$\begin{cases} 2x + y = 5 & \text{Equation 1} \\ 3x - 2y = 4 & \text{Equation 2} \end{cases}$$

A **solution** of this system is an ordered pair that satisfies each equation in the system. For instance, the ordered pair $(2, 1)$ is a solution of this system. To check this, you can substitute 2 for x and 1 for y in *each* equation.

$$\begin{aligned} 2x + y &= 5 && \text{Write Equation 1.} \\ 2(2) + 1 &\stackrel{?}{=} 5 && \text{Substitute 2 for } x \text{ and 1 for } y. \\ 4 + 1 &= 5 && \text{Solution checks in Equation 1.} \checkmark \end{aligned}$$

$$\begin{aligned} 3x - 2y &= 4 && \text{Write Equation 2.} \\ 3(2) - 2(1) &\stackrel{?}{=} 4 && \text{Substitute 2 for } x \text{ and 1 for } y. \\ 6 - 2 &= 4 && \text{Solution checks in Equation 2.} \checkmark \end{aligned}$$

Finding the set of all solutions is called **solving the system of equations.** There are several different ways to solve systems of equations. In this chapter, you will study four of the most common techniques: *the method of substitution*, *the graphical approach*, *the method of elimination*, and *Gaussian elimination*. This section begins with the **method of substitution.**

Method of Substitution

1. *Solve* one of the equations for one variable in terms of the other.
2. *Substitute* the expression found in Step 1 into the other equation to obtain an equation in one variable.
3. *Solve* the equation obtained in Step 2.
4. *Back-substitute* the value found in Step 3 into the expression obtained in Step 1 to find the value of the other variable.
5. *Check* that the solution satisfies *each* of the original equations.

DISCOVERY

Use a graphing utility to graph $y_1 = -x + 4$ and $y_2 = x - 2$ in the same viewing window. Use the *trace* feature to find the coordinates of the point of intersection. Are the coordinates the same as the solution found in Example 1? Explain.

TECHNOLOGY

For instructions on how to use the *trace* feature, see Appendix A;for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

STUDY TIP

Because many steps are required to solve a system of equations, it is easy to make errors in arithmetic. So, you should always check your solution by substituting it into *each* equation in the original system.

When using the method of substitution to solve a system of equations, it does not matter which variable you solve for first. You will obtain the same solution regardless. When making your choice, you should choose the variable that is easier to work with. For instance, solve for a variable that has a coefficient of 1 or -1 to avoid working with fractions.

Example 1 Solving a System of Two Equations by Substitution

Solve the system of equations.

$$\begin{cases} x + y = 4 & \text{Equation 1} \\ x - y = 2 & \text{Equation 2} \end{cases}$$

SOLUTION Begin by solving for y in Equation 1.

$$y = 4 - x \qquad \text{Revised Equation 1}$$

Next, substitute this expression for y into Equation 2 and solve the resulting single-variable equation for x.

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ x - (4 - x) &= 2 && \text{Substitute } 4 - x \text{ for } y. \\ x - 4 + x &= 2 && \text{Distributive Property} \\ 2x &= 6 && \text{Combine like terms.} \\ x &= 3 && \text{Divide each side by 2.} \end{aligned}$$

Finally, you can solve for y by *back-substituting* $x = 3$ into the equation $y = 4 - x$, to obtain

$$\begin{aligned} y &= 4 - x && \text{Write revised Equation 1.} \\ y &= 4 - 3 && \text{Substitute 3 for } x. \\ y &= 1 && \text{Solve for } y. \end{aligned}$$

The solution is the ordered pair $(3, 1)$. You can check this as follows.

CHECK

Substitute $(3, 1)$ into Equation 1:

$$\begin{aligned} x + y &= 4 && \text{Write Equation 1.} \\ 3 + 1 &\stackrel{?}{=} 4 && \text{Substitute for } x \text{ and } y. \\ 4 &= 4 && \text{Solution checks in Equation 1.} \checkmark \end{aligned}$$

Substitute $(3, 1)$ into Equation 2:

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ 3 - 1 &\stackrel{?}{=} 2 && \text{Substitute for } x \text{ and } y. \\ 2 &= 2 && \text{Solution checks in Equation 2.} \end{aligned}$$

The term *back-substitution* implies that you work *backwards.* First you solve for one of the variables, and then you substitute that value *back* into one of the equations in the system to find the value of the other variable.

✓CHECKPOINT 1

Solve the system of equations.

$$\begin{cases} x + y = 6 \\ x - y = 4 \end{cases}$$

Example 2 Solving a System by Substitution

A total of $2,000 is invested in two funds paying 9% and 11% simple interest. (Recall that the formula for simple interest is $I = Prt$, where P is the principal, r is the annual interest rate, and t is time.) The total annual interest is $180. How much is invested at each rate?

SOLUTION

Verbal Model:

$$\begin{array}{l} \text{9\% fund} + \text{11\% fund} = \text{Total investment} \\ \text{9\% interest} + \text{11\% interest} = \text{Total interest} \end{array}$$

Labels:

Amount in 9% fund	$= x$	(dollars)
Interest for 9% fund	$= 0.09x$	(dollars)
Amount in 11% fund	$= y$	(dollars)
Interest for 11% fund	$= 0.11y$	(dollars)
Total investment	$= 12{,}000$	(dollars)
Total interest	$= 1180$	(dollars)

System:

$$\begin{cases} x + y = 12{,}000 & \text{Equation 1} \\ 0.09x + 0.11y = 1180 & \text{Equation 2} \end{cases}$$

To begin, it is convenient to multiply each side of Equation 2 by 100. This eliminates the need to work with decimals.

$$100(0.09x + 0.11y) = 100(1180) \qquad \text{Multiply each side by 100.}$$

$$9x + 11y = 118{,}000 \qquad \text{Revised Equation 2}$$

To solve this system, you can begin by solving for x in Equation 1.

$$x = 12{,}000 - y \qquad \text{Revised Equation 1}$$

Then, substitute this expression for x into revised Equation 2 and solve the resulting equation for y.

$$9x + 11y = 118{,}000 \qquad \text{Write revised Equation 2.}$$

$$9(12{,}000 - y) + 11y = 118{,}000 \qquad \text{Substitute } 12{,}000 - y \text{ for } x.$$

$$108{,}000 - 9y + 11y = 118{,}000 \qquad \text{Distributive Property}$$

$$2y = 10{,}000 \qquad \text{Combine like terms.}$$

$$y = 5000 \qquad \text{Divide each side by 2.}$$

Next, back-substitute the value $y = 5000$ to solve for x.

$$x = 12{,}000 - y \qquad \text{Write revised Equation 1.}$$

$$x = 12{,}000 - 5000 \qquad \text{Substitute 5000 for } y.$$

$$x = 7000 \qquad \text{Solve for } x.$$

The solution is (7000, 5000). So, $7000 is invested at 9% and $5000 is invested at 11%. Check this in the original system.

✓CHECKPOINT 2

In Example 2, suppose a total of $5,000 is invested in the same two funds. The total annual interest is $420. How much is invested at each rate? ■

The equations in Examples 1 and 2 are linear. The method of substitution can also be used to solve systems in which one or both of the equations are nonlinear.

Example 3 Substitution: Two-Solution Case

Solve the system of equations.

$$\begin{cases} x^2 - x - y = 1 & \text{Equation 1} \\ -x + y = -1 & \text{Equation 2} \end{cases}$$

SOLUTION Begin by solving for y in Equation 2 to obtain $y = x - 1$. Next, substitute this expression for y into Equation 1 and solve for x.

$$\begin{aligned} x^2 - x - y &= 1 && \text{Write Equation 1.} \\ x^2 - x - (x - 1) &= 1 && \text{Substitute for } y. \\ x^2 - 2x + 1 &= 1 && \text{Simplify.} \\ x^2 - 2x &= 0 && \text{General form} \\ x(x - 2) &= 0 && \text{Factor.} \\ x &= 0 && \text{Set 1st factor equal to 0.} \\ x - 2 = 0 \quad &\Rightarrow \quad x = 2 && \text{Set 2nd factor equal to 0.} \end{aligned}$$

Back-substituting these values of x to solve for the corresponding values of y produces the two solutions $(0, -1)$ and $(2, 1)$. Check these solutions in the original system.

✓CHECKPOINT 3

Solve the system of equations.

$$\begin{cases} x^2 + 4x - y = 7 \\ 2x - y = -1 \end{cases}$$

When using the method of substitution, you may encounter an equation that has no solution, as shown in Example 4.

Example 4 Substitution: No-Real-Solution Case

Solve the system of equations.

$$\begin{cases} -x + y = 4 & \text{Equation 1} \\ x^2 + y = 3 & \text{Equation 2} \end{cases}$$

SOLUTION Begin by solving for y in Equation 1 to obtain $y = x + 4$. Next, substitute this expression for y into Equation 2 and solve for x.

$$\begin{aligned} x^2 + y &= 3 && \text{Write Equation 2.} \\ x^2 + (x + 4) &= 3 && \text{Substitute } x + 4 \text{ for } y. \\ x^2 + x + 1 &= 0 && \text{Simplify.} \\ x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} && \text{Use the Quadratic Formula.} \\ x &= \frac{-1 \pm \sqrt{-3}}{2} && \text{Simplify.} \end{aligned}$$

Because the discriminant is negative, the equation $x^2 + x + 1 = 0$ has no (real) solution. So, this system has no (real) solution.

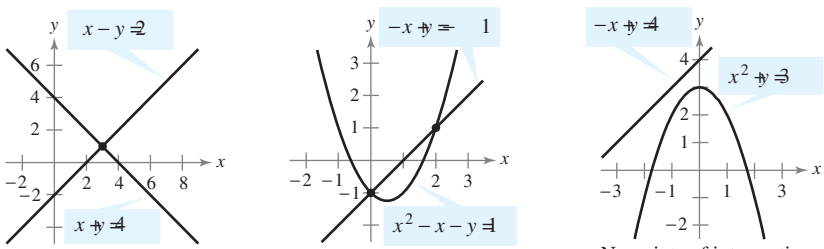
✓CHECKPOINT 4

Solve the system of equations.

$$\begin{cases} 2x^2 - y = 1 \\ x + y = -2 \end{cases}$$

Graphical Approach to Finding Solutions

From Examples 1, 3, and 4, you can see that a system of two equations in two unknowns can have exactly one solution, more than one solution, or no solution. In practice, you can gain insight about the location and number of solutions of a system of equations by graphing each of the equations in the same coordinate plane. The solution(s) of the system correspond to the **point(s) of intersection** of the graphs. For instance, the graph of the system in Example 1 is two lines with a *single point* of intersection, as shown in Figure 5.1(a). The graph of the system in Example 3 is a parabola and a line with *two points* of intersection, as shown in Figure 5.1(b). The graph of the system in Example 4 is a line and a parabola that have *no points* of intersection, as shown in Figure 5.1(c).



(a) One solution (b) Two solutions (c) No solution

FIGURE 5.1

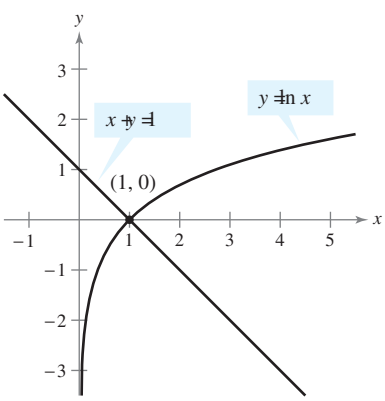


FIGURE 5.2

Example 5 Solving a System of Equations Graphically

Solve the system of equations.

$$\begin{cases} y = \ln x & \text{Equation 1} \\ x + y = 1 & \text{Equation 2} \end{cases}$$

SOLUTION The graph of each equation is shown in Figure 5.2. From the graph, you can see that there is only one point of intersection. So, it appears that $(1, 0)$ is the solution point. You can confirm this by substituting 1 for x and 0 for y in *both* equations.

CHECK

Equation 1: $0 = \ln 1$ ✓

Equation 2: $1 + 0 = 1$ ✓

✓CHECKPOINT 5

Solve the system of equations.

$$\begin{cases} 2x + 2y = 8 \\ y = \ln \frac{1}{4}x \end{cases}$$ ■

TECHNOLOGY

Your graphing utility may have an *intersect* feature that approximates the point(s) of intersection of two graphs. Use the *intersect* feature to verify the solution of Example 5. For instructions on how to use the *intersect* feature, see Appendix A;for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*



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In 2005, the average price for cross-training shoes in the United States was $46.34. *(Source: National Sporting Goods Association)*

Applications

The total cost C of producing x units of a product typically has two components—the initial cost and the cost per unit. When enough units have been sold so that the total revenue R equals the total cost C, the sales are said to have reached the **break-even point.** You will find that the break-even point corresponds to the point of intersection of the cost and revenue graphs.

Example 6 Break–Even Analysis

A shoe company invests $00,000 in equipment to produce cross-training shoes. Each pair of shoes costs $ to produce and is sold for $0. How many pairs of shoes must be sold before the business breaks even?

SOLUTION The total cost of producing x units is

$$\begin{array}{|l|}\hline \text{Total cost} \\ \hline \end{array} = \begin{array}{|l|}\hline \text{Cost per unit} \\ \hline \end{array} \cdot \begin{array}{|l|}\hline \text{Number of units} \\ \hline \end{array} + \begin{array}{|l|}\hline \text{Initial cost} \\ \hline \end{array}$$

$$C = 3x + 300{,}000.$$

The total revenue obtained by selling x units is

$$\begin{array}{|l|}\hline \text{Total revenue} \\ \hline \end{array} = \begin{array}{|l|}\hline \text{Price per unit} \\ \hline \end{array} \cdot \begin{array}{|l|}\hline \text{Number of units} \\ \hline \end{array}$$

$$R = 60x.$$

Because the break-even point occurs when $R = C$, you have $C = 60x$, and the system of equations to solve is

$$\begin{cases} C = 3x + 300{,}000 & \text{Equation 1} \\ C = 60x & \text{Equation 2} \end{cases}$$

Now you can solve by substitution.

$60x = 3x + 300{,}000$ Substitute $60x$ for C in Equation 1.

$57x = 300{,}000$ Subtract $3x$ from each side.

$x = \dfrac{300{,}000}{57}$ Divide each side by 57.

$x \approx 5263$ Use a calculator.

The company must sell about 5263 pairs of shoes to break even. Note in Figure 5.3 that sales less than the break-even point correspond to an overall loss, whereas sales greater than the break-even point correspond to a profit.

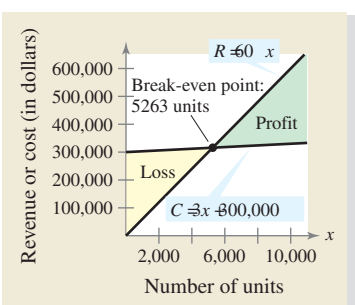


FIGURE 5.3

✓CHECKPOINT 6

In Example 6, suppose each pair of shoes costs $ to produce. How many pairs of shoes must be sold before the business breaks even? ■

Another way to view the solution in Example 6 is to consider the profit function $P = R - C$. The break-even point occurs when the profit is 0, which is the same as saying that $R = C$.

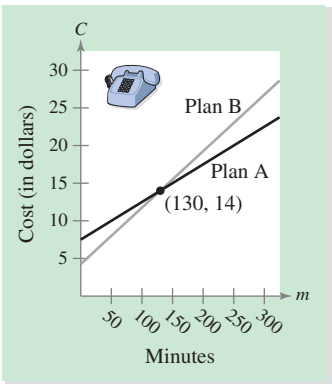


FIGURE 5.4

Example 7
MAKE A DECISION **Long–Distance Phone Plans**

You are choosing between two long-distance telephone plans. Plan A charges ¢0.05 per minute plus a basic monthly fee of $3.50. Plan B charges ¢0.075 per minute plus a basic monthly fee of $4.25. After how many long-distance minutes are the costs of the two plans equal? Which plan should you choose if you use 100 long-distance minutes each month?

SOLUTION Models for each long-distance phone plan are

$$C = 0.05m + 7.5 \qquad \text{Plan A}$$

$$C = 0.075m + 4.25 \qquad \text{Plan B}$$

where C is the monthly phone cost and m is the number of monthly long-distance minutes used. (See Figure 5.4.) Because the first equation has already been solved for C in terms of m, substitute this value into the second equation and solve for m, as follows.

$$\begin{aligned} 0.05m + 7.5 &= 0.075m + 4.25 \\ 0.05m - 0.075m &= 4.25 - 7.5 \\ -0.025m &= -3.25 \\ m &= 130 \end{aligned}$$

So, the costs of the two plans are equal after 130 long-distance minutes. Because Plan B costs less than Plan A when you use less than 130 long-distance minutes, you should choose Plan B.

✓CHECKPOINT 7

In Example 7, suppose Plan A charges ¢0.045 per minute plus a basic monthly fee of $3.49. Which plan should you choose if you use 150 long-distance minutes each month? ■

CONCEPT CHECK

1. The ordered pair $(2, -3)$ is a solution to $x - 2y = 8$. Give values of a, b, and c so that $(2, -3)$ is a solution of the system

$$\begin{cases} x - 2y = 8 \\ ax + by = c \end{cases}$$

where a, b, and c are real numbers.

2. When solving a system of quadratic equations using substitution, the resulting equation is not factorable. Explain your next step.
3. A system of equations consists of a linear equation and a cubic equation. what is the greatest number of possible solutions? Explain.
4. Explain why you can set a cost equation equal to a revenue equation when finding the break-even point.

Skills Review 5.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.5, 1.1, 1.3, and 2.1.

In Exercises 1–4, sketch the graph of the equation.

1. $y = -\frac{1}{3}x + 6$

2. $y = 2(x - 3)$

3. $x^2 + y^2 = 4$

4. $y = 5 - (x - 3)^2$

In Exercises 5–8, perform the indicated operations and simplify.

5. $(3x + 2y) - 2(x + y)$

6. $(-10u + 3v) + 5(2u - 8v)$

7. $x^2 + (x - 3)^2 + 6x$

8. $y^2 - (y + 1)^2 + 2y$

In Exercises 9 and 10, solve the equation.

9. $3x + (x - 5) = 15 + 4$

10. $y^2 + (y - 2)^2 = 2$

Exercises 5.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, determine whether each ordered pair is a solution of the system of equations.

1. $\begin{cases} x + 4y = -3 \\ 5x - y = 6 \end{cases}$

(a) $(-1, -1)$

(b) $(1, -1)$

2. $\begin{cases} 2x - y = 2 \\ x + 3y = 8 \end{cases}$

(a) $(2, 1)$

(b) $(2, 2)$

3. $\begin{cases} 2x + 5y = -5 \\ 2x - y^2 = 1 \end{cases}$

(a) $(5, -3)$

(b) $(0, -1)$

4. $\begin{cases} 4x^2 + y = 3 \\ -x - y = 11 \end{cases}$

(a) $(-2, -9)$

(b) $(2, -13)$

5. $\begin{cases} y = -2e^x \\ 3x - y = 2 \end{cases}$

(a) $(-2, 0)$

(b) $(-1, 2)$

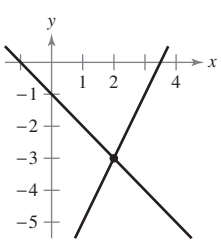
6. $\begin{cases} -\log_{10} x + 3 = y \\ \frac{1}{9}x + y = \frac{28}{9} \end{cases}$

(a) $(1, 3)$

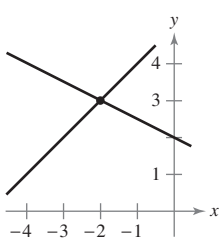
(b) $\left(9, \frac{37}{9}\right)$

In Exercises 7–16, solve the system by the method of substitution. Then use the graph to confirm your solution.

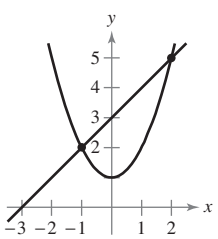
7. $\begin{cases} x + y = -1 \\ -2x + y = -7 \end{cases}$



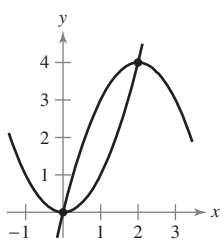
8. $\begin{cases} x - y = -5 \\ x + 2y = 4 \end{cases}$



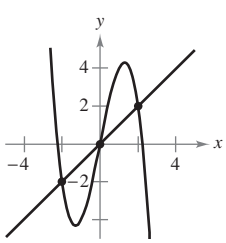
9. $\begin{cases} x - y = -3 \\ x^2 - y = -1 \end{cases}$



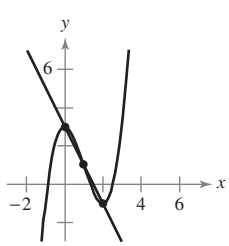
10. $\begin{cases} x^2 - y = 0 \\ x^2 - 4x + y = 0 \end{cases}$



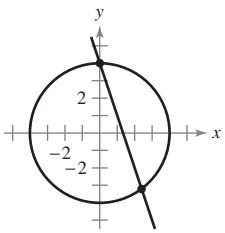
11. $\begin{cases} x - y = 0 \\ x^3 - 5x + y = 0 \end{cases}$



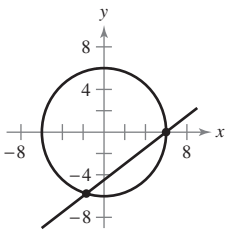
12. $\begin{cases} y = x^3 - 3x^2 + 3 \\ 2x + y = 3 \end{cases}$



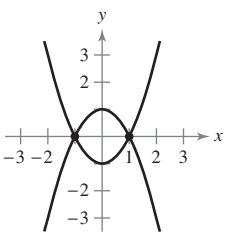
13. $\begin{cases} 3x + y = 4 \\ x^2 + y^2 = 16 \end{cases}$



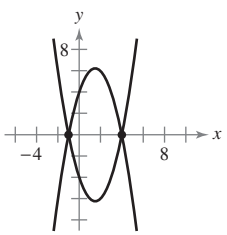
14. $\begin{cases} 3x - 4y = 18 \\ x^2 + y^2 = 36 \end{cases}$



15. $\begin{cases} y = -x^2 + 1 \\ y = x^2 - 1 \end{cases}$



16. $\begin{cases} y = x^2 - 3x - 4 \\ y = -x^2 + 3x + 4 \end{cases}$



In Exercises 17–38, solve the system by the method of substitution.

17. $\begin{cases} 2x - y = -3 \\ -3x - 4y = -1 \end{cases}$

18. $\begin{cases} x + 2y = 1 \\ 5x - 4y = -23 \end{cases}$

19. $\begin{cases} 2x - y + 2 = 0 \\ 4x + y - 5 = 0 \end{cases}$

20. $\begin{cases} 6x - 3y - 4 = 0 \\ x + 2y - 4 = 0 \end{cases}$

21. $\begin{cases} x - y = 7 \\ 2x + y = 23 \end{cases}$

22. $\begin{cases} x - 2y = -2 \\ 3x - y = 6 \end{cases}$

23. $\begin{cases} 0.3x - 0.4y - 0.33 = 0 \\ 0.1x + 0.2y - 0.21 = 0 \end{cases}$

24. $\begin{cases} 1.5x + 0.8y = 2.3 \\ 0.3x - 0.2y = 0.1 \end{cases}$

25. $\begin{cases} \frac{1}{5}x + \frac{1}{2}y = 8 \\ x + y = 20 \end{cases}$

26. $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = 10 \\ \frac{3}{2}x - y = 4 \end{cases}$

27. $\begin{cases} 6x + 5y = -3 \\ -x - \frac{5}{6}y = -7 \end{cases}$

28. $\begin{cases} -\frac{2}{3}x + y = 2 \\ 2x - 3y = 6 \end{cases}$

29. $\begin{cases} y = 2x \\ y = x^2 - 1 \end{cases}$

30. $\begin{cases} x + y = 4 \\ x^2 - y = 2 \end{cases}$

31. $\begin{cases} 3x - 7y + 6 = 0 \\ x^2 - y^2 = 4 \end{cases}$

32. $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$

33. $\begin{cases} x - 2y = 4 \\ x^2 - y = 0 \end{cases}$

34. $\begin{cases} x^2 + y^2 = 9 \\ x - y = -5 \end{cases}$

35. $\begin{cases} y = x^4 - 2x^2 + 1 \\ y = 1 - x^2 \end{cases}$

36. $\begin{cases} y = x^3 - 2x^2 + x - 1 \\ y = -x^2 + 3x - 1 \end{cases}$

37. $\begin{cases} xy - 2 = 0 \\ y = \sqrt{x - 1} \end{cases}$

38. $\begin{cases} xy = 3 \\ y = \sqrt{x - 2} \end{cases}$

In Exercises 39–50, solve the system graphically.

39. $\begin{cases} -x + 2y = 2 \\ 3x + y = 15 \end{cases}$

40. $\begin{cases} x + y = 0 \\ 3x - 2y = 10 \end{cases}$

41. $\begin{cases} x - 3y = -2 \\ 5x + 3y = 17 \end{cases}$

42. $\begin{cases} -x + 2y = 1 \\ x - y = 2 \end{cases}$

43. $\begin{cases} x + y = 4 \\ x^2 + y^2 - 4x = 0 \end{cases}$

44. $\begin{cases} -x + y = 3 \\ x^2 - 6x - 27 + y^2 = 0 \end{cases}$

45. $\begin{cases} x - y + 3 = 0 \\ x^2 - 4x + 7 = y \end{cases}$

46. $\begin{cases} y^2 - 4x + 11 = 0 \\ -\frac{1}{2}x + y = -\frac{1}{2} \end{cases}$

47. $\begin{cases} 7x + 8y = 24 \\ x - 8y = 8 \end{cases}$

48. $\begin{cases} x - y = 0 \\ 5x - 2y = 6 \end{cases}$

49. $\begin{cases} 3x - 2y = 0 \\ x^2 - y^2 = 4 \end{cases}$

50. $\begin{cases} 2x - y + 3 = 0 \\ x^2 + y^2 - 4x = 0 \end{cases}$

In Exercises 51–56, use a graphing utility to determine whether the system of equations has *one solution*, *two solutions*, or *no solution*.

51. $\begin{cases} y = -5x + 1 \\ y = x + 3 \end{cases}$

52. $\begin{cases} -\frac{1}{2}x + y = -1 \\ 7x + y = 2 \end{cases}$

53. $\begin{cases} y = x^2 + 2x - 1 \\ y = 2x + 5 \end{cases}$

54. $\begin{cases} x^2 + 3x + y = 4 \\ 3x + y = -5 \end{cases}$

55. $\begin{cases} y = x^2 + 3x + 7 \\ y = -x^2 - 3x + 1 \end{cases}$

56. $\begin{cases} -10x + y = 2 \\ -10x + y = -3 \end{cases}$

In Exercises 57–64, use a graphing utility to find the point(s) of intersection of the graphs. Then confirm your solution algebraically.

57. $\begin{cases} y = x^2 + 3x - 1 \\ y = -x^2 - 2x + 2 \end{cases}$

58. $\begin{cases} y = -2x^2 + x - 1 \\ y = x^2 - 2x - 1 \end{cases}$

59. $\begin{cases} x - y + 3 = 0 \\ x^2 - 4x + 7 = y \end{cases}$

60. $\begin{cases} x - y = 3 \\ x - y^2 = 1 \end{cases}$

61. $\begin{cases} y = e^x \\ x - y + 1 = 0 \end{cases}$

62. $\begin{cases} y = \sqrt{x} \\ y = x \end{cases}$

63. $\begin{cases} 4x^2 - y^2 - 32x - 2y = -59 \\ 2x + y - 7 = 0 \end{cases}$

64. $\begin{cases} x^2 + y^2 = 8 \\ y = x^2 + 4 \end{cases}$

Break-Even Analysis In Exercises 65–68, find the sales necessary to break even ($R = C$) for the cost C of producing x units and the revenue R obtained by selling x units. (Round your answer to the nearest whole unit.)

65. $C = 8650x + 250{,}000; R = 9950x$

66. $C = 5.5\sqrt{x} + 10{,}000; R = 3.29x$

67. $C = 2.65x + 350{,}000; R = 4.15x$

68. $C = 0.08x + 50{,}000; R = 0.25x$

69. Break-Even Analysis You invest $8,000 in equipment to make CDs. The CDs can be produced for $.95 each and will be sold for $3.95 each. How many CDs must you sell to break even?

70. Break-Even Analysis You invest $000 in a fishing lure business. A lure costs $.06 to produce and will be sold for $.86. How many lures must you sell to break even?

71. Comparing Populations From 1995 to 2005, the population of Kentucky grew more slowly than that of Colorado. Models that represent the populations of the two states are given by

$$\begin{cases} P = 27.9t + 3757 & \text{Kentucky} \\ P = 86.1t + 3425 & \text{Colorado} \end{cases}$$

where P is the population (in thousands) and t represents the year, with $t = 5$ corresponding to 1995. Use the models to estimate when the population of Colorado first exceeded the population of Kentucky. *(Source: U.S. Census Bureau)*

72. Comparing Populations From 1995 to 2005, the population of Maryland grew more slowly than that of Arizona. Models that represent the populations of the two states are given by

$$\begin{cases} P = 55.6t + 4771 & \text{Maryland} \\ P = 145.9t + 3703 & \text{Arizona} \end{cases}$$

where P is the population (in thousands) and t represents the year, with $t = 5$ corresponding to 1995. Use the models to estimate when the population of Arizona first exceeded the population of Maryland. *(Source: U.S. Census Bureau)*

73. Body Mass Index Body mass index (BMI) is a measure of body fat based on height and weight. The 75th percentile BMI for females, ages 9 to 20, grew more slowly than that of males of the same age range. Models that represent the 75th percentile BMI for males and females, ages 9 to 20, are given by

$$\begin{cases} B = 0.73a + 11 & \text{Males} \\ B = 0.61a + 12.8 & \text{Females} \end{cases}$$

where B is the BMI (kg/m^2) and a represents the age, with $a = 9$ corresponding to 9 years old. Use a graphing utility to determine whether the BMI for males will exceed the BMI for females. *(Source: National Center for Health Statistics)*

74. Clothing Sales From 1996 to 2005, the sales of Abercrombie & Fitch Company grew faster than those of Timberland Company. Models that represent the sales of the two companies are given by

$$\begin{cases} S = 235.1t - 1126 & \text{Abercrombie \& Fitch Company} \\ S = 97.7t + 88 & \text{Timberland Company} \end{cases}$$

where S is the sales (in millions) and t represents the year, with $t = 6$ corresponding to 1996. Use a graphing utility to determine whether the sales of Abercrombie & Fitch Company will exceed the sales of Timberland Company. *(Source: Abercrombie & Fitch Company and Timberland Company)*

75. A total of $5,000 is invested in two funds paying 8.5% and 12% simple interest. The total annual interest is $675. How much is invested at each rate?

76. A total of $5,000 is invested in two funds paying 8% and 10.5% simple interest. The total annual interest is $275. How much is invested at each rate?

77. Job Choices You are offered two different jobs. Company A offers an annual salary of $0,000 plus a year-end bonus of 2.5% of your total sales. Company B offers a salary of $4,000 plus a year-end bonus of 6.5% of your total sales. What is the amount you must sell in one year to earn the same salary working for either company?

78. Camping You are choosing between camping outfitters. Outfitter A charges a reservation fee of $50 plus a daily guide fee of $0. Outfitter B charges a reservation fee of $5 plus a daily guide fee of $0. Estimate when the cost of Outfitter A equals the cost of Outfitter B.

79. Financial Aid The average award for Federal Pell Grants and Federal Perkins Loans from 1995 to 2005 can be approximated by

$$\begin{cases} A = -2.051t^3 + 56.87t^2 - 376.7t + 2238 & \text{Federal Pell Grant} \\ A = -1.810t^3 + 56.64t^2 - 476.4t + 2711 & \text{Federal Perkins Loan} \end{cases}$$

where A is the award (in dollars) and t represents the year, with $t = 5$ corresponding to 1995. Use a graphing utility to determine whether Federal Perkins Loan awards will exceed Federal Pell Grant awards. Do you think these models will continue to be accurate? Explain your reasoning. *(Source: U.S. Department of Education)*

80. SAT or ACT? The number of participants in SAT and ACT testing from 1995 to 2005 can be approximated by

$$\begin{cases} y = 0.68t^2 + 28.1t + 903 & \text{SAT} \\ y = -0.485t^3 + 14.88t^2 - 115.1t + 1201 & \text{ACT} \end{cases}$$

where y is the number of participants (in thousands) and t represents the year, with $t = 5$ corresponding to 1995. Use a graphing utility to determine whether the number of participants in ACT testing will exceed the number of participants in SAT testing. Do you think these models will continue to be accurate? Explain your reasoning. *(Source: College Board; ACT, Inc.)*

Section 5.2

Solving Systems Using Elimination

- Solve a linear system by the method of elimination.
- Interpret the solution of a linear system graphically.
- Construct and use a linear system to solve an application problem.

The Method of Elimination

In Section 5.1, you studied two methods for solving a system of equations: substitution and graphing. In this section, you will study a third method called the **method of elimination.** The key step in the method of elimination is to obtain, for one of the variables, coefficients that differ only in sign, so that *adding* the two equations eliminates this variable. The following system provides an example.

$$\begin{aligned} 3x + 5y &= 7 && \text{Equation 1} \\ -3x - 2y &= -1 && \text{Equation 2} \\ \hline 3y &= 6 && \text{Add equations.} \end{aligned}$$

Note that by adding the two equations, you eliminate the variable x and obtain a single equation in y. Solving this equation for y produces $y = 2$, which you can then back-substitute into one of the original equations to solve for x.

Example 1 The Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 3x + 2y = 4 & \text{Equation 1} \\ 5x - 2y = 8 & \text{Equation 2} \end{cases}$$

SOLUTION Because the coefficients of the y-terms differ only in sign, you can eliminate the y-terms by adding the two equations. This leaves you with a single equation in x.

$$\begin{aligned} 3x + 2y &= 4 && \text{Write Equation 1.} \\ 5x - 2y &= 8 && \text{Write Equation 2.} \\ \hline 8x &= 12 && \text{Add equations.} \end{aligned}$$

So, $x = \frac{3}{2}$. By back-substituting this value into Equation 1, you can solve for y.

$$\begin{aligned} 3x + 2y &= 4 && \text{Write Equation 1.} \\ 3\left(\tfrac{3}{2}\right) + 2y &= 4 && \text{Substitute } \tfrac{3}{2} \text{ for } x. \\ y &= -\tfrac{1}{4} && \text{Solve for } y. \end{aligned}$$

The solution is $\left(\frac{3}{2}, -\frac{1}{4}\right)$. Check this in the original system.

STUDY TIP

The method of substitution can also be used to solve the system in Example 1. Use substitution to solve the system. Which method do you think is easier?

✓CHECKPOINT 1

Solve the system of linear equations.

$$\begin{cases} 2x - 3y = 5 \\ 5x + 3y = 9 \end{cases}$$

To obtain coefficients (for one of the variables) that differ only in sign, you may need to multiply one or both of the equations by a suitable constant, as demonstrated in Example 2.

Example 2 The Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 2x - 3y = -7 & \text{Equation 1} \\ 3x + y = -5 & \text{Equation 2} \end{cases}$$

SOLUTION For this system, you can obtain coefficients that differ only in sign by multiplying Equation 2 by 3. Then, by adding the two equations, you can eliminate the y-terms. This leaves you with a single equation in x.

$$\begin{array}{rcl} 2x - 3y = -7 & \Rightarrow & 2x - 3y = -7 \quad \text{Write Equation 1.} \\ \underline{3x + y = -5} & \Rightarrow & \underline{9x + 3y = -15} \quad \text{Multiply Equation 2 by 3.} \\ & & 11x = -22 \quad \text{Add equations.} \end{array}$$

By dividing each side by 11, you can see that $x = -2$. By back-substituting this value of x into Equation 1, you can solve for y.

$$\begin{array}{rl} 2x - 3y = -7 & \text{Write Equation 1.} \\ 2(-2) - 3y = -7 & \text{Substitute } -2 \text{ for } x. \\ -3y = -3 & \text{Add 4 to each side.} \\ y = 1 & \text{Solve for } y. \end{array}$$

The solution is $(-2, 1)$. Check this in the original system, as follows.

CHECK

$$\begin{array}{rl} 2(-2) - 3(1) \stackrel{?}{=} -7 & \text{Substitute into Equation 1.} \\ -4 - 3 = -7 & \text{Equation 1 checks. } \checkmark \\ 3(-2) + 1 \stackrel{?}{=} -5 & \text{Substitute into Equation 2.} \\ -6 + 1 = -5 & \text{Equation 2 checks. } \checkmark \end{array}$$

✓CHECKPOINT 2

Solve the system of linear equations.

$$\begin{cases} 3x - 5y = -1 \\ x - 2y = -1 \end{cases}$$

In Example 2, the two systems of linear equations

$$\begin{cases} 2x - 3y = -7 \\ 3x + y = -5 \end{cases} \quad \text{and} \quad \begin{cases} 2x - 3y = -7 \\ 9x + 3y = -15 \end{cases}$$

are called **equivalent systems** because they have precisely the same solution set. The operations that can be performed on a system of linear equations to produce an equivalent system are (1) interchanging any two equations, (2) multiplying an equation by a nonzero constant, and (3) adding a multiple of one equation to any other equation in the system.

The Method of Elimination

To use the **method of elimination** to solve a system of two linear equations in x and y, use the following steps.

1. Examine the system to determine which variable is easiest to eliminate.
2. Obtain coefficients of x (or y) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.
3. Add the equations to eliminate one variable and solve the resulting equation.
4. Back-substitute the value obtained in Step 3 into either of the original equations and solve for the other variable.
5. Check your solution in both of the original equations.

Example 3 The Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 5x + 3y = 9 & \text{Equation 1} \\ 2x - 4y = 14 & \text{Equation 2} \end{cases}$$

SOLUTION You can obtain coefficients of y that differ only in sign by multiplying Equation 1 by 4 and multiplying Equation 2 by 3.

$$\begin{array}{rcl} 5x + 3y = 9 & \Rightarrow & 20x + 12y = 36 \quad \text{Multiply Equation 1 by 4.} \\ 2x - 4y = 14 & \Rightarrow & 6x - 12y = 42 \quad \text{Multiply Equation 2 by 3.} \\ & & 26x \quad\quad = 78 \quad \text{Add equations.} \end{array}$$

From this equation, you can see that $x = 3$. By back-substituting this value of x into Equation 2, you can solve for y, as follows.

$$\begin{array}{rl} 2x - 4y = 14 & \text{Write Equation 2.} \\ 2(3) - 4y = 14 & \text{Substitute 3 for } x. \\ -4y = 8 & \text{Subtract 6 from each side.} \\ y = -2 & \text{Solve for } y. \end{array}$$

The solution is $(3, -2)$. Check this in the original system.

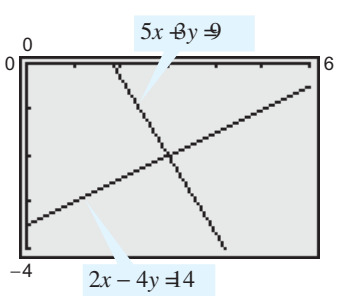


FIGURE 5.5

✓CHECKPOINT 3

Solve the system of linear equations.

$$\begin{cases} -2x + 7y = -15 \\ 3x - 5y = 6 \end{cases}$$

Remember that you can check the solution of a system of equations graphically. For instance, to check the solution found in Example 3, graph both equations in the same viewing window, as shown in Figure 5.5. Notice that the two lines intersect at $(3, -2)$.

Example 4 illustrates a strategy for solving a system of linear equations that has decimal coefficients.

TECHNOLOGY

The general solution of the linear system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

is

$$x = (ce - bf)/(ae - bd)$$

and

$$y = (af - cd)/(ae - bd).$$

If $ae - bd = 0$, the system does not have a unique solution. Graphing utility programs for solving such a system can be found at the text website at *college.hmco.com/info/larsonapplied*. Try using one of these programs to check the solution of the system in Example 4.

Example 4 A Linear System Having Decimal Coefficients

Solve the system of linear equations.

$$\begin{cases} 0.02x - 0.05y = -0.38 & \text{Equation 1} \\ 0.03x + 0.04y = 1.04 & \text{Equation 2} \end{cases}$$

SOLUTION Because the coefficients in this system have two decimal places, you can begin by multiplying each equation by 100. (This produces a system in which the coefficients are all integers.)

$$\begin{cases} 2x - 5y = -38 & \text{Revised Equation 1} \\ 3x + 4y = 104 & \text{Revised Equation 2} \end{cases}$$

Now, to obtain coefficients of x that differ only in sign, multiply revised Equation 1 by 3 and revised Equation 2 by -2.

$$\begin{array}{rcrl} 2x - 5y = -38 & \Rightarrow & 6x - 15y = -114 & \text{Multiply by 3.} \\ \underline{3x + 4y = 104} & \Rightarrow & \underline{-6x - 8y = -208} & \text{Multiply by } -2. \\ & & -23y = -322 & \text{Add equations.} \end{array}$$

So, you can conclude that $y = \dfrac{-322}{-23} = 14$. Now, back-substitute $y = 14$ into any of the original or revised equations of the system that contain the variable y. Back-substituting this value into revised Equation 2 produces the following.

$$\begin{array}{rl} 3x + 4y = 104 & \text{Write revised Equation 2.} \\ 3x + 4(14) = 104 & \text{Substitute 14 for } y. \\ 3x = 48 & \text{Subtract 56 from each side.} \\ x = 16 & \text{Solve for } x. \end{array}$$

The solution is (16, 14). Check this in the original system.

✓CHECKPOINT 4

Solve the system of linear equations.

$$\begin{cases} 0.03x + 0.04y = -0.13 \\ -0.04x + 0.05y = -0.24 \end{cases}$$

■

DISCOVERY

Rewrite each system of equations in slope-intercept form and graph the system using a graphing utility. What is the relationship between the slopes of the two lines and the number of points of intersection?

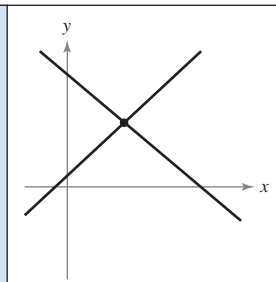
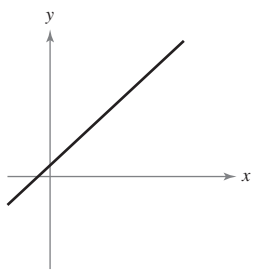
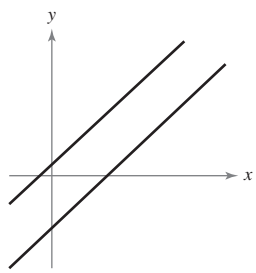
a. $\begin{cases} 2x + 4y = 8 \\ 4x - 3y = -6 \end{cases}$ **b.** $\begin{cases} -x + 5y = 15 \\ 2x - 10y = -7 \end{cases}$ **c.** $\begin{cases} x - y = 9 \\ 2x - 2y = 18 \end{cases}$

STUDY TIP

Keep in mind that the terminology and methods discussed in this section and the following section apply only to systems of linear equations.

Graphical Interpretation of Solutions

It is possible for a *general* system of equations to have exactly one solution, two or more solutions, or no solution. If a system of *linear* equations has two different solutions, it must have an *infinite* number of solutions. To see why this is true, consider the following graphical interpretations of systems of two linear equations in two variables. (Remember that the graph of a linear equation in two variables is a line.)

Graph			
Graphical Interpretation	The two lines intersect.	The two lines coincide (are identical).	The two lines are parallel.
Intersection	Single point of intersection	Infinitely many points of intersection	No point of intersection
Slopes of Lines	Slopes are not equal.	Slopes are equal.	Slopes are equal.
Number of Solutions	Exactly one solution	Infinitely many solutions	No solution
Type of System	**Independent (consistent) system**	**Dependent (consistent) system**	**Inconsistent system**

A system of linear equations is *consistent* if it has at least one solution. A consistent system with exactly one solution is *independent*, whereas a consistent system with infinitely many solutions is *dependent*. A system is *inconsistent* if it has no solution.

From the graphs above, you can see that a comparison of the slopes and y-intercepts of two lines is helpful in determining the number of solutions of the corresponding system of equations. For instance:

Independent (consistent) systems have lines with slopes that are not equal.
Dependent (consistent) systems have lines with equal slopes and the same y-intercept.
Inconsistent systems have lines with equal slopes, but different y-intercepts.

So, when solving a system of linear equations graphically, it is helpful to know the slope of each line. Writing each linear equation in the slope-intercept form

$$y = mx + b \qquad \text{Slope-intercept form}$$

enables you to identify the slopes quickly.

In Examples 5 and 6, note how you can use the method of elimination to determine that a linear system has no solution or infinitely many solutions.

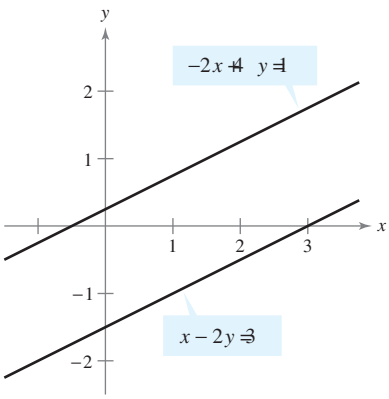


FIGURE 5.6 No Solution

Example 5 The Method of Elimination: No-Solution Case

Solve the system of linear equations.

$$\begin{cases} x - 2y = 3 & \text{Equation 1} \\ -2x + 4y = 1 & \text{Equation 2} \end{cases}$$

SOLUTION Obtain coefficients that differ only in sign, as follows.

$$\begin{array}{rcll} x - 2y = 3 & \Rightarrow & 2x - 4y = 6 & \text{Multiply Equation 1 by 2.} \\ -2x + 4y = 1 & \Rightarrow & -2x + 4y = 1 & \text{Write Equation 2.} \\ & & 0 = 7 & \text{False statement} \end{array}$$

Because there are no values of x and y for which $0 = 7$, you can conclude that the system is inconsistent and has no solution. The graphs of the equations are shown in Figure 5.6. Note that the two lines have equal slopes, but different y-intercepts. Therefore, the lines are parallel and have no point of intersection.

In Example 5, note that the occurrence of a false statement, such as $0 = 7$, indicates that the system has no solution. In the next example, note that the occurrence of a statement that is true for all values of the variables, such as $0 = 0$, indicates that the system has infinitely many solutions.

Example 6 The Method of Elimination: Many-Solutions Case

Solve the system of linear equations.

$$\begin{cases} 2x - y = 1 & \text{Equation 1} \\ 4x - 2y = 2 & \text{Equation 2} \end{cases}$$

SOLUTION Obtain coefficients that differ only in sign, as follows.

$$\begin{array}{rcll} 2x - y = 1 & & 2x - y = 1 & \text{Write Equation 1.} \\ 4x - 2y = 2 & \Rightarrow & -2x + y = -1 & \text{Multiply Equation 2 by } -\tfrac{1}{2}. \\ & & 0 = 0 & \text{Add equations.} \end{array}$$

Because the two equations are equivalent (have the same solution set), you can conclude that the system is consistent and has infinitely many solutions. The solution set consists of all points (x, y) lying on the line $2x - y = 1$, as shown in Figure 5.7. To represent the solution set as an ordered pair, let $x = a$, where a is any real number. Then $y = 2a - 1$ and the solution set can be written as $(a, 2a - 1)$.

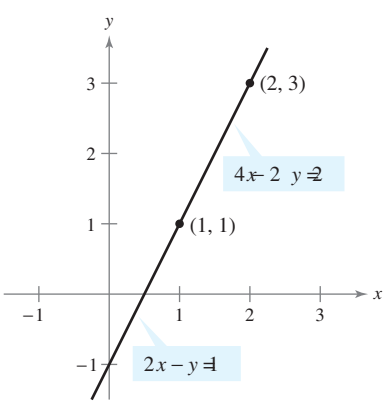


FIGURE 5.7 Infinite Number of Solutions

✓CHECKPOINT 5

Solve the system of linear equations.

$$\begin{cases} -3x + 6y = 5 \\ x - 2y = 2 \end{cases}$$ ■

✓CHECKPOINT 6

Solve the system of linear equations.

$$\begin{cases} -x - y = 5 \\ 4x + 4y = -20 \end{cases}$$ ■

Applications

At this point, you may be asking the question, "How can I tell which application problems can be solved using a system of linear equations?" The answer comes from the following considerations.

1. Does the problem involve more than one unknown quantity?

2. Are there two (or more) equations or conditions to be satisfied?

If one or both of these conditions occur, the appropriate mathematical model for the problem may be a system of linear equations. Example 7 shows how to construct such a model.

Example 7 An Application of a Linear System

An airplane flying into a headwind travels the 2000-mile flying distance between Wilmington, Delaware and Tucson, Arizona in 4 hours and 24 minutes. On the return flight, the same distance is traveled in 4 hours. Find the air speed of the plane and the speed of the wind, assuming that both remain constant.

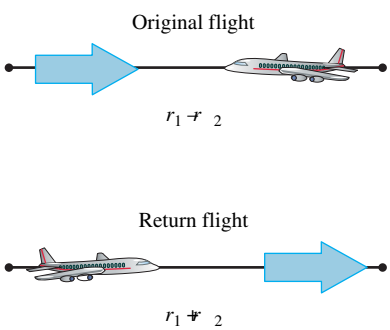


FIGURE 5.8

SOLUTION The two unknown quantities are the speeds of the wind and the plane. If r_1 is the air speed of the plane and r_2 is the speed of the wind, then

$r_1 - r_2$ = speed of the plane *against* the wind

$r_1 + r_2$ = speed of the plane *with* the wind

as shown in Figure 5.8. Using the formula distance = (rate)(time) for these two speeds, you obtain the following equations.

$$2000 = (r_1 - r_2)\left(4 + \frac{24}{60}\right)$$

$$2000 = (r_1 + r_2)(4)$$

These two equations simplify as follows.

$$\begin{cases} 5000 = 11r_1 - 11r_2 & \text{Equation 1} \\ 500 = r_1 + r_2 & \text{Equation 2} \end{cases}$$

To solve this system by elimination, multiply Equation 2 by 11.

$$\begin{array}{rcl} 5000 = 11r_1 - 11r_2 & \Rightarrow & 5000 = 11r_1 - 11r_2 \quad \text{Write Equation 1.} \\ 500 = r_1 + r_2 & \Rightarrow & 5500 = 11r_1 + 11r_2 \quad \text{Multiply Equation 2 by 11.} \\ & & \hline 10{,}500 = 22r_1 \quad \text{Add equations.} \end{array}$$

The solution is

$$r_1 = \frac{10{,}500}{22} = \frac{5250}{11} \approx 477.27$$

$$r_2 = 500 - \frac{5250}{11} = \frac{250}{11} \approx 22.73$$

So, the air speed of the plane is about 477.27 miles per hour and the speed of the wind is about 22.73 miles per hour. Check this solution in the original statement of the problem.

✓CHECKPOINT 7

In Example 7, suppose the return flight takes 4 hours and 6 minutes. Find the air speed of the plane and the speed of the wind, assuming that both remain constant. ■

In a free market, the demands for many products are related to the prices of the products. As the prices decrease, the demands by consumers increase and the amounts that producers are able or willing to supply decrease.

Example 8 Finding the Point of Equilibrium

The demand and supply equations for a DVD are given by

$$\begin{cases} p = 35 - 0.0001x & \text{Demand equation} \\ p = 8 + 0.0001x & \text{Supply equation} \end{cases}$$

where p is the price (in dollars) and x represents the number of DVDs. For how many units will the quantity demanded equal the quantity supplied? What price corresponds to this value?

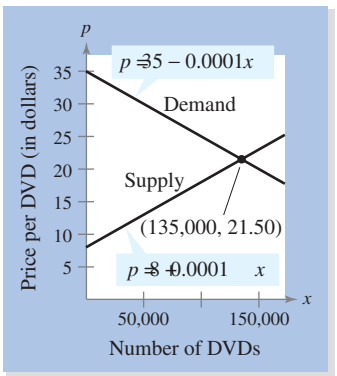


FIGURE 5.9

SOLUTION To obtain coefficients of p that differ only in sign, multiply the demand equation by -1.

$$\begin{aligned} p &= 35 - 0.0001x \quad \Rightarrow \quad -p = -35 + 0.0001x && \text{Multiply demand equation by } -1. \\ p &= 8 + 0.0001x \quad \Rightarrow \quad p = 8 + 0.0001x && \text{Write supply equation.} \\ & \qquad\qquad\quad 0 = -27 + 0.0002x && \text{Add equations.} \end{aligned}$$

By solving the equation $0 = -27 + 0.0002x$, you get $x = 135{,}000$. So, the quantity demanded equals the quantity supplied for 135,000 units (see Figure 5.9). The price that corresponds to this x-value is obtained by back-substituting $x = 135{,}000$ into either of the original equations. For instance, back-substituting into the demand equation produces

$$p = 35 - 0.0001(135{,}000) = 35 - 13.5 = \$21.50.$$

Back-substitute $x = 135{,}000$ into the supply equation to see that you obtain the same price. The solution (135,000, 21.50) is called the *point of equilibrium.* The **point of equilibrium** is the price p and the number of units x that satisfy both the demand and supply equations.

✓CHECKPOINT 8

In Example 8, suppose the supply equation is $p = 9 + 0.0001x$. Find the point of equilibrium. ■

CONCEPT CHECK

1. Two systems have infinitely many solutions. Are the systems equivalent? Explain.
2. Using the method of elimination, you reduce a system to $0 = 5$. What can you conclude about the system?
3. Can the graphs of the equations in an inconsistent system intersect? Explain.
4. Can the graphs of the equations in an independent (consistent) system have the same y-intercept? Can they have different y-intercepts? Explain.

Skills Review 5.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 2.2.

In Exercises 1 and 2, sketch the graph of the equation.

1. $2x + y = 4$

2. $5x - 2y = 3$

In Exercises 3 and 4, find an equation of the line passing through the two points.

3. $(-1, 3), (4, 8)$

4. $(2, 6), (5, 1)$

In Exercises 5 and 6, determine the slope of the line.

5. $3x + 6y = 4$

6. $7x - 4y = 10$

In Exercises 7–10, determine whether the lines represented by the pair of equations are parallel, perpendicular, or neither.

7. $2x - 3y = -10$
$3x + 2y = 11$

8. $4x - 12y = 5$
$-2x + 6y = 3$

9. $5x + y = 2$
$3x + 2y = 1$

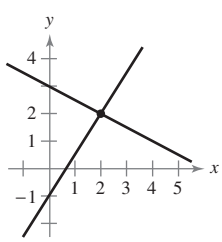
10. $x - 3y = 2$
$6x + 2y = 4$

Exercises 5.2

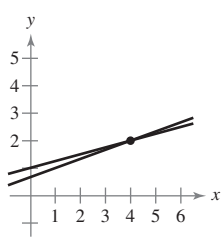
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, solve the system by elimination. Then use the graph to confirm your solution. Copy the graph and label each line with the appropriate equation.

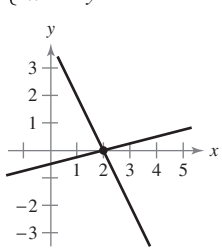
1. $\begin{cases} 3x - 2y = 2 \\ x + 2y = 6 \end{cases}$



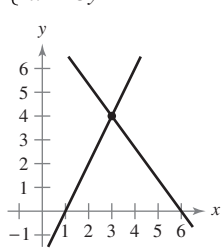
2. $\begin{cases} -x + 3y = 2 \\ x - 4y = -4 \end{cases}$



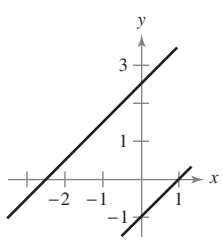
3. $\begin{cases} x - 4y = 2 \\ 2x + y = 4 \end{cases}$



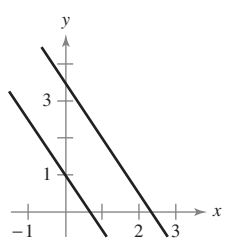
4. $\begin{cases} 2x - y = 2 \\ 4x + 3y = 24 \end{cases}$



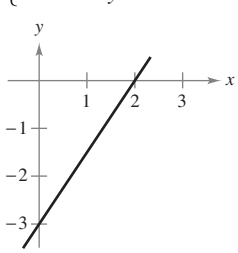
5. $\begin{cases} x - y = 1 \\ -2x + 2y = 5 \end{cases}$



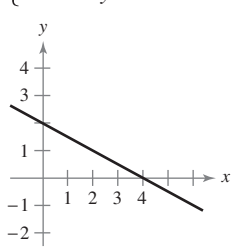
6. $\begin{cases} 3x + 2y = 2 \\ 6x + 4y = 14 \end{cases}$



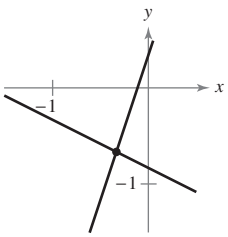
7. $\begin{cases} 3x - 2y = 6 \\ -6x + 4y = -12 \end{cases}$



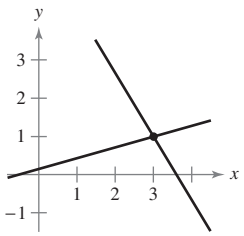
8. $\begin{cases} 2x + 4y = 8 \\ 6x + 12y = 24 \end{cases}$



9. $\begin{cases} 9x - 3y = -1 \\ 3x + 6y = -5 \end{cases}$



10. $\begin{cases} 5x + 3y = 18 \\ 2x - 7y = -1 \end{cases}$



In Exercises 11–32, solve the system by elimination. Then state whether the system is consistent or inconsistent.

11. $\begin{cases} x + 2y = 3 \\ x - 2y = 1 \end{cases}$

12. $\begin{cases} 2x - 3y = 4 \\ -2x - y = 4 \end{cases}$

13. $\begin{cases} 4x - 3y = 11 \\ -6x + 3y = 3 \end{cases}$

14. $\begin{cases} 3x - 5y = 2 \\ 2x + 5y = 13 \end{cases}$

15. $\begin{cases} 3x - y = 17 \\ 5x + 5y = -5 \end{cases}$

16. $\begin{cases} x + 7y = 12 \\ 3x - 5y = 10 \end{cases}$

17. $\begin{cases} 3x + 2y = 10 \\ 2x + 5y = 3 \end{cases}$

18. $\begin{cases} 8r + 16s = 20 \\ 16r + 50s = 55 \end{cases}$

19. $\begin{cases} 2u + v = 120 \\ u + 2v = 120 \end{cases}$

20. $\begin{cases} 5u + 6v = 24 \\ 3u + 5v = 18 \end{cases}$

21. $\begin{cases} 4b + 3m = 3 \\ 3b + 11m = 13 \end{cases}$

22. $\begin{cases} 3b + 3m = 7 \\ 3b + 5m = 3 \end{cases}$

23. $\begin{cases} 6r - 5s = 3 \\ -1.2r + s = 0.5 \end{cases}$

24. $\begin{cases} 1.8x + 1.2y = 4 \\ 9x + 6y = 3 \end{cases}$

25. $\begin{cases} \frac{x}{4} + \frac{y}{6} = 1 \\ x - y = 3 \end{cases}$

26. $\begin{cases} \frac{1}{6}x - \frac{2}{3}y = 3 \\ 3x + y = 15 \end{cases}$

27. $\begin{cases} \frac{x+3}{4} + \frac{y-1}{3} = 1 \\ x - y = 3 \end{cases}$

28. $\begin{cases} \frac{x-1}{2} + \frac{y+2}{3} = 4 \\ x - 2y = 5 \end{cases}$

29. $\begin{cases} 2.5x - 3y = 1.5 \\ 10x - 12y = 6 \end{cases}$

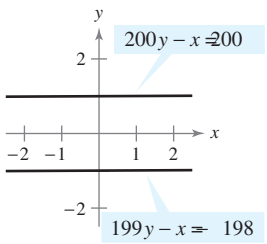
30. $\begin{cases} 1.5x + 2y = 3.75 \\ 7.5x + 10y = 18.75 \end{cases}$

31. $\begin{cases} 0.05x - 0.03y = 0.21 \\ 0.07x + 0.02y = 0.16 \end{cases}$

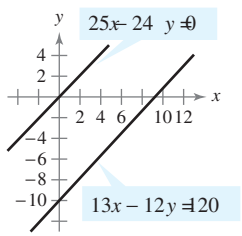
32. $\begin{cases} 0.02x - 0.05y = -0.19 \\ 0.03x + 0.04y = 0.52 \end{cases}$

In Exercises 33 and 34, the graphs of the two equations appear to be parallel. Are they? Justify your answer by using elimination to solve the system.

33. $\begin{cases} 200y - x = 200 \\ 199y - x = -198 \end{cases}$



34. $\begin{cases} 25x - 24y = 0 \\ 13x - 12y = 120 \end{cases}$



In Exercises 35–38, use the given statements to write a system of equations. Solve the system by elimination.

35. The sum of a number x and a number y is 13. The difference of x and y is 3.

36. The sum of a number a and a number b is 43. The difference of a and b is -27.

37. The sum of twice a number r and a number s is 8. The difference of r and s is 7.

38. The difference of a number m and twice a number n is 1. The sum of two times m and n is 22.

39. Airplane Speed An airplane flying into a headwind travels the 1800-mile flying distance between Los Angeles, California and South Bend, Indiana in 3 hours and 36 minutes. On the return flight, the distance is traveled in 3 hours. Find the air speed of the plane and the speed of the wind, assuming that both remain constant.

40. Airplane Speed Two planes start from the same airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 50 miles per hour faster. Find the air speed of each plane if, 2 hours after the first plane departs, the planes are 2000 miles apart.

41. Acid Mixture Ten gallons of a 30% acid solution is obtained by mixing a 20% solution with a 50% solution. How much of each solution is required to obtain the specified concentration of the final mixture?

42. Fuel Mixture Five hundred gallons of 89-octane gasoline is obtained by mixing 87-octane gasoline with 92-octane gasoline. How much of each type of gasoline is required to obtain the specified mixture? (Octane ratings can be interpreted as percents.)

43. Investment Portfolio A total of $35,000 is invested in two corporate bonds that pay 9.5% and 14% simple interest. The total annual interest is $4050. How much is invested in each bond?

44. Investment Portfolio A total of $10,000 is invested in two municipal bonds that pay 6.75% and 8.25% simple interest. The total annual interest is $900. How much is invested in each bond?

45. Ticket Sales You are the manager of a theater. On Saturday morning you are going over the ticket sales for Friday evening. A total of 740 tickets were sold. The tickets for adults and children sold for $8.50 and $4.00, respectively, and the total receipts for the performance were $4688. However, your assistant manager did not record how many of each type of ticket were sold. From the information you have, can you determine how many of each type were sold? Explain your reasoning.

46. Shoe Sales You are the manager of a shoe store. On Sunday morning you are going over the receipts for the previous week's sales. A total of 320 pairs of cross-training shoes were sold. One style sold for $6.95 and the other sold for $2.95. The total receipts were $1,024. The cash register that was supposed to keep track of the number of each type of shoe sold malfunctioned. Can you recover the information? If so, how many of each type were sold?

Supply and Demand In Exercises 47–50, find the point of equilibrium for the pair of demand and supply equations.

	Demand	*Supply*
47.	$p = 56 - 0.0001x$	$p = 22 + 0.00001x$
48.	$p = 60 - 0.00001x$	$p = 15 + 0.00004x$
49.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$
50.	$p = 400 - 0.0002x$	$p = 225 + 0.0005x$

51. Restaurants The total sales y (in billions of dollars) for fast-food and full-service restaurants for the years 1999 to 2005 are shown in the table. *(Source: National Restaurant Association)*

Year	Fast-food	Full-service
1999	103.0	125.4
2000	107.1	133.8
2001	111.6	139.9
2002	115.1	141.9
2003	120.5	148.3
2004	129.4	157.0
2005	135.6	164.9

(a) Use a spreadsheet software program to create a scatter plot of the data for fast-food sales and use the *regression* feature to find a linear model. Let x represent the year, with $x = 9$ corresponding to 1999. Repeat the procedure for the data for full-service sales.

(b) Assuming that the amounts for the given 7 years are representative of future years, will fast-food sales ever equal full-service sales?

52. Prescriptions The numbers of prescriptions y (in thousands) filled at two pharmacies in the years 2002 to 2008 are shown in the table.

Year	Pharmacy A	Pharmacy B
2002	18.1	19.5
2003	18.6	19.9
2004	19.2	20.4
2005	19.6	20.8
2006	20.0	21.1
2007	20.4	21.4
2008	21.3	22.0

(a) Use a spreadsheet software program to create a scatter plot of the data for pharmacy A and use the *regression* feature to find a linear model. Let x represent the year, with $x = 2$ corresponding to 2002. Repeat the procedure for the data for pharmacy B.

(b) Assuming the amounts for the given 7 years are representative of future years, will the number of prescriptions filled at pharmacy A ever exceed the number of prescriptions filled at pharmacy B?

53. Supply and Demand The supply and demand equations for a small LCD television are given by

$$\begin{cases} p + 0.53x = 1542 & \text{Demand} \\ p - 0.37x = 300 & \text{Supply} \end{cases}$$

where p is the price (in dollars) and x represents the number of televisions. For how many units will the quantity demanded equal the quantity supplied? What price corresponds to this value?

54. Supply and Demand The supply and demand equations for a microscope are given by

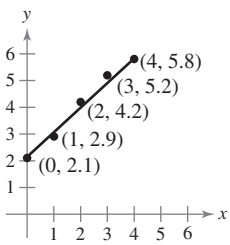
$$\begin{cases} p + 0.85x = 650 & \text{Demand} \\ p - 0.4x = 75 & \text{Supply} \end{cases}$$

where p is the price (in dollars) and x represents the number of microscopes. For how many units will the quantity demanded equal the quantity supplied? What price corresponds to this value?

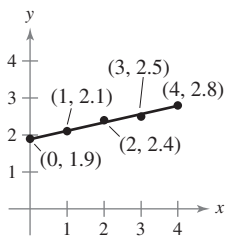
Fitting a Line to Data In Exercises 55–60, find the least squares regression line $y = ax + b$ for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by solving the system for a and b. (If you are unfamiliar with summation notation, look at the discussion in Section 7.1.)

$$\begin{cases} nb + \left(\sum_{i=1}^{n} x_i\right)a = \sum_{i=1}^{n} y_i \\ \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} x_i y_i \end{cases}$$

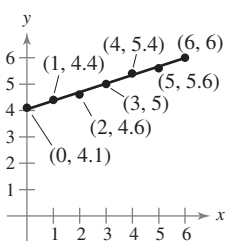
55. $\begin{cases} 5b + 10a = 20.2 \\ 10b + 30a = 50.1 \end{cases}$



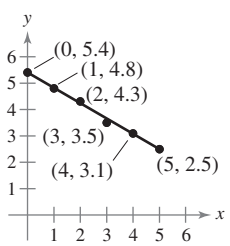
56. $\begin{cases} 5b + 10a = 11.7 \\ 10b + 30a = 25.6 \end{cases}$



57. $\begin{cases} 7b + 21a = 35.1 \\ 21b + 91a = 114.2 \end{cases}$



58. $\begin{cases} 6b + 15a = 23.6 \\ 15b + 55a = 48.8 \end{cases}$



59. (0, 4), (1, 3), (1, 1), (2, 0)

60. (1, 0), (2, 0), (3, 0), (3, 1), (4, 1), (4, 2), (5, 2), (6, 2)

61. Classic Cars The numbers of cars y sold at Barrett-Jackson Collector Car Auction in Scottsdale in the years 2003 to 2007 are shown in the table. *(Source: Barrett-Jackson Auction Company)*

Year	2003	2004	2005	2006	2007
t	0	1	2	3	4
Cars, y	655	727	877	1105	1271

(a) Solve the following system for a and b to find the least squares regression line $y = at + b$ for the data. Let t represent the year, with $t = 0$ corresponding to 2003.

$$\begin{cases} 5b + 10a = 4635 \\ 10b + 30a = 10{,}880 \end{cases}$$

(b) Use a graphing utility to graph the regression line and estimate the number of cars that will be sold in 2009.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data. Compare this model with the one you found in part (a).

62. Atmosphere The concentration y (in parts per million) of carbon dioxide in the atmosphere is measured at the Mauna Loa Observatory in Hawaii. The greatest monthly carbon dioxide concentrations for the years 2002 to 2006 are shown in the table. *(Source: Scripps CO2 Program)*

Year	t	Concentration, y
2002	0	375.55
2003	1	378.35
2004	2	380.63
2005	3	382.26
2006	4	384.92

(a) Solve the following system for a and b to find the least squares regression line $y = at + b$ for the data. Let t represent the year, with $t = 0$ corresponding to 2002.

$$\begin{cases} 5b + 10a = 1901.71 \\ 10b + 30a = 3826.07 \end{cases}$$

(b) Use a graphing utility to graph the regression line and predict the largest monthly carbon dioxide concentration in 2012.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data. Compare this model with the one you found in part (a).

63. Reasoning Design a system of two linear equations with infinitely many solutions. Solve the system algebraically and explain how the solution indicates that there are infinitely many solutions.

64. Reasoning Design a system of two linear equations with no solution. Solve the system algebraically and explain how the solution indicates that there is no solution.

65. Think About It For the system below, find the value(s) of k for which the system is (a) inconsistent and (b) consistent (dependent). Explain how you found your answers.

$$\begin{cases} 3x - 12y = 9 \\ x - 4y = k \end{cases}$$

66. Think About It For the system in Exercise 65, can you find a value of k for which the system is consistent (independent)? Explain.

Section 5.3

Linear Systems in Three or More Variables

- Solve a linear system in row-echelon form using back-substitution.
- Use Gaussian elimination to solve a linear system.
- Solve a nonsquare linear system.
- Construct and use a linear system in three or more variables to solve an application problem.
- Find the equation of a circle or a parabola using a linear system in three or more variables.

Row-Echelon Form and Back-Substitution

The method of elimination can be applied to a system of linear equations in more than two variables. In fact, this method easily adapts to computer use for solving linear systems with dozens of variables.

When elimination is used to solve a system of linear equations, the goal is to rewrite the system in a form to which back-substitution can be applied. To see how this works, consider the following two systems of linear equations.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$ System of Three Linear Equations in Three Variables

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$ Equivalent System in Row-Echelon Form

The second system is said to be in **row-echelon form,** which means that it has a "stair-step" pattern with leading coefficients of 1. After comparing the two systems, it should be clear that it is easier to solve the second system.

Example 1 Using Back-Substitution

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$ Equation 1; Equation 2; Equation 3

SOLUTION From Equation 3, you know the value of z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

$$y + 3(2) = 5 \quad \Rightarrow \quad y = -1.$$

Finally, substitute $y = -1$ and $z = 2$ into Equation 1 to obtain

$$x - 2(-1) + 3(2) = 9 \quad \Rightarrow \quad x = 1.$$

The solution is $x = 1$, $y = -1$, and $z = 2$, which can be written as the **ordered triple** $(1, -1, 2)$. Check this in the original system of equations.

✓CHECKPOINT 1

Solve the system of linear equations.

$$\begin{cases} 2x + y - 3z = 10 \\ y + z = 4 \\ z = 2 \end{cases}$$

Gaussian Elimination

Two systems of equations are **equivalent** if they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that is in row-echelon form. To see how this is done, let's take another look at the method of elimination, as applied to a system of two linear equations.

Example 2 The Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 3x - 2y = -1 & \text{Equation 1} \\ x - y = 0 & \text{Equation 2} \end{cases}$$

SOLUTION An easy way of obtaining a leading coefficient of 1 is to interchange the two equations.

$$\begin{cases} x - y = 0 \\ 3x - 2y = -1 \end{cases}$$ Interchange two equations in the system.

$$\begin{cases} -3x + 3y = 0 \\ 3x - 2y = -1 \end{cases}$$ Multiply the first equation by -3.

$$\begin{array}{r} -3x + 3y = 0 \\ 3x - 2y = -1 \\ \hline y = -1 \end{array}$$ Add the multiple of the first equation to the second equation to obtain a new second equation.

$$\begin{cases} x - y = 0 \\ y = -1 \end{cases}$$ New system in row-echelon form

Now, using back-substitution, you can determine that the solution is $y = -1$ and $x = -1$, which can be written as the ordered pair $(-1, -1)$. Check this in the original system of equations.

✓CHECKPOINT 2

Solve the system of linear equations.

$$\begin{cases} 2x + 4y = 1 \\ x + y = 0 \end{cases}$$ ■

The process of rewriting a system of equations in row-echelon form by using the three basic row operations is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss. Example 2 shows the chain of equivalent systems used to solve a linear system in two variables.

Operations That Produce Equivalent Systems

Each of the following **row operations** on a system of linear equations produces an *equivalent* system of linear equations.

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one of the equations to another equation to replace the latter equation.

Example 3 Using Elimination to Solve a System

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ -x + 3y = -4 & \text{Equation 2} \\ 2x - 5y + 5z = 17 & \text{Equation 3} \end{cases}$$

SOLUTION Because the leading coefficient of Equation 1 is 1, you can begin by saving the x in the upper left position and eliminating the other x-terms from the first column.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that all but the first x have been eliminated from the first column, work on the second column. (You need to eliminate y from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases}$$

Adding the second equation to the third equation produces a new third equation.

Finally, you need a coefficient of 1 for z in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system that was solved in Example 1, and, as in that example, you can conclude that the solution is

$$x = 1, \quad y = -1, \quad \text{and} \quad z = 2.$$

✓CHECKPOINT 3

Solve the system of linear equations.

$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ 3x - z = 0 \end{cases}$$

In Example 3, you can check the solution by substituting $x = 1$, $y = -1$, and $z = 2$ into each original equation, as follows.

Equation 1: $(1) - 2(-1) + 3(2) = 9$ ✓

Equation 2: $-(1) + 3(-1) = -4$ ✓

Equation 3: $2(1) - 5(-1) + 5(2) = 17$ ✓

The next example involves an inconsistent system~~one~~ that has no solution. The key to recognizing an inconsistent system is that at some stage in the elimination process, you obtain a false statement such as $0 = -2$.

Example 4 An Inconsistent System

Solve the system of linear equations.

$$\begin{cases} x - 3y + z = 1 & \text{Equation 1} \\ 2x - y - 2z = 2 & \text{Equation 2} \\ x + 2y - 3z = -1 & \text{Equation 3} \end{cases}$$

SOLUTION

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ x + 2y - 3z = -1 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 5y - 4z = -2 \end{cases}$$

Adding -1 times the first equation to the third equation produces a new third equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 0 = -2 \end{cases}$$

Adding -1 times the second equation to the third equation produces a new third equation.

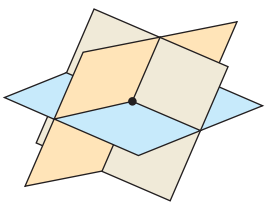
Because $0 = -2$ is a false statement, you can conclude that this system is inconsistent and therefore has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

✓CHECKPOINT 4

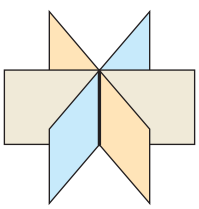
Solve the system of linear equations.

$$\begin{cases} 2x + y - z = 7 \\ x - 2y + 2z = -9 \\ 3x - y + z = 5 \end{cases}$$ ■

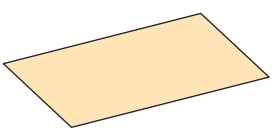
As with a system of linear equations in two variables, the solution(s) of a system of linear equations in more than two variables must fall into one of three categories. Because an equation in three variables represents a plane in space, the possible solutions can be shown graphically. See Figure 5.10.



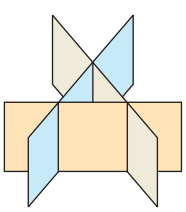
(a) Solution: one point



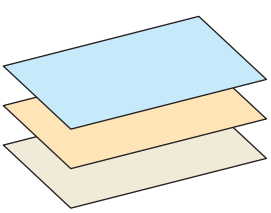
(b) Solution: one line



(c) Solution: one plane



(d) Solution: none



(e) Solution: none

FIGURE 5.10

The Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

1. There is exactly one solution. [See Figure 5.10(a).]
2. There are infinitely many solutions. [See Figures 5.10(b) and (c).]
3. There is no solution. [See Figures 5.10(d) and (e).]

Example 5 A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{cases} x + y - 3z = -1 & \text{Equation 1} \\ y - z = 0 & \text{Equation 2} \\ -x + 2y = 1 & \text{Equation 3} \end{cases}$$

SOLUTION

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 3y - 3z = 0 \end{cases}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 0 = 0 \end{cases}$$

Adding -3 times the second equation to the third equation produces a new third equation.

This means that Equation 3 depends on Equations 1 and 2 in the sense that it gives us no additional information about the variables. Because $0 = 0$ is a true statement, you can conclude that this system has infinitely many solutions. So, the original system is equivalent to the system

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \end{cases}.$$

In this last equation, solve for y in terms of z to obtain $y = z$. Back-substituting for y into the previous equation produces $x = 2z - 1$. Finally, letting $z = a$, the solutions to the original system are all of the form

$$x = 2a - 1, \quad y = a, \quad \text{and} \quad z = a$$

where a is a real number. So, every ordered triple of the form

$$(2a - 1, a, a), \quad a \text{ is a real number}$$

is a solution of the system.

✓CHECKPOINT 5

Solve the system of linear equations.

$$\begin{cases} 2x + y + 3z = 1 \\ 2x + 6y + 12z = 3 \\ 6x + 8y + 18z = 5 \end{cases}$$

In Example 5, there are other ways to write the same infinite set of solutions. For instance, the solutions could have been written as

$$\left(b, \tfrac{1}{2}(b + 1), \tfrac{1}{2}(b + 1)\right), \quad b \text{ is a real number.}$$

To convince yourself that this description produces the same set of solutions, consider the following.

Substitution	*Solution*
$a = 0$	$(2(0) - 1, 0, 0) = (-1, 0, 0)$
$b = -1$	$\left(-1, \tfrac{1}{2}(-1 + 1), \tfrac{1}{2}(-1 + 1)\right) = (-1, 0, 0)$
$a = 1$	$(2(1) - 1, 1, 1) = (1, 1, 1)$
$b = 1$	$\left(1, \tfrac{1}{2}(1 + 1), \tfrac{1}{2}(1 + 1)\right) = (1, 1, 1)$

In both cases, you obtain the same ordered triples. So, when comparing descriptions of an infinite solution set, keep in mind that there is more than one way to describe the set.

Nonsquare Systems

So far, each system of linear equations you have looked at has been **square,** which means that the number of equations is equal to the number of variables. In a **nonsquare** system, the number of equations differs from the number of variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables in the system.

Example 6 A System with Fewer Equations than Variables

Solve the system of linear equations.

$$\begin{cases} x - 2y + z = 2 & \text{Equation 1} \\ 2x - y - z = 1 & \text{Equation 2} \end{cases}$$

SOLUTION Begin by rewriting the system in row-echelon form, as follows.

$$\begin{cases} x - 2y + z = 2 \\ 3y - 3z = -3 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + z = 2 \\ y - z = -1 \end{cases}$$

Multiplying the second equation by $\frac{1}{3}$ produces a new second equation.

Solving for y in terms of z, you obtain $y = z - 1$. Back-substitution into Equation 1 yields

$$\begin{aligned} x - 2(z - 1) + z &= 2 \\ x - 2z + 2 + z &= 2 \\ x &= z. \end{aligned}$$

Finally, by letting $z = a$, you have the solution

$$x = a, \quad y = a - 1, \quad \text{and} \quad z = a$$

where a is a real number. So, every ordered triple of the form

$$(a, a - 1, a), \quad a \text{ is a real number}$$

is a solution of the system. Because there were originally three variables and only two equations, the system cannot have a unique solution.

✓CHECKPOINT 6

Solve the system of linear equations.

$$\begin{cases} 2x - 2y + 5z = 2 \\ 4x - z = 0 \end{cases}$$ ■

In Example 6, try choosing some values of a to obtain different solutions of the system, such as $(1, 0, 1)$, $(2, 1, 2)$, and $(3, 2, 3)$. Then check each of the solutions in the original system. For example, you can check the solution $(1, 0, 1)$ as follows.

Equation 1: $1 - 2(0) + 1 = 2$ ✓

Equation 2: $2(1) - 0 - 1 = 1$ ✓

Applications

Example 7
MAKE A DECISION **An Investment Portfolio**

You have a portfolio totaling $450,000 and want to invest in (1) certificates of deposit, (2) municipal bonds, (3) blue-chip stocks, and (4) growth or speculative stocks. The certificates pay 9% simple annual interest, and the municipal bonds pay 6% simple annual interest. You expect the blue-chip stocks to return 10% simple annual interest and the growth stocks to return 15% simple annual interest. You want a combined annual return of 8% and you also want to have only one-third of the portfolio invested in stocks. How much should be allocated to each type of investment?

SOLUTION Let C, M, B, and G represent the amounts in the four types of investments. Because the total investment is $450,000, you can write the equation

$$C + M + B + G = 450{,}000.$$

A second equation can be derived from the fact that the combined annual return should be 8%

$$0.09C + 0.06M + 0.10B + 0.15G = 0.08(450{,}000)$$

Finally, because only one-third of the total investment should be allocated to stocks, you can write

$$B + G = \tfrac{1}{3}(450{,}000).$$

These three equations make up the following system.

$$\begin{cases} C + M + B + G = 450{,}000 & \text{Equation 1} \\ 0.09C + 0.06M + 0.10B + 0.15G = 36{,}000 & \text{Equation 2} \\ B + G = 150{,}000 & \text{Equation 3} \end{cases}$$

Using elimination, you find that the system has infinitely many solutions, which can be written as follows.

$$C = -\tfrac{5}{3}a + 100{,}000$$
$$M = \tfrac{5}{3}a + 200{,}000$$
$$B = -a + 150{,}000$$
$$G = a$$

So, you have many different options. One possible solution is to choose $a = 30{,}000$, which yields the following portfolio.

1. Certificates of deposit: $0,000
2. Municipal bonds: $350,000
3. Blue-chip stocks: $20,000
4. Growth or speculative stocks: $0,000

✓CHECKPOINT 7

In Example 7, suppose the total investment is $60,000. How much should be allocated to each type of investment? ■

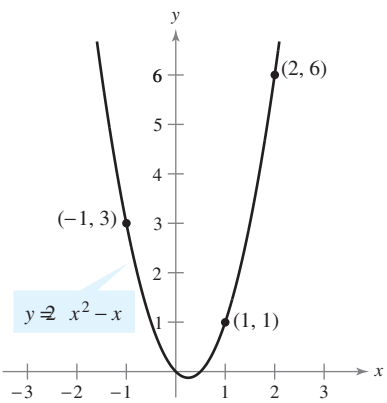


FIGURE 5.11

Example 8 Data Analysis: Curve-Fitting

Find a quadratic equation, $y = ax^2 + bx + c$, whose graph passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$.

SOLUTION Because the graph of $y = ax^2 + bx + c$ passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$, you can write the following.

When $x = -1$, $y = 3$: $a(-1)^2 + b(-1) + c = 3$

When $x = 1$, $y = 1$: $a(1)^2 + b(1) + c = 1$

When $x = 2$, $y = 6$: $a(2)^2 + b(2) + c = 6$

This produces the following system of linear equations.

$$\begin{cases} a - b + c = 3 & \text{Equation 1} \\ a + b + c = 1 & \text{Equation 2} \\ 4a + 2b + c = 6 & \text{Equation 3} \end{cases}$$

The solution of this system is $a = 2$, $b = -1$, and $c = 0$. So, the equation of the parabola is $y = 2x^2 - x$, as shown in Figure 5.11.

✓CHECKPOINT 8

Find a quadratic equation, $y = ax^2 + bx + c$, whose graph passes through the points $(-1, 7)$, $(1, 3)$, and $(2, 7)$. ■

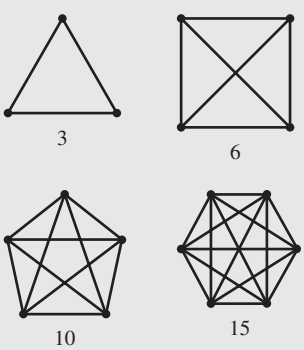
CONCEPT CHECK

1. The ordered triple (a, b, c) is the solution of system A and system B. Are the systems equivalent? Explain.
2. Using Gaussian elimination to solve a system of three linear equations produces $0 = -2$. What does this tell you about the graphs of the equations in the system?
3. Describe the solution set of a system of equations with two equations in three variables.
4. The graph of the quadratic equation $y = 2x^2 + 3x - 1$ passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). Describe the solution set of the given system.

$$\begin{cases} ax_1^2 + bx_1 + c = y_1 \\ ax_2^2 + bx_2 + c = y_2 \\ ax_3^2 + bx_3 + c = y_3 \end{cases}$$

DISCOVERY

The total numbers of sides and diagonals of regular polygons with three, four, and five sides are three, six, and ten, respectively, as shown in the figure.



Find a quadratic function, $y = ax^2 + bx + c$, where y represents the total number of sides and diagonals and x represents the number of sides, that fits these data. Check to see if the quadratic function gives the correct answers for a polygon with six sides.

Skills Review 5.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.2, 2.1, 5.1, and 5.2.

In Exercises 1–4, solve the system of linear equations.

1. $\begin{cases} x + y = 25 \\ y = 10 \end{cases}$

2. $\begin{cases} 2x - 3y = 4 \\ 6x = -12 \end{cases}$

3. $\begin{cases} x + y = 32 \\ x - y = 24 \end{cases}$

4. $\begin{cases} 2r - s = 5 \\ r + 2s = 10 \end{cases}$

In Exercises 5–8, determine whether the ordered triple is a solution of the equation.

5. $5x - 3y + 4z = 2$

$(-1, -2, 1)$

6. $x - 2y + 12z = 9$

$(6, 3, 2)$

7. $2x - 5y + 3z = -9$

$(a - 2, a + 1, a)$

8. $-5x + y + z = 21$

$(a - 4, 4a + 1, a)$

In Exercises 9 and 10, solve for x in terms of a.

9. $x + 2y - 3z = 4$

$y = 1 - a, z = a$

10. $x - 3y + 5z = 4$

$y = 2a + 3, z = a$

Exercises 5.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, match each system of equations with its solution. [The solutions are labeled (a), (b), (c), and (d).]

1. $\begin{cases} -2x + 3y - 2z = 5 \\ 3x - 4y + z = -1 \\ x + 2y + 5z = -11 \end{cases}$

2. $\begin{cases} 5x + 2y - 4z = -17 \\ -8x + y - 5z = -7 \\ 4x + 3y - z = -7 \end{cases}$

3. $\begin{cases} 2x + 5y - 7z = 36 \\ x + 6y - 10z = 38 \\ x - 4y + 8z = -18 \end{cases}$

4. $\begin{cases} -x - 2y + 5z = 23 \\ -3x + y + 6z = 17 \\ 9x + 2y - 7z = -1 \end{cases}$

(a) $(-1, 0, 3)$

(b) $(6, 2, -2)$

(c) $(2, 1, -3)$

(d) $(4, -1, 5)$

In Exercises 5–8, determine whether the system of equations is in row–echelon form. Justify your answer.

5. $\begin{cases} x + 3y - 7z = -11 \\ y - 2z = -3 \\ z = 2 \end{cases}$

6. $\begin{cases} x - y + 3z = -11 \\ y + 8z = -12 \\ z = -2 \end{cases}$

7. $\begin{cases} x - 9y + z = 22 \\ 2y + z = -3 \\ z = 1 \end{cases}$

8. $\begin{cases} x - y - 8z = 12 \\ 2y - 2z = 2 \\ 7z = -7 \end{cases}$

In Exercises 9 and 10, use back-substitution to solve the system of linear equations.

9. $\begin{cases} x - y + z = 4 \\ 2y + z = -6 \\ z = -2 \end{cases}$

10. $\begin{cases} 4x - 2y + z = 8 \\ -y + z = 4 \\ z = 2 \end{cases}$

In Exercises 11–36, solve the system of equations.

11. $\begin{cases} 4x + y - 3z = 11 \\ 2x - 3y + 2z = 9 \\ x + y + z = -3 \end{cases}$

12. $\begin{cases} 6y + 4z = -12 \\ 3x + 3y = 9 \\ 2x - 3z = 10 \end{cases}$

13. $\begin{cases} 3x + 2z = 13 \\ x + 2y + z = -5 \\ -3y - z = 10 \end{cases}$

14. $\begin{cases} 2x + 3y + z = -4 \\ 2x - 4y + 3z = 18 \\ 3x - 2y + 2z = 9 \end{cases}$

15. $\begin{cases} 3x - 2y + 4z = 1 \\ x + y - 2z = 3 \\ 2x - 3y + 6z = 8 \end{cases}$

16. $\begin{cases} 5x - 3y + 2z = 3 \\ 2x + 4y - z = 7 \\ x - 11y + 4z = 3 \end{cases}$

17. $\begin{cases} 3x + 3y + 5z = 1 \\ 3x + 5y + 9z = 0 \\ 5x + 9y + 17z = 0 \end{cases}$

18. $\begin{cases} 2x + y - z = 13 \\ x + 2y + z = 2 \\ 8x - 3y + 4z = -2 \end{cases}$

19. $\begin{cases} x + 2y - 7z = -4 \\ 2x + y + z = 13 \\ 3x + 9y - 36z = -33 \end{cases}$

20. $\begin{cases} 2x + y - 3z = 4 \\ 4x + 2z = 10 \\ -2x + 3y - 13z = -8 \end{cases}$

21. $\begin{cases} x + 4z = 13 \\ 4x - 2y + z = 7 \\ 2x - 2y - 7z = -19 \end{cases}$

22. $\begin{cases} 4x - y + 5z = 11 \\ x + 2y - z = 5 \\ 5x - 8y + 13z = 7 \end{cases}$

23. $\begin{cases} x + 4z = 1 \\ x + y + 10z = 10 \\ 2x - y + 2z = -5 \end{cases}$

24. $\begin{cases} 3x - 2y - 6z = 4 \\ -3x + 2y + 6z = 1 \\ x - y - 5z = 3 \end{cases}$

25. $\begin{cases} 4x + 3y + 5z = 10 \\ 5x + 2y + 10z = 13 \\ 3x + y - 2z = -9 \end{cases}$

26. $\begin{cases} 2x + 5y = 25 \\ 3x - 2y + 4z = 1 \\ 4x - 3y + z = 9 \end{cases}$

27. $\begin{cases} 2x + 3y - z = 1 \\ x - 2y + z = 7 \\ 3x + y + 2z = 12 \end{cases}$

28. $\begin{cases} 2x + 3y = 0 \\ 4x + 3y - z = 0 \\ 8x + 3y + 3z = 0 \end{cases}$

29. $\begin{cases} 12x + 5y + z = 0 \\ 12x + 4y - z = 0 \end{cases}$

30. $\begin{cases} x - 2y + 5z = 2 \\ 3x + 2y - z = -2 \end{cases}$

31. $\begin{cases} x - 3y + 2z = 18 \\ 5x - 13y + 12z = 80 \end{cases}$

32. $\begin{cases} 2x - 3y + z = -2 \\ -4x + 9y = 7 \end{cases}$

33. $\begin{cases} 3x - 3y + 6z = 7 \\ -x + y - 2z = 3 \\ 2x + 3y - 4z = 8 \end{cases}$

34. $\begin{cases} 4x + 3y = 7 \\ x - 2y + z = 0 \\ -2x + 4y - 2z = 13 \end{cases}$

35. $\begin{cases} x + 3w = 4 \\ 2y - z - w = 0 \\ 3y - 2w = 1 \\ 2x - y + 4z = 5 \end{cases}$

36. $\begin{cases} x + y + z + w = 6 \\ 2x + 3y - w = 0 \\ -3x + 4y + z + 2w = 4 \\ x + 2y - z + w = 0 \end{cases}$

In Exercises 37–40, find two systems of equations that have the ordered triple as a solution. (There are many correct answers.)

37. $(3, -1, 2)$

38. $\left(-\frac{1}{2}, -2, 4\right)$

39. $(1, -5, -3)$

40. $\left(0, 2, \frac{1}{2}\right)$

In Exercises 41–44, write three ordered triples of the given form.

41. $\left(a, a - 5, \frac{2}{3}a + 1\right)$

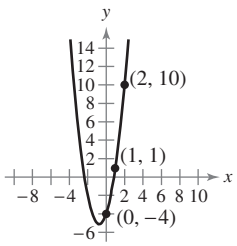
42. $(3a, 5 - a, a)$

43. $\left(\frac{1}{2}a, 3a, 5\right)$

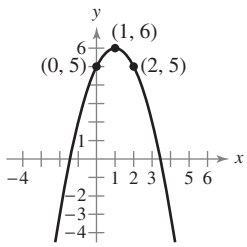
44. $\left(-\frac{1}{2}a + 5, -1, a\right)$

In Exercises 45–48, find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points.

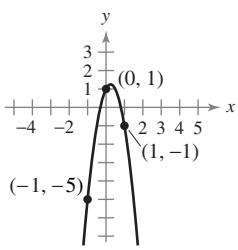
45.



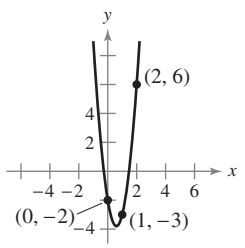
46.



47.

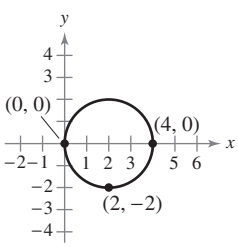


48.

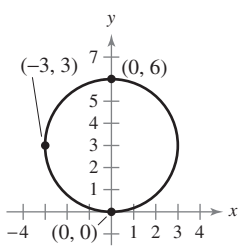


In Exercises 49–52, find the equation of the circle $x^2 + y^2 + Dx + Ey + F = 0$ that passes through the points.

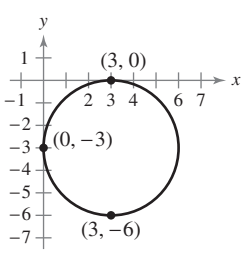
49.



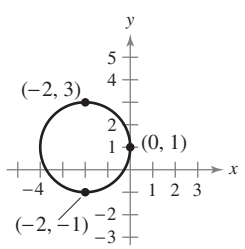
50.



51.



52.



53. Investment A real estate company borrows $500,000. Some of the money is borrowed at 7% some at 8% and some at 10% simple annual interest. How much is borrowed at each rate when the total annual interest is $47,000 and the amount borrowed at 8% is the same as the amount borrowed at 10%

54. Investment A clothing company borrows $300,000. Some of the money is borrowed at 8% some at 9% and some at 10% simple annual interest. How much is borrowed at each rate when the total annual interest is $30,500 and the amount borrowed at 8% is three times the amount borrowed at 10%

55. Candles A candle company sells three types of candles for \$5, \$0, and \$ per unit. In one year, the total revenue for the three products was \$50,000, which corresponded to the sale of 50,000 units. The company sold half as many units of the \$5 candles as units of the \$0 candles. How many units of each type of candle were sold?

56. Hair Products A hair product company sells three types of hair products for \$0, \$0, and \$0 per unit. In one year, the total revenue for the three products was \$00,000, which corresponded to the sale of 40,000 units. The company sold half as many units of the \$0 product as units of the \$0 product. How many units of each product were sold?

57. Crop Spraying A mixture of 5 gallons of chemical A, 8 gallons of chemical B, and 12 gallons of chemical C is required to kill a crop destroying insect. Commercial spray X contains 1, 2, and 3 parts of these chemicals, respectively. Commercial spray Y contains only chemical C. Commercial spray Z contains chemicals A, B, and C in equal amounts. How much of each type of commercial spray is needed to obtain the desired mixture?

58. Acid Mixture A chemist needs 10 liters of a 25% acid solution. The solution is to be mixed from three solutions whose acid concentrations are 10%20%and 50% How many liters of each solution should the chemist use to satisfy the following?

(a) Use as little as possible of the 50%olution.

(b) Use as much as possible of the 50%olution.

(c) Use 2 liters of the 50%olution.

MAKE A DECISION: INVESTMENT PORTFOLIO In Exercises 59 and 60, you have a total of \$500,000 that is to be invested in (1) certificates of deposit, (2) municipal bonds, (3) blue-chip stocks, and (4) growth or speculative stocks. How much should be put in each type of investment?

59. The certificates of deposit pay 2.5%imple annual interest, and the municipal bonds pay 10%imple annual interest. Over a five-year period, you expect the blue-chip stocks to return 12%imple annual interest and the growth stocks to return 18%imple annual interest. You want a combined annual return of 10%and you also want to have only one-fourth of the portfolio invested in stocks.

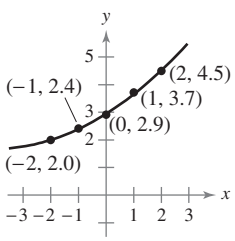
60. The certificates of deposit pay 3%imple annual interest, and the municipal bonds pay 10%imple annual interest. Over a five-year period, you expect the blue-chip stocks to return 12%imple annual interest and the growth stocks to return 15%imple annual interest. You want a combined annual return of 10%and you also want to have only one-fourth of the portfolio invested in stocks.

Fitting a Parabola to Data In Exercises 61–64, find the least squares regression parabola

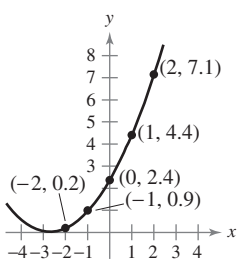
$y = ax^2 + bx + c$

for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by solving the system of linear equations for a, b, and c.

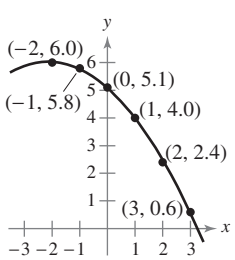
61.
$$\begin{cases} 5c + 10a = 15.5 \\ 10b = 6.3 \\ 10c + 34a = 32.1 \end{cases}$$



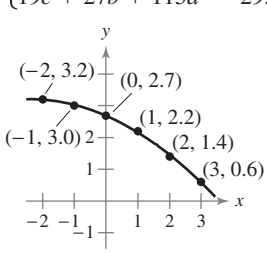
62.
$$\begin{cases} 5c + 10a = 15.0 \\ 10b = 17.3 \\ 10c + 34a = 34.5 \end{cases}$$



63.
$$\begin{cases} 6c + 3b + 19a = 23.9 \\ 3c + 19b + 27a = -7.2 \\ 19c + 27b + 115a = 48.8 \end{cases}$$



64.
$$\begin{cases} 6c + 3b + 19a = 13.1 \\ 3c + 19b + 27a = -2.6 \\ 19c + 27b + 115a = 29.0 \end{cases}$$



65. Sailboats The total numbers y (in thousands) of sailboats purchased in the United States in the years 2001 to 2005 are shown in the table. In the table, x represents the year, with $x = 0$ corresponding to 2003. *(Source: National Marine Manufacturers Association)*

Year, x	Number, y
-2	18.6
-1	15.8
0	15.0
1	14.3
2	14.4

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the following system.

$$\begin{cases} 5c + 10a = 78.1 \\ 10b = -9.9 \\ 10c + 34a = 162.1 \end{cases}$$

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Compare the quadratic model with the model found in part (a).

66. Genetically Modified Soybeans The global areas y (in millions of hectares) of genetically modified soybean crops planted in the years 2002 to 2006 are shown in the table. In the table, x represents the year, with $x = 0$ corresponding to 2004. *(Source: ISAAA, Clive James, 2006)*

Year, x	Area, y
-2	36.5
-1	41.4
0	48.4
1	54.4
2	58.6

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the following system.

$$\begin{cases} 5c + 10a = 239.3 \\ 10b = 57.2 \\ 10c + 34a = 476.2 \end{cases}$$

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Compare the quadratic model with the model found in part (a).

67. Federal Debt The values of the federal debt of the United States as percents of the Gross Domestic Product (GDP) for the years 2001 to 2005 are shown in the table. In the table, x represents the year, with $x = 0$ corresponding to 2002. *(Source: U.S. Office of Management and Budget)*

Year, x	%f GDP
-1	57.4
0	59.7
1	62.6
2	63.7
3	64.3

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the following system.

$$\begin{cases} 5c + 5b + 15a = 307.7 \\ 5c + 15b + 35a = 325.5 \\ 15c + 35b + 99a = 953.5 \end{cases}$$

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Compare the quadratic model with the model found in part (a).

(c) Use either model to predict the federal debt as a percent of the GDP in 2007.

68. Revenues Per Share The revenues per share (in dollars) for Panera Bread Company for the years 2002 to 2006 are shown in the table. In the table, x represents the year, with $x = 0$ corresponding to 2003. *(Source: Panera Bread Company)*

Year, x	Revenues per share
-1	9.47
0	11.85
1	15.72
2	20.49
3	26.11

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the following system.

$$\begin{cases} 5c + 5b + 15a = 83.64 \\ 5c + 15b + 35a = 125.56 \\ 15c + 35b + 99a = 342.14 \end{cases}$$

(T) (b) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Compare the quadratic model with the model found in part (a).

(c) Use either model to predict the revenues per share in 2008 and 2009.

69. MAKE A DECISION: STOPPING DISTANCE In testing of the new braking system of an automobile, the speed (in miles per hour) and the stopping distance (in feet) were recorded in the table below.

Speed, x	Stopping distance, y
30	54
40	116
50	203
60	315
70	452

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the following system.

$$\begin{cases} 5c + 250b + 13{,}500a = 1140 \\ 250c + 13{,}500b + 775{,}000a = 66{,}950 \\ 13{,}500c + 775{,}000b + 46{,}590{,}000a = 4{,}090{,}500 \end{cases}$$

(T) (b) Use the *regression* feature of a graphing utility to check your answer to part (a).

(c) A car design specification requires the car to stop within 520 feet when traveling 75 miles per hour. Does the new braking system meet this specification?

70. Sound Recordings The percents of sound recordings purchased over the Internet (not including digital downloads) in the years 1999 to 2005 are shown in the table. In the table, x represents the year, with $x = 0$ corresponding to 2000. *(Source: The Recording Industry Association of America)*

Year, x	Percent of sound recordings, y
-1	2.4
0	3.2
1	2.9
2	3.4
3	5.0
4	5.9
5	8.2

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the following system.

$$\begin{cases} 7c + 14b + 56a = 31.0 \\ 14c + 56b + 224a = 86.9 \\ 56c + 224b + 980a = 363.3 \end{cases}$$

(T) (b) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Compare the quadratic model with the model found in part (a).

(c) Use either model to predict the percent of Internet sales in 2008. Does your result seem reasonable? Explain.

71. Reasoning Is it possible for a square linear system to have no solution? Explain.

72. Reasoning Is it possible for a square linear system to have infinitely many solutions? Explain.

73. One solution for Exercise 30 is $(-a, 2a - 1, a)$. A student gives $(b, -2b - 1, -b)$ as a solution to the same exercise. Explain why both solutions are correct.

74. Extended Application To work an extended application analyzing the sales per share of Wal-Mart, visit this text's website at *college.hmco.com*. *(Source: Wal-Mart Stores, Inc.)*

Business Capsule



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SAS is a leader in business software and services. Using SAS forecasting technologies, customers can accurately analyze and forecast processes that take place over time. SAS/ETS software contains popular forecasting methods such as regression analysis and trend extrapolation.

75. Research Project Use your campus library, the Internet, or some other reference source to find information about a company or small business that generates software which uses regression analysis to predict trends. Write a brief paper about the company or small business.

Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, solve the system algebraically. Use a graphing utility to verify your solution.

1. $\begin{cases} 3x + y = 11 \\ x - 2y = -8 \end{cases}$

2. $\begin{cases} 4x + 8y = 8 \\ x + 2y = 6 \end{cases}$

3. $\begin{cases} x + y = 4 \\ y = 2\sqrt{x} + 1 \end{cases}$

4. $\begin{cases} x^2 + y^2 = 9 \\ y = 2x + 1 \end{cases}$

In Exercises 5 and 6, find the number of units x that need to be sold to break even.

5. $C = 10.50x + 9000, R = 16.50x$

6. $C = 3.79x + 400{,}000, R = 4.59x$

In Exercises 7 and 8, solve the system by substitution or elimination.

7. $\begin{cases} 2.5x - y = 6 \\ 3x + 4y = 2 \end{cases}$

8. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 1 \\ x - 2y = -2 \end{cases}$

9. Find the point of equilibrium for the pair of supply and demand equations. Verify your solution graphically.

Demand: $p = 50 - 0.002x$

Supply: $p = 20 + 0.004x$

10. The total numbers y (in millions) of Medicare enrollees in the years 2001 to 2005 are shown in the table at the left. In the table, x represents the year, with $x = 0$ corresponding to 2001. Solve the following system for a and b to find the least squares regression line $y = ax + b$ for the data. *(Source: U.S. Centers for Medicare and Medicaid Services)*

$$\begin{cases} 5b + 10a = 206.2 \\ 10b + 30a = 418.6 \end{cases}$$

Year, x	Number, y
0	40.1
1	40.5
2	41.2
3	41.9
4	42.5

Table for 10

In Exercises 11–13, solve the system of equations.

11. $\begin{cases} 2x + 3y - z = -7 \\ x + 3z = 10 \\ 2y + z = -1 \end{cases}$

12. $\begin{cases} x + y - 2z = 12 \\ 2x - y - z = 6 \\ y - z = 6 \end{cases}$

13. $\begin{cases} 3x + 2y + z = 17 \\ -x + y + z = 4 \\ x - y - z = 3 \end{cases}$

14. The average prices y (in dollars) of retail prescription drugs for the years 2001 to 2005 are shown in the table at the left. In the table, x represents the year, with $x = 0$ corresponding to 2003. Solve the following system for a, b, and c to find the least squares regression parabola $y = ax^2 + bx + c$ for the data. *(Source: National Association of Chain Drug Stores)*

$$\begin{cases} 5c + 10a = 293.40 \\ 10b = 37.82 \\ 10c + 34a = 578.64 \end{cases}$$

Year, x	Average price, y
-2	50.06
-1	55.37
0	59.52
1	63.59
2	64.86

Table for 14

Section 5.4

Systems of Inequalities

- Sketch the graph of an inequality in two variables.
- Solve a system of inequalities.
- Construct and use a system of inequalities to solve an application problem.

The Graph of an Inequality

The following statements are inequalities in two variables:

$$3x - 2y < 6 \quad \text{and} \quad 2x^2 + 3y^2 \geq 6.$$

An ordered pair (a, b) is a **solution of an inequality** in x and y if the inequality is true when a and b are substituted for x and y, respectively. The **graph of an inequality** is the collection of all solutions of the inequality. To sketch the graph of an inequality, begin by sketching the graph of the *corresponding equation.* The graph of the equation will normally separate the plane into two or more regions. In each such region, one of the following must be true.

1. *All* points in the region are solutions of the inequality.

2. *No* point in the region is a solution of the inequality.

So, you can determine whether the points in an entire region satisfy the inequality simply by testing *one* point in the region.

Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign by an equal sign, and sketch the graph of the resulting equation. (Use a dashed line for $<$ or $>$ and a solid line for $\leq$ or $\geq$.)

2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, shade the entire region to denote that every point in the region satisfies the inequality.

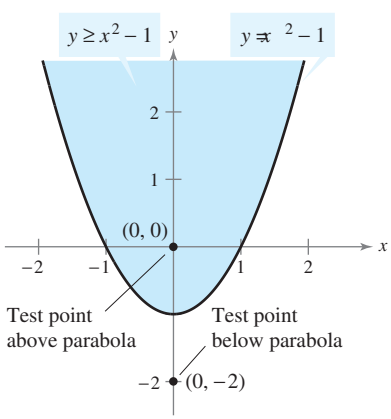


FIGURE 5.12

Example 1 Sketching the Graph of an Inequality

Sketch the graph of the inequality $y \geq x^2 - 1$.

SOLUTION The graph of the corresponding *equation* $y = x^2 - 1$ is a parabola, as shown in Figure 5.12. By testing a point *above* the parabola $(0, 0)$ and a point *below* the parabola $(0, -2)$, you can see that the points that satisfy the inequality are those lying above (or on) the parabola.

✓CHECKPOINT 1

Sketch the graph of $y < x^2 + 2$. ■

The inequality in Example 1 is a nonlinear inequality in two variables. Most of the following examples involve **linear inequalities** such as $ax + by < c$ (a and b are not both zero). The graph of a linear inequality is a half-plane lying on one side of the line $ax + by = c$. The simplest linear inequalities are those corresponding to horizontal or vertical lines, as shown in Example 2.

Example 2 Sketching the Graph of a Linear Inequality

Sketch the graph of each linear inequality.

a. $x > -2$ **b.** $y \leq 3$

SOLUTION

a. The graph of the corresponding equation $x = -2$ is a vertical line. The points that satisfy the inequality $x > -2$ are those lying to the right of this line, as shown in Figure 5.13.

b. The graph of the corresponding equation $y = 3$ is a horizontal line. The points that satisfy the inequality $y \leq 3$ are those lying below (or on) this line, as shown in Figure 5.14.

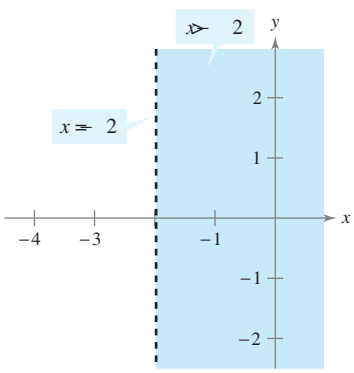


FIGURE 5.13

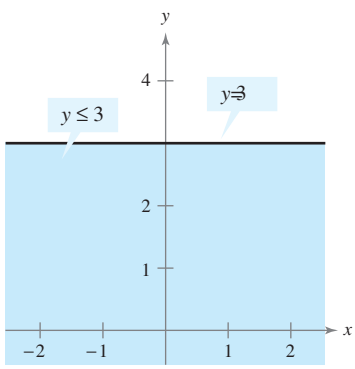


FIGURE 5.14

STUDY TIP

To graph a linear inequality, it can help to write the inequality in slope-intercept form. For instance, by writing $x - y < 2$ in the form

$$y > x - 2$$

you can see that the solution points lie above the line $x - y = 2$ (or $y = x - 2$), as shown in Figure 5.15.

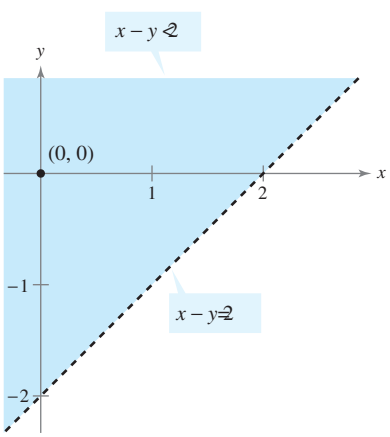


FIGURE 5.15

✓CHECKPOINT 2

Sketch the graph of $x \geq -1$. ■

Example 3 Sketching the Graph of a Linear Inequality

Sketch the graph of $x - y < 2$.

SOLUTION The graph of the corresponding equation $x - y = 2$ is a line, as shown in Figure 5.15. Because the origin $(0, 0)$ satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line. Regardless of which point you choose, you will see that it does not satisfy the inequality.)

✓CHECKPOINT 3

Sketch the graph of $x + y \leq 1$. ■

Systems of Inequalities

A **solution** of a system of inequalities in x and y is a point (x, y) that satisfies each inequality in the system. To solve a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is *common* to every graph in the system. This region represents the **solution set** of the system. For systems of *linear* inequalities, it is helpful to find the vertices of the solution region.

Example 4 Solving a System of Inequalities

Solve the system of linear inequalities.

$$\begin{cases} x - y < 2 & \text{Inequality 1} \\ x > -2 & \text{Inequality 2} \\ y \leq 3 & \text{Inequality 3} \end{cases}$$

SOLUTION To solve the system, sketch the graph of the solution set. The graphs of these inequalities are shown in Figures 5.15, 5.13, and 5.14 on page 442. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate plane, as shown in Figure 5.16. To find the vertices of the region, solve the three systems of equations obtained by taking the *pairs* of equations representing the boundaries of the individual regions.

Vertex A: $(-2, -4)$
Obtained by solving the system

$$\begin{cases} x - y = 2 \\ x = -2 \end{cases}$$

Vertex B: $(5, 3)$
Obtained by solving the system

$$\begin{cases} x - y = 2 \\ y = 3 \end{cases}$$

Vertex C: $(-2, 3)$
Obtained by solving the system

$$\begin{cases} x = -2 \\ y = 3 \end{cases}$$

STUDY TIP

Using different colored pencils to shade the solutions of the inequalities in a system makes identifying the solution of the system of inequalities easier. The region common to every graph in the system is where all the shaded regions overlap.

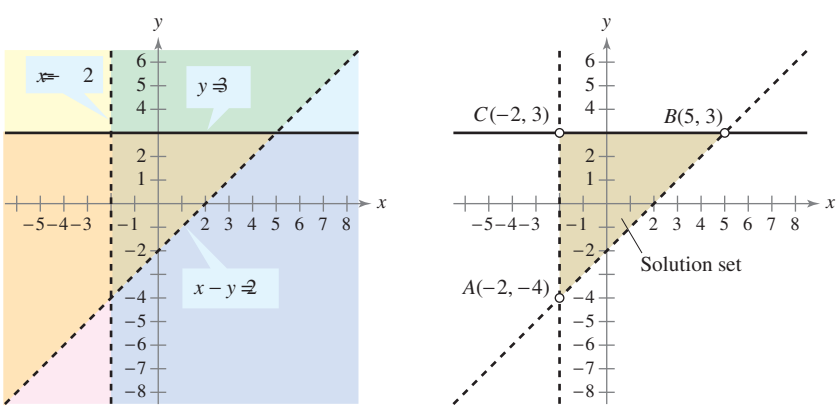


FIGURE 5.16

✓CHECKPOINT 4

Solve the system of linear inequalities.

$$\begin{cases} 2x - y > -3 \\ x \leq 2 \\ y > -2 \end{cases}$$

For the triangular region shown in Figure 5.16, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is *not* a vertex of the region, as shown in Figure 5.17. In order to determine which points of intersection are actually vertices of the region, you should sketch the region and refer to your sketch as you find each point of intersection.

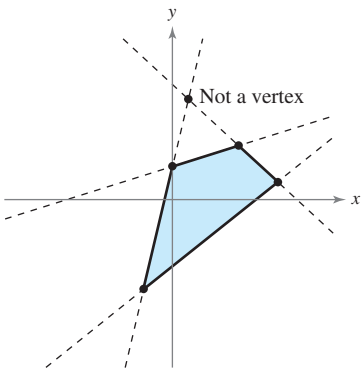


FIGURE 5.17 Boundary lines can intersect at a point that is not a vertex.

Example 5 Solving a System of Inequalities

Solve the system of inequalities.

$$\begin{cases} x^2 - y \le 1 & \text{Inequality 1} \\ -x + y \le 1 & \text{Inequality 2} \end{cases}$$

SOLUTION To solve the system, sketch the graph of the solution set. As shown in Figure 5.18, the points that satisfy the inequality $x^2 - y \le 1$ are the points lying above (or on) the parabola given by

$$y = x^2 - 1. \qquad \text{Parabola}$$

The points satisfying the inequality $-x + y \le 1$ are the points lying below (or on) the line given by

$$y = x + 1. \qquad \text{Line}$$

To find the points of intersection of the parabola and the line, solve the system of corresponding equations.

$$\begin{cases} x^2 - y = 1 \\ -x + y = 1 \end{cases}$$

Using the method of substitution, you can find the points of intersection to be $(-1, 0)$ and $(2, 3)$. The graph of the solution set of the system is shown in Figure 5.18.

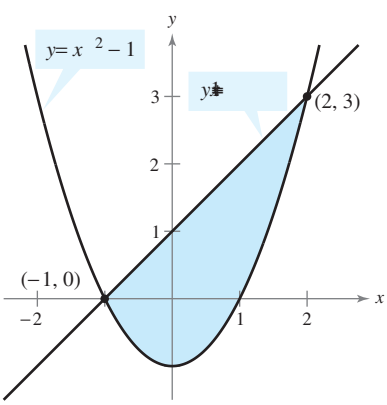


FIGURE 5.18

✓CHECKPOINT 5

Solve the system of linear inequalities.

$$\begin{cases} x^2 + y < 3 \\ x + y > -3 \end{cases}$$

When solving a system of inequalities, you should be aware that the system might have no solution. For instance, the system

$$\begin{cases} x + y > 3 \\ x + y < -1 \end{cases}$$

has no solution points, because the quantity $(x + y)$ cannot be less than -1 *and* greater than 3, as shown in Figure 5.19.

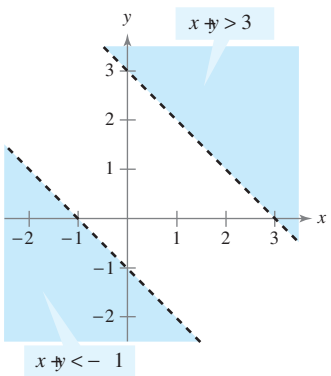


FIGURE 5.19 No Solution

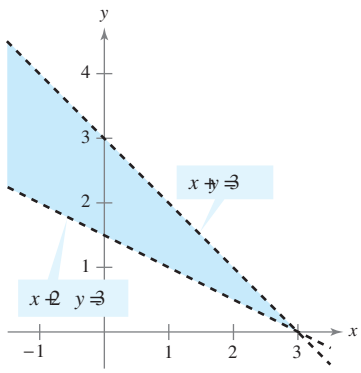


FIGURE 5.20 Unbounded Region

Another possibility is that the solution set of a system of inequalities can be unbounded. For instance, the solution set of

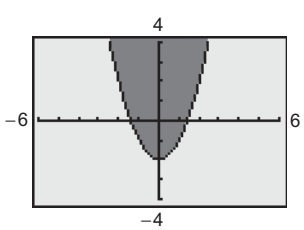
$$\begin{cases} x + y < 3 \\ x + 2y > 3 \end{cases}$$

forms an *infinite wedge,* as shown in Figure 5.20.

TECHNOLOGY

Inequalities and Graphing Utilities

A graphing utility can be used to graph an inequality. The graph of $y \geq x^2 - 2$ is shown below.



Use your graphing utility to graph each inequality given below. For specific keystrokes on how to graph inequalities and systems of inequalities, go to the text website at *college.hmco.com/info/larsonapplied*.

a. $y \leq 2x + 2$ **b.** $y \geq \frac{1}{2}x^2 - 4$ **c.** $y \leq x^3 - 4x^2 + 4$

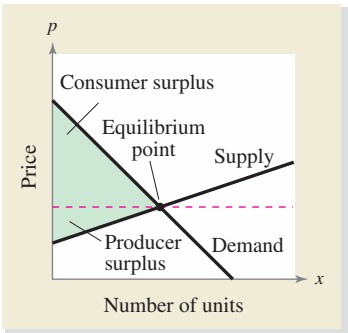


FIGURE 5.21

Applications

Example 8 in Section 5.2 discussed the *point of equilibrium* for a demand function and a supply function. The next example discusses two related concepts that economists call **consumer surplus** and **producer surplus.** As shown in Figure 5.21, the consumer surplus is defined as the area of the region that lies *below* the demand graph, *above* the horizontal line passing through the equilibrium point, and to the right of the p-axis. Similarly, the producer surplus is defined as the area of the region that lies *above* the supply graph, *below* the horizontal line passing through the equilibrium point, and to the right of the p-axis. The consumer surplus is a measure of the amount that consumers would have been willing to pay *above what they actually paid,* whereas the producer surplus is a measure of the amount that producers would have been willing to receive *below what they actually received.*

Example 6 Consumer and Producer Surpluses

The demand and supply equations for a DVD are given by

$$\begin{cases} p = 35 - 0.0001x & \text{Demand equation} \\ p = 8 + 0.0001x & \text{Supply equation} \end{cases}$$

where p is the price (in dollars) and x represents the number of DVDs. Find the consumer surplus and producer surplus for these two equations.

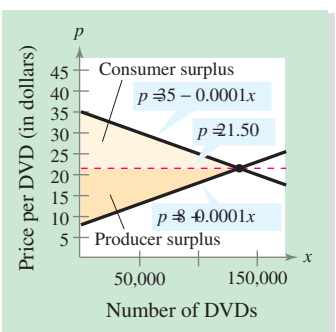


FIGURE 5.22

SOLUTION In Example 8 in Section 5.2, you saw that the point of equilibrium for these equations is

$$(135{,}000,\ 21.50).$$

So, the horizontal line passing through this point is $p = 21.50$. Now you can determine that the consumer surplus and producer surplus are the areas of the triangular regions given by the following systems of inequalities, respectively.

Consumer Surplus

$$\begin{cases} p \le 35 - 0.0001x \\ p \ge 21.50 \\ x \ge 0 \end{cases}$$

Producer Surplus

$$\begin{cases} p \ge 8 + 0.0001x \\ p \le 21.50 \\ x \ge 0 \end{cases}$$

In Figure 5.22, you can see that the consumer and producer surpluses are defined as the areas of the shaded triangles. The base of the triangle representing the consumer surplus is 135,000 because the x-value of the point of equilibrium is 135,000. To find the height of this triangle, subtract the p-value of the point of equilibrium, 21.50, from the p-intercept of the demand equation, 35, to obtain 13.50. You can find the base and height of the triangle representing the producer surplus in a similar manner.

$$\text{Consumer surplus} = \tfrac{1}{2}(\text{base})(\text{height}) = \tfrac{1}{2}(135{,}000)(13.50) = \$11{,}250$$

$$\text{Producer surplus} = \tfrac{1}{2}(\text{base})(\text{height}) = \tfrac{1}{2}(135{,}000)(13.50) = \$11{,}250.$$

✓CHECKPOINT 6

In Example 6, suppose the supply equation is given by $p = 9 + 0.0001x$ and the new point of equilibrium is (130,000, 22). Find the consumer surplus and producer surplus. ■



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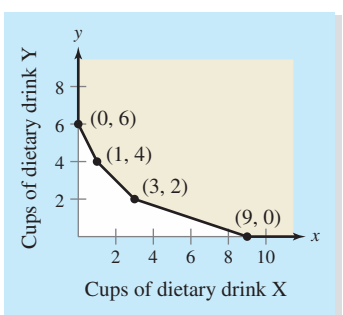


FIGURE 5.23

Example 7 Nutrition

The liquid portion of a diet requires at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C.

a. Set up a system of linear inequalities that describes how many cups of each drink should be consumed each day to meet or exceed the minimum daily requirements for calories and vitamins.

b. A nutritionist normally gives a patient 6 cups of dietary drink X and 1 cup of dietary drink Y per day. Supplies on dietary drink X are running low. Use the graph of the system of linear inequalities to determine other combinations of drinks X and Y that can be given that will meet the minimum daily requirements.

SOLUTION

a. Begin by letting x represent the number of cups of drink X and letting y be the number of cups of drink Y. To meet or exceed the minimum daily requirements, the following inequalities must be satisfied.

$$\begin{cases} 60x + 60y \geq 300 & \text{Calories} \\ 12x + 6y \geq 36 & \text{Vitamin A} \\ 10x + 30y \geq 90 & \text{Vitamin C} \\ x \geq 0 & \\ y \geq 0 & \end{cases}$$

The last two inequalities are included because x and y cannot be negative. The graph of this system of inequalities is shown in Figure 5.23. (More is said about this application in Example 5 in Section 5.5.)

b. From Figure 5.23, there are many different possible substitutions that the nutritionist can make. Because supplies are running low on dietary drink X, the nutritionist should choose a combination that contains a small amount of drink X. For instance, 1 cup of dietary drink X and 4 cups of dietary drink Y will also meet the minimum requirements.

✓CHECKPOINT 7

In Example 7, should the nutritionist give a patient 4 cups of dietary drink X and 1 cup of dietary drink Y? Explain. ■

CONCEPT CHECK

1. How do solid lines and dashed lines differ in representing the solution set of the graph of an inequality?
2. When sketching the graph of $y > x$, does testing the point (1, 1) help you determine which region to shade? Explain.
3. Under what circumstances are the vertices of the graph of a solution set represented by open dots?
4. Can a system of inequalities have only one solution? Justify your answer.

Skills Review 5.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.1, 2.2, 3.1, 5.1, and 5.2.

In Exercises 1–6, classify the graph of the equation as a line, a parabola, or a circle.

1. $x + y = 3$

2. $y = x^2 - 4$

3. $x^2 + y^2 = 9$

4. $y = -x^2 + 1$

5. $4x - y = 8$

6. $y^2 = 16 - x^2$

In Exercises 7–10, solve the system of equations.

7. $\begin{cases} x + 2y = 3 \\ 4x - 7y = -3 \end{cases}$

8. $\begin{cases} 2x - 3y = 4 \\ x + 5y = 2 \end{cases}$

9. $\begin{cases} x^2 + y = 5 \\ 2x - 4y = 0 \end{cases}$

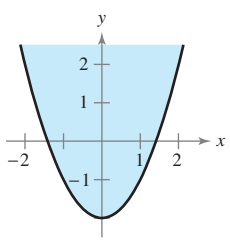
10. $\begin{cases} x^2 + y^2 = 13 \\ x + y = 5 \end{cases}$

Exercises 5.4

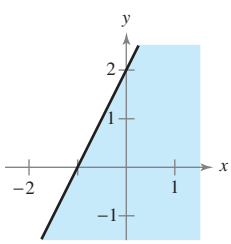
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

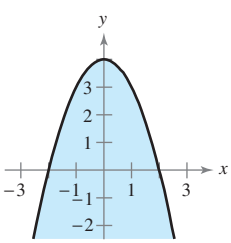
(a)



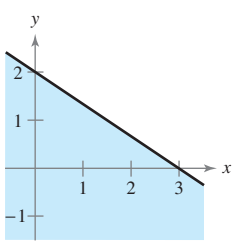
(b)



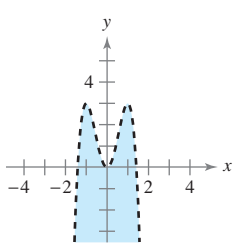
(c)



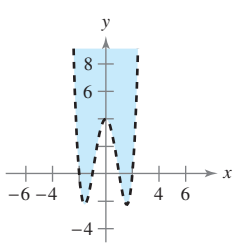
(d)



(e)



(f)



1. $2x + 3y \leq 6$

2. $2x - y \geq -2$

3. $x^2 - y \leq 2$

4. $y \leq 4 - x^2$

5. $y > x^4 - 5x^2 + 4$

6. $3x^4 + y < 6x^2$

In Exercises 7–22, sketch the graph of the inequality.

7. $x \geq 2$

8. $x < 4$

9. $y + 2x^2 > 0$

10. $y^2 - x < 0$

11. $y > -1$

12. $y \leq 3$

13. $y < 2 - x$

14. $y > 2x - 4$

15. $2y - x \geq 4$

16. $5x + 3y \geq -15$

17. $y < \ln x$

18. $y \geq -\ln x + 1$

19. $x^2 + y^2 \leq 4$

20. $x^2 + y^2 > 4$

21. $x^2 + (y - 2)^2 < 16$

22. $y - (x - 3)^3 \geq 0$

In Exercises 23–44, graph the solution set of the system of inequalities.

23. $\begin{cases} x + y \leq 2 \\ -x + y \leq 2 \\ y \geq 0 \end{cases}$

24. $\begin{cases} 3x + 2y < 6 \\ x > 1 \\ y > 0 \end{cases}$

25. $\begin{cases} x + y \leq 5 \\ x \geq 2 \\ y \geq 0 \end{cases}$

26. $\begin{cases} 2x + y \geq 2 \\ x \leq 2 \\ y \leq 1 \end{cases}$

27. $\begin{cases} -3x + 2y < 6 \\ x + 4y < -2 \\ 2x + y < 3 \end{cases}$

28. $\begin{cases} x - 7y > -36 \\ 5x + 2y > 5 \\ 6x - 5y > 6 \end{cases}$

29. $\begin{cases} x^2 + y \leq 6 \\ x \geq -1 \\ y \geq 0 \end{cases}$

30. $\begin{cases} 2x^2 + y > 4 \\ x < 0 \\ y < 2 \end{cases}$

31. $\begin{cases} 2x + y > 2 \\ 6x + 3y < 2 \end{cases}$

32. $\begin{cases} 5x - 3y > -6 \\ 5x - 3y < -9 \end{cases}$

33. $\begin{cases} y \geq -3 \\ y \leq 1 - x^2 \end{cases}$

34. $\begin{cases} x - y^2 > 0 \\ y > (x - 3)^2 - 4 \end{cases}$

35. $\begin{cases} x^2 + y^2 \leq 16 \\ x^2 + y^2 < 1 \end{cases}$

36. $\begin{cases} x^2 + y^2 \leq 25 \\ x^2 + y^2 \geq 9 \end{cases}$

37. $\begin{cases} x > y^2 \\ x < y + 2 \end{cases}$

38. $\begin{cases} x < 2y - y^2 \\ 0 < x + y \end{cases}$

39. $\begin{cases} y \leq \sqrt{3x} + 1 \\ y \geq x + 1 \end{cases}$

40. $\begin{cases} y < \sqrt{2x} + 3 \\ y > x + 3 \end{cases}$

41. $\begin{cases} y < x^3 - 2x + 1 \\ y > -2x \\ x \leq 1 \end{cases}$

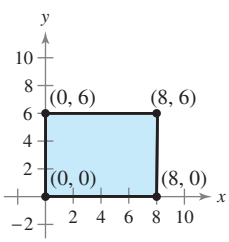
42. $\begin{cases} x^2 + y \leq 4 \\ y \geq 2x \\ x \geq -1 \end{cases}$

43. $\begin{cases} y \leq e^x \\ y \geq \ln x \\ x \geq \frac{1}{2} \\ x \leq 2 \end{cases}$

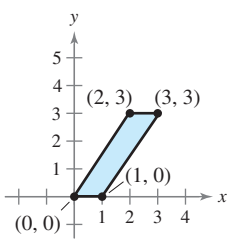
44. $\begin{cases} y \leq e^{-x^2/2} \\ y \geq 0 \\ x \geq -1 \\ x \leq 0 \end{cases}$

In Exercises 45–50, write a system of inequalities that corresponds to the solution set shown in the graph.

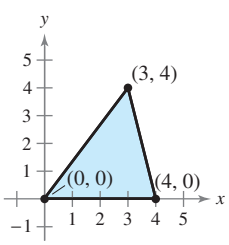
45. Rectangle



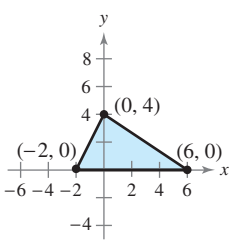
46. Parallelogram



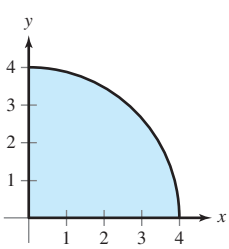
47. Triangle



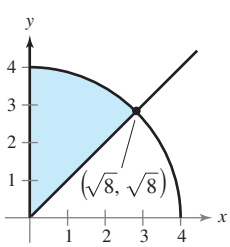
48. Triangle



49. Sector of a circle



50. Sector of a circle



51. Furniture Production A furniture company produces tables and chairs. Each table requires 2 hours in the assembly center and $1\frac{1}{2}$ hours in the finishing center. Each chair requires $1\frac{1}{2}$ hours in the assembly center and $1\frac{1}{2}$ hours in the finishing center. The company's assembly center is available 18 hours per day, and its finishing center is available 15 hours per day. Let x and y be the numbers of tables and chairs produced per day, respectively. (a) Find a system of inequalities describing all possible production levels, and (b) sketch the graph of the system.

52. Kayak Inventory A store sells two models of kayaks. Because of the demand, it is necessary to stock at least twice as many units of model A as units of model B. The costs to the store for the two models are $00 and $00, respectively. The management does not want more than $0,000 in kayak inventory at any one time, and it wants at least six model A kayaks and three model B kayaks in inventory at all times. (a) Find a system of inequalities describing all possible inventory levels, and (b) sketch the graph of the system.

Consumer and Producer Surpluses In Exercises 53–56, find the consumer surplus and producer surplus for the pair of demand and supply equations.

	Demand	*Supply*
53.	$p = 56 - 0.0001x$	$p = 22 + 0.00001x$
54.	$p = 60 - 0.00001x$	$p = 15 + 0.00004x$
55.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$
56.	$p = 600 - 0.0002x$	$p = 125 + 0.0006x$

57. Think About It Under what circumstances are the consumer surplus and producer surplus equal for a pair of linear supply and demand equations? Explain.

58. Think About It Under what circumstances is the consumer surplus greater than the producer surplus for a pair of linear supply and demand equations? Explain.

59. Investment You plan to invest up to $0,000 in two different interest-bearing accounts. Each account is to contain at least $000. Moreover, one account should have at least twice the amount that is in the other account. (a) Find a system of inequalities that describes the amounts that you can invest in each account, and (b) sketch the graph of the system.

60. Concert Ticket Sales Two types of tickets are to be sold for a concert. One type costs $0 per ticket and the other type costs $0 per ticket. The promoter of the concert must sell at least 20,000 tickets, including at least 8000 of the $0 tickets and at least 5000 of the $0 tickets. Moreover, the gross receipts must total at least $80,000 in order for the concert to be held. (a) Find a system of inequalities describing the different numbers of tickets that must be sold, and (b) sketch the graph of the system.

61. ***MAKE A DECISION: DIET SUPPLEMENT*** A dietitian designs a special diet supplement using two different foods. Each ounce of food X contains 20 units of calcium, 10 units of iron, and 15 units of vitamin B. Each ounce of food Y contains 15 units of calcium, 20 units of iron, and 20 units of vitamin B. The minimum daily requirements for the diet are 400 units of calcium, 250 units of iron, and 220 units of vitamin B.

(a) Find a system of inequalities describing the different amounts of food X and food Y that the dietitian can use in the diet.

(b) Sketch the graph of the system.

(c) A nutritionist normally gives a patient 18 ounces of food X and 3.5 ounces of food Y per day. Supplies of food X are running low. What other combinations of foods X and Y can be given to the patient to meet the minimum daily requirements?

62. ***MAKE A DECISION: DIET SUPPLEMENT*** A dietitian designs a special diet supplement using two different foods. Each ounce of food X contains 12 units of calcium, 10 units of iron, and 20 units of vitamin B. Each ounce of food Y contains 15 units of calcium, 20 units of iron, and 12 units of vitamin B. The minimum daily requirements for the diet are 300 units of calcium, 280 units of iron, and 300 units of vitamin B.

(a) Find a system of inequalities describing the different amounts of food X and food Y that the dietitian can use in the diet.

(b) Sketch the graph of the system.

(c) A nutritionist normally gives a patient 10 ounces of food X and 12 ounces of food Y per day. Supplies of food Y are running low. What other combinations of foods X and Y can be given to the patient to meet the minimum daily requirements?

63. **Health** A person's maximum heart rate is $220 - x$, where x is the person's age in years for $20 \le x \le 70$. When a person exercises, it is recommended that the person strive for a heart rate that is at least 50% of the maximum and at most 75% of the maximum. *(Source: American Heart Association)*

(a) Write a system of inequalities that describes the exercise target heart rate region. Let y represent a person's heart rate.

(b) Sketch a graph of the region in part (a).

(c) Find two solutions to the system and interpret their meanings in the context of the problem.

64. **Peregrine Falcons** The numbers of nesting pairs y of peregrine falcons in Yellowstone National Park from 2001 to 2005 can be approximated by the linear model

$$y = 3.4t + 13, \quad 1 \le t \le 5$$

where t represents the year, with $t = 1$ corresponding to 2001. *(Source: Yellowstone Bird Report 2005)*

(a) The *total* number of nesting pairs during this five-year period can be approximated by finding the area of the trapezoid represented by the following system.

$$\begin{cases} y \le 3.4t + 13 \\ y \ge 0 \\ t \ge 0.5 \\ t \le 5.5 \end{cases}$$

Graph this region using a graphing utility.

(b) Use the formula for the area of a trapezoid to approximate the total number of nesting pairs.

65. **Computers** The sales y (in billions of dollars) for Dell Inc. from 1996 to 2005 can be approximated by the linear model

$$y = 5.07t - 22.4, \quad 6 \le t \le 15$$

where t represents the year, with $t = 6$ corresponding to 1996. *(Source: Dell Inc.)*

(a) The *total* sales during this ten-year period can be approximated by finding the area of the trapezoid represented by the following system.

$$\begin{cases} y \le 5.07t - 22.4 \\ y \ge 0 \\ t \ge 5.5 \\ t \le 15.5 \end{cases}$$

Graph this region using a graphing utility.

(b) Use the formula for the area of a trapezoid to approximate the total sales.

66. Write a system of inequalities whose graphed solution set is a right triangle.

67. Write a system of inequalities whose graphed solution set is an isosceles triangle.

68. **Writing** Explain the difference between the graphs of the inequality $x \le 4$ on the real number line and on the rectangular coordinate system.

69. **Graphical Reasoning** Two concentric circles have radii x and y, where $y > x$. The area between the circles must be at least 10 square units.

(a) Find a system of inequalities describing the constraints on the circles.

(b) Use a graphing utility to graph the system of inequalities in part (a). Graph the line $y = x$ in the same viewing window.

(c) Identify the graph of the line in relation to the boundary of the inequality. Explain its meaning in the context of the problem.

Section 5.5

Linear Programming

- Use linear programming to minimize or maximize an objective function.
- Use linear programming to optimize an application.

Linear Programming: A Graphical Approach

Many applications in business and economics involve a process called **optimization,** in which you are asked to find the minimum cost, the maximum profit, or the minimum use of resources. In this section you will study an optimization strategy called **linear programming.**

A two-dimensional linear programming problem consists of a linear **objective function** and a system of linear inequalities called **constraints.** The objective function gives the quantity that is to be maximized (or minimized), and the constraints determine the set of **feasible solutions.** For example, consider a linear programming problem in which you are asked to maximize the value of

$$z = ax + by \qquad \text{Objective function}$$

subject to a set of constraints that determines the region in Figure 5.24. Because every point in the region satisfies each constraint, it is not clear how you should go about finding the point that yields a maximum value of z Fortunately, it can be shown that if there is an optimal solution, it must occur at one of the vertices of the region. This means that *you can find the maximum value by testing z at each of the vertices.*

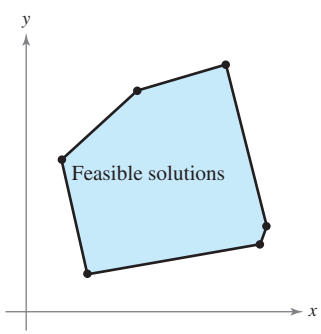


FIGURE 5.24

Optimal Solution of a Linear Programming Problem

If a linear programming problem has a solution, it must occur at a vertex of the set of feasible solutions. If the problem has more than one solution, then at least one solution must occur at a vertex of the set of feasible solutions. In either case, the value of the objective function is unique.

The process for solving a linear programming problem in two variables is shown in Example 1 on the next page.

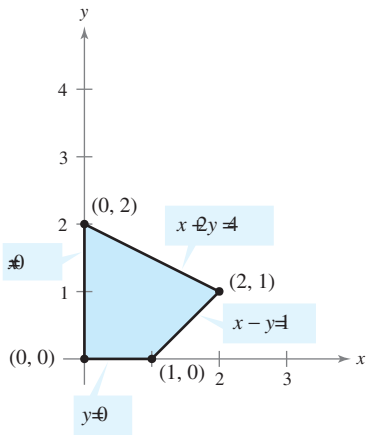


FIGURE 5.25

Example 1 Solving a Linear Programming Problem

Find the maximum value of

$$z = 3x + 2y$$ Objective function

subject to the following constraints.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}\right\}$$ Constraints

SOLUTION The constraints form the region shown in Figure 5.25. At the four vertices of this region, the objective function has the following values.

At $(0, 0)$: $z = 3(0) + 2(0) = 0$

At $(1, 0)$: $z = 3(1) + 2(0) = 3$

At $(2, 1)$: $z = 3(2) + 2(1) = 8$ Maximum value of z

At $(0, 2)$: $z = 3(0) + 2(2) = 4$

So, the maximum value of z is 8, which occurs when $x = 2$ and $y = 1$.

✓CHECKPOINT 1

Find the maximum value of

$$z = 2x + 3y$$

subject to the following constraints.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 3 \\ x - y &\leq 2 \end{aligned}\right\}$$

In Example 1, try testing some of the *interior* points of the region. You will see that the corresponding values of z are less than 8. Here are some examples.

At $(1, 1)$: $z = 3(1) + 2(1) = 5$

At $\left(1, \frac{1}{2}\right)$: $z = 3(1) + 2\left(\frac{1}{2}\right) = 4$

To see why the maximum value of the objective function in Example 1 must occur at a vertex, consider writing the objective function in slope-intercept form

$$y = -\frac{3}{2}x + \frac{z}{2}$$ Family of lines

where $z/2$ is the y-intercept of the objective function. This equation represents a family of lines, each of slope $-\frac{3}{2}$. Of these infinitely many lines, you want the one that has the largest z-value while still intersecting the region determined by the constraints. In other words, of all the lines whose slope is $-\frac{3}{2}$, you want the one that has the largest y-intercept *and* intersects the given region, as shown in Figure 5.26. It should be clear that such a line will pass through one (or more) of the vertices of the region.

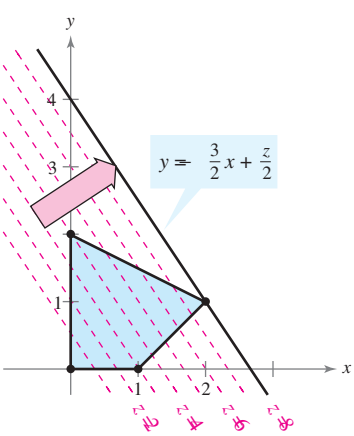


FIGURE 5.26

STUDY TIP

Remember that a vertex of a region can be found using a system of linear equations. The system will consist of the equations of the lines passing through the vertex.

Solving a Linear Programming Problem

To solve a linear programming problem involving two variables by the graphical method, use the following steps.

1. Sketch the region corresponding to the system of constraints.
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and a maximum value will exist. (For an unbounded region, *if* an optimal solution exists, it will occur at a vertex.)

The guidelines above will work whether the objective function is to be maximized or minimized. For instance, the same test used in Example 1 to find the maximum value of z can be used to conclude that the minimum value of z is 0 and that this value occurs at the vertex $(0, 0)$.

Example 2 Solving a Linear Programming Problem

Find (a) the maximum value and (b) the minimum value of

$$z = 4x + 6y$$ Objective function

subject to the following constraints.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 11 \\ x + y &\leq 27 \\ 2x + 5y &\leq 90 \end{aligned}\right\}$$ Constraints

SOLUTION

a. The region bounded by the constraints is shown in Figure 5.27. By testing the objective function at each vertex, you obtain the following.

At $(0, 0)$: $z = 4(0) + 6(0) = 0$

At $(0, 11)$: $z = 4(0) + 6(11) = 66$

At $(5, 16)$: $z = 4(5) + 6(16) = 116$

At $(15, 12)$: $z = 4(15) + 6(12) = 132$ Maximum value of z

At $(27, 0)$: $z = 4(27) + 6(0) = 108$

So, the maximum value of z is 132, which occurs when $x = 15$ and $y = 12$.

b. Using the values of z at the vertices in part (a), you can conclude that the minimum value of z is 0, and that this value occurs when $x = 0$ and $y = 0$.

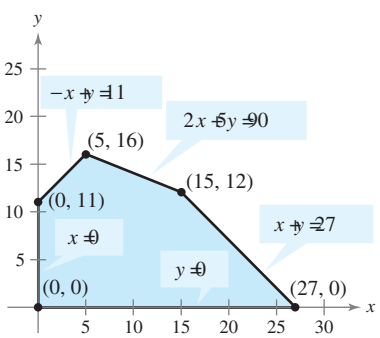


FIGURE 5.27

STUDY TIP

The steps used to find the minimum and maximum values of an objective function are precisely the same. In other words, once you have evaluated the objective function at the vertices of the set of feasible solutions, you simply choose the largest value as the maximum and the smallest value as the minimum.

✓CHECKPOINT 2

Find (a) the maximum value and (b) the minimum value of $z = 5x + 2y$ subject to the same constraints as in Example 2. ■

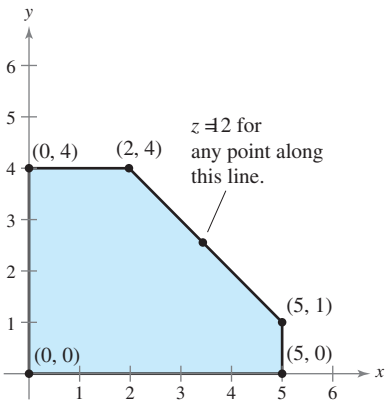


FIGURE 5.28

It is possible for the maximum (or minimum) value in a linear programming problem to occur at *two* different vertices. For instance, at the vertices of the region shown in Figure 5.28, the objective function

$$z = 2x + 2y$$ Objective function

has the following values.

At $(0, 0)$: $z = 2(0) + 2(0) = 0$

At $(0, 4)$: $z = 2(0) + 2(4) = 8$

At $(2, 4)$: $z = 2(2) + 2(4) = 12$ Maximum value of z

At $(5, 1)$: $z = 2(5) + 2(1) = 12$ Maximum value of z

At $(5, 0)$: $z = 2(5) + 2(0) = 10$

In this case, you can conclude that the objective function has a maximum value not only at the vertices $(2, 4)$ and $(5, 1)$; it also has a maximum value (of 12) at *any point on the line segment connecting these two vertices.* Note that the objective function $y = -x + \frac{1}{2}z$ has the same slope as the line through the vertices $(2, 4)$ and $(5, 1)$.

Some linear programming problems have no optimal solution. This can occur if the region determined by the constraints is *unbounded.* Example 3 illustrates such a problem.

Example 3 An Unbounded Region

Find the maximum value of

$$z = 4x + 2y$$ Objective function

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} x + 2y &\geq 4 \\ 3x + y &\geq 7 \\ -x + 2y &\leq 7 \end{aligned}\right\}$$ Constraints

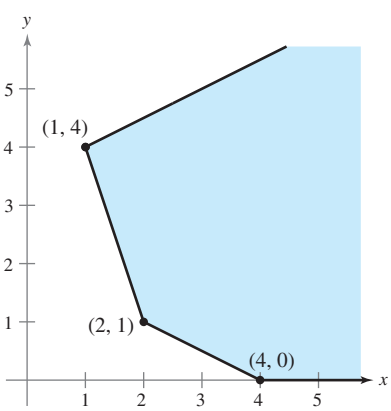


FIGURE 5.29

SOLUTION The region determined by the constraints is shown in Figure 5.29. For this unbounded region, there is no maximum value of z. To see this, note that the point $(x, 0)$ lies in the region for all values of $x \geq 4$. By choosing x to be large, you can obtain values of

$$z = 4(x) + 2(0) = 4x$$

that are as large as you want. So, there is no maximum value of z. For this problem, there *is* a minimum value of z, $z = 10$, which occurs at the vertex $(2, 1)$, as shown below.

At $(1, 4)$: $z = 4(1) + 2(4) = 12$

At $(2, 1)$: $z = 4(2) + 2(1) = 10$ Minimum value of z

At $(4, 0)$: $z = 4(4) + 2(0) = 16$

CHECKPOINT 3

Find the maximum value of the objective function $z = x + 8y$ where $x \geq 0$ and $y \geq 0$, subject to the same constraints as in Example 3. ■

Applications

Example 4 shows how linear programming can be used to find the maximum profit in a business application.

Example 4 Optimal Profit

A manufacturer wants to maximize the profit for two laboratory products. Product I yields a profit of $.50 per unit, and product II yields a profit of $.00 per unit. Market tests and available resources have indicated the following constraints.

1. The combined production level should not exceed 1200 units per month.
2. The demand for product II is no more than half the demand for product I.
3. The production level of product I is less than or equal to 600 units plus three times the production level of product II.

What is the optimal production level for each product?

SOLUTION Let x be the number of units of product I and let y be the number of units of product II. The objective function (for the combined profit) is given by

$$P = 1.5x + 2y.$$ Objective function

The three constraints translate into the following linear inequalities.

1. $x + y \leq 1200 \Rightarrow x + y \leq 1200$
2. $y \leq \frac{1}{2}x \Rightarrow -x + 2y \leq 0$
3. $x \leq 3y + 600 \Rightarrow x - 3y \leq 600$

Because neither x nor y can be negative, you also have the two additional constraints of $x \geq 0$ and $y \geq 0$. Figure 5.30 shows the region determined by the constraints. To find the maximum profit, test the value of P at each vertex of the region.

At (0, 0): $P = 1.5(0) + 2(0) = 0$

At (800, 400): $P = 1.5(800) + 2(400) = 2000$ Maximum profit

At (1050, 150): $P = 1.5(1050) + 2(150) = 1875$

At (600, 0): $P = 1.5(600) + 2(0) = 900$

So, the maximum profit is $000, and it occurs when the monthly production levels are 800 units of product I and 400 units of product II.

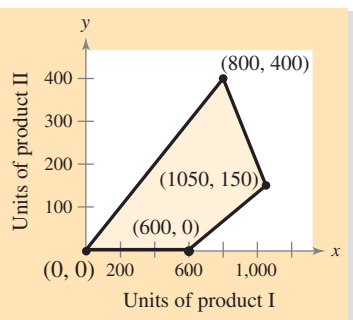


FIGURE 5.30

✓CHECKPOINT 4

In Example 4, suppose the manufacturer improved the production of product I so that it yielded a profit of $.50 per unit. How would this improvement affect the optimal number of units the manufacturer should sell in order to obtain a maximum profit? ■

Example 5 shows how linear programming can be used to find the optimal cost in a real-life application.

Example 5
MAKE A DECISION **Optimal Cost**

The liquid portion of a diet requires at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X costs ₿0.12 and provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y costs ₿0.15 and provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. How many cups of each drink should be consumed each day to obtain an optimal cost and still meet the daily requirements?

SOLUTION As in Example 7 on page 447, let x be the number of cups of dietary drink X and let y be the number of cups of dietary drink Y.

$$\left.\begin{array}{lrcl} \text{For calories:} & 60x + 60y & \geq & 300 \\ \text{For vitamin A:} & 12x + 6y & \geq & 36 \\ \text{For vitamin C:} & 10x + 30y & \geq & 90 \\ & x & \geq & 0 \\ & y & \geq & 0 \end{array}\right\} \quad \text{Constraints}$$

The cost C is given by

$$C = 0.12x + 0.15y. \quad \text{Objective function}$$

The graph of the region corresponding to the constraints is shown in Figure 5.31. Because you want to incur as little cost as possible, you want to determine the *minimum* cost. To determine the minimum cost, test C at each vertex of the region, as follows.

At (0, 6): $C = 0.12(0) + 0.15(6) = 0.90$

At (1, 4): $C = 0.12(1) + 0.15(4) = 0.72$

At (3, 2): $C = 0.12(3) + 0.15(2) = 0.66$ Minimum value of C

At (9, 0): $C = 0.12(9) + 0.15(0) = 1.08$

So, the minimum cost is ₿0.66 per day, and this cost occurs when 3 cups of drink X and 2 cups of drink Y are consumed each day.

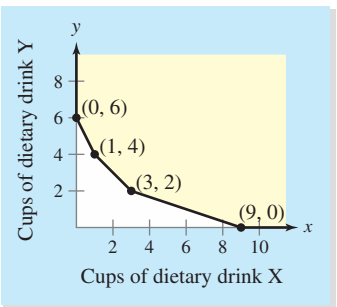


FIGURE 5.31

✓CHECKPOINT 5

In Example 5, suppose a cup of dietary drink Y costs ₿0.11. How would this affect the number of cups of each drink that should be consumed each day to obtain an optimal cost and still meet the daily requirements? ■

CONCEPT CHECK

1. Does every linear programming problem have an optimal solution? Explain.
2. Can a linear programming problem have a maximum value but no minimum value? Explain.
3. Can a linear programming problem have a minimum value at two vertices and a maximum value at two vertices? Justify your answer.
4. Suppose that a linear programming problem has a minimum value of a and a maximum value of b. Write an interval for the value of the objective function at any point in the interior of the region determined by the constraints. Explain your reasoning.

Skills Review 5.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 2.2, 5.1, and 5.4.

In Exercises 1–4, sketch the graph of the linear equation.

1. $y + x = 3$

2. $y - x = 12$

3. $x = 0$

4. $y = 4$

In Exercises 5–8, find the point of intersection of the two lines.

5. $x + y = 4$
$x = 0$

6. $x + 2y = 12$
$y = 0$

7. $x + y = 4$
$2x + 3y = 9$

8. $x + 2y = 12$
$2x + y = 9$

In Exercises 9 and 10, sketch the graph of the inequality.

9. $2x + 3y \geq 18$

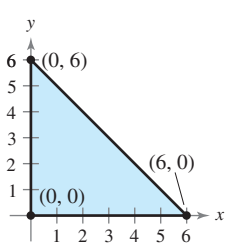
10. $4x + 3y \geq 12$

Exercises 5.5

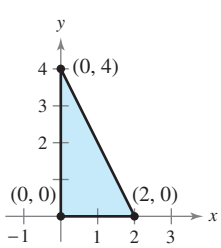
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints. (For each exercise, the graph of the region determined by the constraints is provided.)

1. *Objective function:*
$z = 6x + 5y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + y \leq 6$



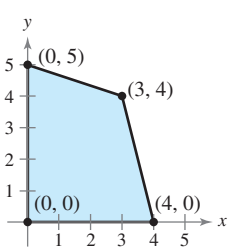
2. *Objective function:*
$z = 2x + 8y$
Constraints:
$x \geq 0$
$y \geq 0$
$2x + y \leq 4$



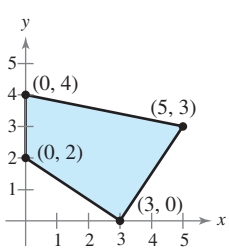
3. *Objective function:*
$z = 8x + 7y$
Constraints:
See Exercise 1.

4. *Objective function:*
$z = 7x + 3y$
Constraints:
See Exercise 2.

5. *Objective function:*
$z = 3x + 2y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + 3y \leq 15$
$4x + y \leq 16$



6. *Objective function:*
$z = 5x + 4y$
Constraints:
$x \geq 0$
$y \geq 0$
$2x + 3y \geq 6$
$3x - 2y \leq 9$
$x + 5y \leq 20$



7. *Objective function:*
$z = 5x + 0.5y$
Constraints:
See Exercise 5.

8. *Objective function:*
$z = x + 6y$
Constraints:
See Exercise 6.

In Exercises 9–20, sketch the region determined by the constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints.

9. *Objective function:*

$z = 6x + 10y$

Constraints:

$x \geq 0$

$y \geq 0$

$3x + 5y \leq 15$

10. *Objective function:*

$z = 7x + 8y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 2y \leq 8$

11. *Objective function:*

$z = 9x + 4y$

Constraints:

See Exercise 9.

12. *Objective function:*

$z = 7x + 2y$

Constraints:

See Exercise 10.

13. *Objective function:*

$z = 4x + 5y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + y \geq 8$

$3x + 5y \geq 30$

14. *Objective function:*

$z = 4x + 5y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + y \leq 5$

$x + 2y \leq 6$

15. *Objective function:*

$z = 2x + 7y$

Constraints:

See Exercise 13.

16. *Objective function:*

$z = 2x - y$

Constraints:

See Exercise 14.

17. *Objective function:*

$z = x + 2y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 2y \leq 40$

$x + y \leq 30$

$2x + 3y \leq 65$

18. *Objective function:*

$z = x$

Constraints:

$x \geq 0$

$y \geq 0$

$2x + 3y \leq 60$

$2x + y \leq 28$

$4x + y \leq 48$

19. *Objective function:*

$z = x + y$

Constraints:

See Exercise 17.

20. *Objective function:*

$z = y$

Constraints:

See Exercise 18.

In Exercises 21–24, maximize the objective function subject to the constraints $3x + y \leq 15$, $4x + 3y \leq 30$, $x \geq 0$, and $y \geq 0$.

21. $z = 2x + y$

22. $z = 5x + y$

23. $z = 4x + 3y$

24. $z = 3x + y$

In Exercises 25–28, maximize the objective function subject to the constraints $x + 4y \leq 20$, $x + y \leq 8$, $3x + 2y \leq 21$, $x \geq 0$, and $y \geq 0$.

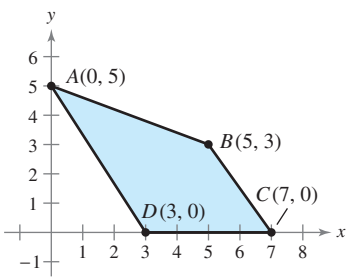
25. $z = 2x + 5y$

26. $z = 3x + 5y$

27. $z = 12x + 5y$

28. $z = 15x + 8y$

Think About It In Exercises 29–36, find an objective function that has a maximum or minimum value at the indicated vertex of the constraint region shown. (There are many correct answers.)



29. The maximum occurs at vertex A.

30. The maximum occurs at vertex B.

31. The minimum occurs at vertex C.

32. The minimum occurs at vertex D.

33. The maximum occurs at vertices A and B.

34. The maximum occurs at vertices B and C.

35. The minimum occurs at vertices A and D.

36. The minimum occurs at vertices C and D.

37. Optimal Profit A fruit grower raises crops A and B. The profit is $85 per acre for crop A and $45 per acre for crop B. Research and available resources indicate the following constraints.

- The fruit grower has 150 acres of land for raising the crops.
- It takes 1 day to trim an acre of crop A and 2 days to trim an acre of crop B, and there are 240 days per year available for trimming.
- It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking.

What is the optimal acreage for each fruit? What is the optimal profit?

38. Optimal Profit The costs to a store for two models of Global Positioning System (GPS) receivers are $80 and $100. The $80 model yields a profit of $35 and the $100 model yields a profit of $50. Market tests and available resources indicate the following constraints.

- The merchant estimates that the total monthly demand will not exceed 200 units.
- The merchant does not want to invest more than $8,000 in GPS receiver inventory.

What is the optimal inventory level for each model? What is the optimal profit?

39. Optimal Cost A farming cooperative mixes two brands of cattle feed. Brand X costs $40 per bag, and brand Y costs $25 per bag. Research and available resources have indicated the following constraints.

- Brand X contains two units of nutritional element A, two units of element B, and two units of element C.
- Brand Y contains one unit of nutritional element A, nine units of element B, and three units of element C.
- The minimum requirements for nutrients A, B, and C are 12 units, 36 units, and 24 units, respectively.

What is the optimal number of bags of each brand that should be mixed? What is the optimal cost?

40. Optimal Cost A humanitarian agency can use two models of vehicles for a refugee rescue mission. Each model A vehicle costs $000 and each model B vehicle costs $500. Mission strategies and objectives indicate the following constraints.

- A total of at least 20 vehicles must be used.
- A model A vehicle can hold 45 boxes of supplies. A model B vehicle can hold 30 boxes of supplies. The agency must deliver at least 690 boxes of supplies to the refugee camp.
- A model A vehicle can hold 20 refugees. A model B vehicle can hold 32 refugees. The agency must rescue at least 520 refugees.

What is the optimal number of vehicles of each model that should be used? What is the optimal cost?

41. Optimal Profit A manufacturer produces two models of bicycles. The times (in hours) required for assembling, painting, and packaging each model are shown in the table.

Process	Model A	Model B
Assembling	2	2.5
Painting	4	1
Packaging	1	0.75

The total times available for assembling, painting, and packaging are 4000 hours, 4800 hours, and 1500 hours, respectively. The profits per unit are $50 for model A and $75 for model B. What is the optimal production level for each model? What is the optimal profit?

42. Optimal Profit A company makes two models of doghouses. The times (in hours) required for assembling, painting, and packaging are shown in the table.

Process	Model A	Model B
Assembling	2.5	3
Painting	2	1
Packaging	0.75	1.25

The total times available for assembling, painting, and packaging are 4000 hours, 2500 hours, and 1500 hours, respectively. The profits per unit are $60 for model A and $75 for model B. What is the optimal production level for each model? What is the optimal profit?

43. Optimal Revenue An accounting firm charges $500 for an audit and $50 for a tax return. Research and available resources have indicated the following constraints.

- The firm has 900 hours of staff time available each week.
- The firm has 155 hours of review time available each week.
- Each audit requires 75 hours of staff time and 10 hours of review time.
- Each tax return requires 12.5 hours of staff time and 2.5 hours of review time.

What numbers of audits and tax returns will bring in an optimal revenue?

44. Optimal Revenue The accounting firm in Exercise 43 lowers its charge for an audit to $000. What numbers of audits and tax returns will bring in an optimal revenue?

45. Media Selection A company has budgeted a maximum of $,000,000 for national advertising of an allergy medication. Each minute of television time costs $00,000 and each one-page newspaper ad costs $0,000. Each television ad is expected to be viewed by 20 million viewers, and each newspaper ad is expected to be seen by 5 million readers. The company's market research department recommends that at most 80% of the advertising budget be spent on television ads. What is the optimal amount that should be spent on each type of ad? What is the optimal total audience?

46. Optimal Profit A fruit juice company makes two drinks by blending apple and pineapple juices. The percents of apple juice and pineapple juice in each drink are shown in the table.

Mixture	Drink A	Drink B
Apple juice	30%	60%
Pineapple juice	70%	40%

There are 1000 liters of apple juice and 1500 liters of pineapple juice available. The profit for drink A is $0.70 per liter and the profit for drink B is $0.60 per liter. What is the optimal production level for each type of drink? What is the optimal profit?

47. Investments An investor has up to $50,000 to invest in two types of investments. Type A investments pay 7% annually and type B pay 12% annually. To have a well-balanced portfolio, the investor imposes the following conditions. At least one-fourth of the total portfolio is to be allocated to type A investments and at least one-fourth is to be allocated to type B investments. What is the optimal amount that should be invested in each type of investment? What is the optimal return?

48. Investments An investor has up to $50,000 to invest in two types of investments. Type A investments pay 8% annually and type B pay 14% annually. To have a well-balanced portfolio, the investor imposes the following conditions. At least one-half of the total portfolio is to be allocated to type A investments and at least one-fourth is to be allocated to type B investments. What is the optimal amount that should be invested in each type of investment? What is the optimal return?

49. Optimal Profit A company makes two models of a patio furniture set. The times for assembling, finishing, and packaging model A are 3 hours, 2.5 hours, and 0.6 hour, respectively. The times for model B are 2.75 hours, 1 hour, and 1.25 hours. The total times available for assembling, finishing, and packaging are 3000 hours, 2400 hours, and 1200 hours, respectively. The profit per unit for model A is $00 and the profit per unit for model B is $5. What is the optimal production level for each model? What is the optimal profit?

50. Optimal Profit A manufacturer produces two models of elliptical cross-training exercise machines. The times for assembling, finishing, and packaging model A are 3 hours, 3 hours, and 0.8 hour, respectively. The times for model B are 4 hours, 2.5 hours, and 0.4 hour. The total times available for assembling, finishing, and packaging are 6000 hours, 4200 hours, and 950 hours, respectively. The profits per unit are $00 for model A and $75 for model B. What is the optimal production level for each model? What is the optimal profit?

In Exercises 51–56, the given linear programming problem has an unusual characteristic. Sketch a graph of the solution region for the problem and describe the unusual characteristic. Find the maximum value of the objective function and where it occurs.

51. *Objective function:*

$z = 2.5x + y$

Constraints:

$x \geq 0$

$y \geq 0$

$3x + 5y \leq 15$

$5x + 2y \leq 10$

52. *Objective function:*

$z = x + y$

Constraints:

$x \geq 0$

$y \geq 0$

$-x + y \leq 1$

$-x + 2y \leq 4$

53. *Objective function:*

$z = -x + 2y$

Constraints:

$x \geq 0$

$y \geq 0$

$x \leq 10$

$x + y \leq 7$

54. *Objective function:*

$z = x + y$

Constraints:

$x \geq 0$

$y \geq 0$

$-x + y \leq 1$

$-3x + y \geq 3$

55. *Objective function:*

$z = 3x + 4y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + y \leq 1$

$2x + y \leq 4$

56. *Objective function:*

$z = x + 2y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 2y \leq 4$

$2x + y \leq 4$

57. Reasoning An objective function has a maximum value at the vertices (0, 14) and (3, 8).

(a) Can you conclude that it also has a maximum value at the point (1, 12)? Explain.

(b) Can you conclude that it also has a maximum value at the point (4, 6)? Explain.

(c) Find another point that maximizes the objective function.

58. Reasoning An objective function has a minimum value at the vertex (20, 0). Can you conclude that it also has a minimum value at the point (0, 0)? Explain.

59. Reasoning When solving a linear programming problem, you find that the objective function has a maximum value at more than one vertex. Can you assume that there are an infinite number of points that will produce the maximum value? Explain your reasoning.

Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 462.
Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 5.1	Review Exercises
■ Solve a system of equations by the method of substitution.	*1–6*
■ Solve a system of equations graphically.	*7, 8, 12*
■ Construct and use a system of equations to solve an application problem.	*9–12*
Section 5.2	
■ Solve a linear system by the method of elimination.	*13–20, 41*
■ Interpret the solution of a linear system graphically.	*21, 22*
■ Construct and use a linear system to solve an application problem.	*23–26*
Section 5.3	
■ Solve a linear system in row-echelon form using back-substitution.	*27, 28*
■ Use Gaussian elimination to solve a linear system.	*29, 30, 33, 34*
■ Solve a nonsquare linear system.	*31, 32*
■ Construct and use a linear system in three or more variables to solve an application problem.	*39, 40, 43*
■ Find the equation of a circle or a parabola using a linear system in three or more variables.	*35–38, 42*
Section 5.4	
■ Sketch the graph of an inequality in two variables.	*44–49*
■ Solve a system of inequalities.	*50–57*
■ Construct and use a system of inequalities to solve an application problem.	*58–63*
Section 5.5	
■ Use linear programming to minimize or maximize an objective function.	*64–71*
■ Use linear programming to optimize an application.	*72–76*

Study Strategies

■ **Units of Variables in Applied Problems** When using systems of equations to solve real-life applications, be sure to keep track of the unit(s) assigned to each variable. This will allow you to write correctly each equation of the system based on the constraints given in the application.

*Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, solve the system by the method of substitution.

1. $\begin{cases} x + 3y = 10 \\ 4x - 5y = -28 \end{cases}$

2. $\begin{cases} 3x - y - 13 = 0 \\ 4x + 3y - 26 = 0 \end{cases}$

3. $\begin{cases} \frac{1}{2}x + \frac{3}{5}y = -2 \\ 2x + y = 6 \end{cases}$

4. $\begin{cases} 1.3x + 0.9y = 7.5 \\ 0.4x - 0.5y = -0.8 \end{cases}$

5. $\begin{cases} x^2 + y^2 = 100 \\ x + 2y = 20 \end{cases}$

6. $\begin{cases} y = x^3 - 2x^2 - 2x - 3 \\ y = -x^2 + 4x - 3 \end{cases}$

In Exercises 7 and 8, use a graphing utility to find the point(s) of intersection of the graphs.

7. $\begin{cases} y = x^2 - 3x + 11 \\ y = -x^2 + 2x + 8 \end{cases}$

8. $\begin{cases} y = \sqrt{9 - x^2} \\ y = e^x + 1 \end{cases}$

9. Break-Even Analysis You invest $5000 in a greenhouse. The planter, potting soil, and seed for each plant costs $1.43 and the selling price is $2.68. How many plants must you sell to break even?

10. Break-Even Analysis You are setting up a basket-weaving business and have made an initial investment of $30,000. The cost of each basket is $3.25 and the selling price is $6.95. How many baskets must you sell to break even? (Round to the nearest whole unit.)

11. Choice of Newscasts Television Stations A and B are competing for the 6 P.M. newscast audience. Station A is implementing a new newscast format for the 6 P.M. audience. Models that represent the numbers of 6 P.M. viewers each month for the two stations are given by

$$\begin{cases} y = 950x + 10{,}000 & \text{Station A (new format)} \\ y = -875x + 18{,}000 & \text{Station B} \end{cases}$$

where y is the number of viewers and x represents the month, with $x = 1$ corresponding to the first month of the new format. Use the models to estimate when the number of viewers for Station A's 6 P.M. newscast will exceed the number of viewers for Station B's 6 P.M. newscast.

12. Comparing Populations From 2000 to 2005, the population of Vermont grew more slowly than that of Alaska. Models that represent the populations of the two states are given by

$$\begin{cases} P = 7.7t + 626 & \text{Alaska} \\ P = 2.8t + 610 & \text{Vermont} \end{cases}$$

where P is the population (in thousands) and t represents the year, with $t = 0$ corresponding to 2000. Use a graphing utility to determine whether the population of Vermont will exceed that of Alaska. *(Source: U.S. Census Bureau)*

In Exercises 13–20, solve the system by elimination.

13. $\begin{cases} 2x - 3y = 21 \\ 3x + y = 4 \end{cases}$

14. $\begin{cases} 3u + 5v = 9 \\ 12u + 10v = 22 \end{cases}$

15. $\begin{cases} 4x - 3y = 10 \\ 8x - 6y = 20 \end{cases}$

16. $\begin{cases} 3x + 4y = 18 \\ 6x + 8y = 18 \end{cases}$

17. $\begin{cases} 1.25x - 2y = 3.5 \\ 5x - 8y = 14 \end{cases}$

18. $\begin{cases} 1.5x + 2.5y = 8.5 \\ 6x + 10y = 24 \end{cases}$

19. $\begin{cases} \dfrac{x-2}{3} + \dfrac{y+3}{4} = 5 \\ 2x - y = 7 \end{cases}$

20. $\begin{cases} \frac{3}{5}x + \frac{2}{7}y = 10 \\ x + 2y = 38 \end{cases}$

In Exercises 21 and 22, describe the graph of the solution of the linear system.

21. $\begin{cases} 2x + y = -1 \\ 3x - 2y = -5 \end{cases}$

22. $\begin{cases} x - 2y = -1 \\ -2x + 4y = 2 \end{cases}$

23. Acid Mixture Twelve gallons of a 25% acid solution is obtained by mixing a 10% solution with a 50% solution.

(a) Write a system of equations that represents the problem and use a graphing utility to graph the equations in the same viewing window.

(b) How much of each solution is required to obtain the specified concentration of the final mixture?

24. Acid Mixture Twenty gallons of a 30% acid solution is obtained by mixing a 12% solution with a 60% solution.

(a) Write a system of equations that represents the problem and use a graphing utility to graph the equations in the same viewing window.

(b) How much of each solution is required to obtain the specified concentration of the final mixture?

Supply and Demand In Exercises 25 and 26, find the point of equilibrium for the pair of demand and supply equations.

	Demand	*Supply*
25.	$p = 37 - 0.0002x$	$p = 22 + 0.00001x$
26.	$p = 120 - 0.0001x$	$p = 45 + 0.0002x$

In Exercises 27–34, solve the system of equations.

27. $\begin{cases} 4x - 3y + 2z = 1 \\ 2y - 4z = 2 \\ z = 2 \end{cases}$

28. $\begin{cases} 2x + y - 4z = 6 \\ 3y + z = 2 \\ z = -4 \end{cases}$

29. $\begin{cases} 2x + y + z = 6 \\ x - 4y - z = 3 \\ x + y + z = 4 \end{cases}$

30. $\begin{cases} x + 3y - z = 13 \\ 2x - 5z = 23 \\ 4x - y - 2z = 4 \end{cases}$

31. $\begin{cases} x + y + z = 10 \\ -2x + 3y + 4z = 22 \end{cases}$

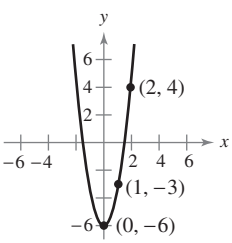
32. $\begin{cases} 5x - 12y + 7z = 16 \\ 3x - 7y + 4z = 9 \end{cases}$

33. $\begin{cases} 2x + 6y - z = 1 \\ x - 3y + z = 2 \\ \frac{3}{2}x + \frac{3}{2}y = 6 \end{cases}$

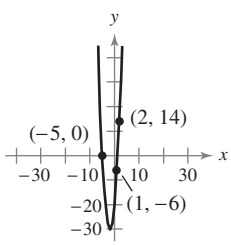
34. $\begin{cases} x + y + z + w = 8 \\ 4y + 5z - 2w = 3 \\ 2x + 3y - z = -2 \\ 3x + 2y - 4w = -20 \end{cases}$

In Exercises 35 and 36, find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points.

35.

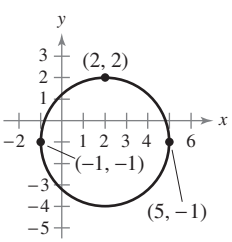


36.

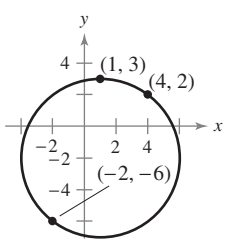


In Exercises 37 and 38, find the equation of the circle $x^2 + y^2 + Dx + Ey + F = 0$ that passes through the points.

37.



38.



39. Investment Portfolio An investor allocates a portfolio totaling $100,000 among the following types of investments: (1) certificates of deposit, (2) municipal bonds, (3) blue-chip stocks, and (4) growth or speculative stocks. The certificates of deposit pay 5% simple annual interest, and the municipal bonds pay 8% simple annual interest. Over a five-year period, the investor expects the blue-chip stocks to return 10% simple annual interest and the growth stocks to return 15% simple annual interest. The investor wishes a combined return of 9.45% and also wants to have only two-fifths of the portfolio invested in stocks. How much should the investor allocate to each type of investment if the amount invested in certificates of deposit is twice that invested in municipal bonds?

40. Investment You receive $580 a year in simple annual interest from three investments. The interest rates for the three investments are 6%, 8%, and 10%. The value of the 10% investment is two times that of the 6% investment, and the 8% investment is $1000 more than the 6% investment. What is the amount of each investment?

41. Fitting a Line to Data Find the least squares regression line $y = ax + b$ for the points $(0, 1.6)$, $(1, 2.4)$, $(2, 3.6)$, $(3, 4.7)$, and $(4, 5.5)$ by solving the following system of linear equations for a and b.

$$\begin{cases} 5b + 10a = 17.8 \\ 10b + 30a = 45.7 \end{cases}$$

42. Fitting a Parabola to Data Find the least squares regression parabola $y = ax^2 + bx + c$ for the points

$(-2, 0.4)$, $(-1, 0.9)$, $(0, 1.9)$, $(1, 2.1)$, and $(2, 3.8)$

by solving the following system of linear equations for a, b, and c.

$$\begin{cases} 5c + 10a = 9.1 \\ 10b = 8.0 \\ 10c + 34a = 19.8 \end{cases}$$

43. Revenue The revenues y (in billions of dollars) for McDonald's Corporation for the years 2001 to 2005 are shown in the table, where t represents the year, with $t = 0$ corresponding to 2002. *(Source: McDonald's Corporation)*

Year, t	Revenue, y
−1	14.9
0	15.4
1	17.1
2	19.1
3	20.5

(a) Use a graphing utility to create a scatter plot of the data.

(b) Solve the following system for a and b to find the least squares regression line $y = at + b$ for the data.

$$\begin{cases} 5b + 5a = 87.0 \\ 5b + 15a = 101.9 \end{cases}$$

(c) Solve the following system for a, b, and c to find the least squares regression parabola $y = at^2 + bt + c$ for the data.

$$\begin{cases} 5c + 5b + 15a = 87.0 \\ 5c + 15b + 35a = 101.9 \\ 15c + 35b + 99a = 292.9 \end{cases}$$

(d) Use the *regression* feature of a graphing utility to find linear and quadratic models for the data. Compare them with the least squares regression models found in parts (b) and (c).

(e) Use a graphing utility to graph the linear and quadratic models. Use the models to predict the revenues in 2006 and 2007. Compare the predictions for each year.

In Exercises 44–49, sketch the graph of the inequality.

44. $x \geq -4$

45. $y < 5$

46. $y \leq 5 - \frac{1}{2}x$

47. $3y - x \geq 7$

48. $y - 4x^2 > -1$

49. $y \leq \dfrac{3}{x^2 + 2}$

In Exercises 50–57, graph the solution set of the system of inequalities.

50. $\begin{cases} 2x + 3y < 9 \\ x > 0 \\ y > 0 \end{cases}$

51. $\begin{cases} 2x - y > 6 \\ x < 5 \\ y \geq -2 \end{cases}$

52. $\begin{cases} 3x - y > -4 \\ 2x + y > -1 \\ 7x + y < 4 \end{cases}$

53. $\begin{cases} x + y > 4 \\ 3x + y < 10 \\ x - y \leq 0 \end{cases}$

54. $\begin{cases} x^2 + y^2 \leq 9 \\ x^2 - x - 2 \leq y \end{cases}$

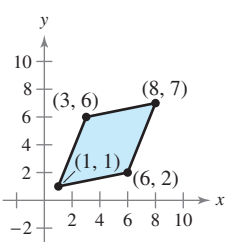
55. $\begin{cases} x^2 + y^2 \leq 4 \\ -2x^2 + 2 < y \end{cases}$

56. $\begin{cases} \ln x < y \\ y > -1 \\ x < 4 \end{cases}$

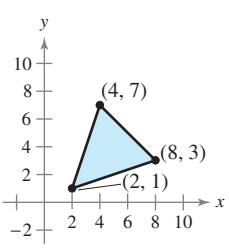
57. $\begin{cases} \ln x \geq y \\ -x + y < -2 \\ x > 2 \end{cases}$

In Exercises 58 and 59, write a system of inequalities that corresponds to the solution set that is shown in the graph.

58. Parallelogram



59. Triangle



Consumer and Producer Surpluses In Exercises 60 and 61, find the consumer surplus and producer surplus for the pair of demand and supply equations.

	Demand	*Supply*
60.	$p = 160 - 0.0001x$	$p = 70 + 0.0002x$
61.	$p = 130 - 0.0002x$	$p = 30 + 0.0003x$

62. Movie Player Inventory A store sells two models of Blu-ray Disc™ players (BDPs). Because of the demand, it is necessary to stock at least twice as many units of model Y as units of model Z. The costs to the store for the two models are $200 and $300, respectively. The management does not want more than $6000 in BDP inventory at any one time, and it wants at least four model Y BDPs and two model Z BDPs in inventory at all times. Find a system of inequalities that describes all possible inventory levels. Sketch the graph of the system.

63. Concert Ticket Sales Two types of tickets are to be sold for a concert. One type costs $20 per ticket and the other type costs $50 per ticket. The promoter of the concert must sell at least 15,000 tickets, including at least 8000 of the $20 tickets and at least 4000 of the $50 tickets. Moreover, the gross receipts must total at least $550,000 in order for the concert to be held. Find a system of inequalities describing the different numbers of tickets that must be sold. Sketch the graph of the system.

In Exercises 64–67, find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints. (For each exercise, the graph of the region determined by the constraints is provided.)

64. *Objective function:*

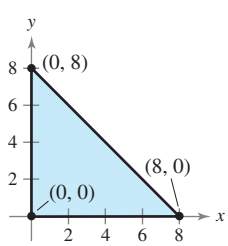
$z = 5x + 6y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + y \leq 8$



65. *Objective function:*

$z = 15x + 12y$

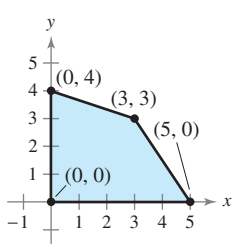
Constraints:

$x \geq 0$

$y \geq 0$

$x + 3y \leq 12$

$3x + 2y \leq 15$



66. *Objective function:*

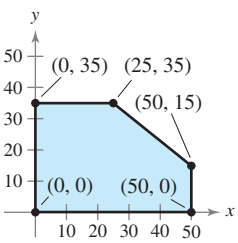
$z = 8x + 10y$

Constraints:

$0 \le x \le 50$

$0 \le y \le 35$

$4x + 5y \le 275$



67. *Objective function:*

$z = 50x + 60y$

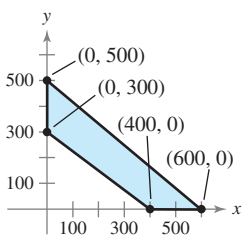
Constraints:

$x \ge 0$

$y \ge 0$

$3x + 4y \ge 1200$

$5x + 6y \le 3000$



In Exercises 68–71, sketch the region determined by the constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints.

68. *Objective function:*

$z = 6x + 8y$

Constraints:

$x \ge 0$

$y \ge 0$

$x + 4y \le 16$

$3x + 2y \le 18$

69. *Objective function:*

$z = 5x + 8y$

Constraints:

$0 \le x \le 5$

$y \ge 0$

$x + 2y \le 12$

$2x + 3y \le 19$

70. *Objective function:*

$z = 8x + 3y$

Constraints:

$0 \le x \le 5$

$0 \le y \le 7$

$x + y \le 9$

$3x + y \le 17$

71. *Objective function:*

$z = 10x + 11y$

Constraints:

$x \ge 0$

$y \ge 0$

$2x + 5y \le 30$

$x + y \ge 3$

$2x + y \le 14$

72. Optimal Profit A company makes two models of desks. The times (in hours) required for assembling, finishing, and packaging each model are shown in the table.

Process	Model A	Model B
Assembling	3.5	8
Finishing	2.5	2
Packaging	1.3	0.7

The total times available for assembling, finishing, and packaging are 5600 hours, 2000 hours, and 910 hours, respectively. The profits per unit are $00 for model A and $50 for model B. What is the optimal production level for each model? What is the optimal profit?

73. Optimal Profit A factory manufactures two television set models: a basic model that yields $00 profit and a deluxe model that yields a profit of $80. The times (in hours) required for assembling, finishing, and packaging each model are shown in the table.

Process	Basic model	Deluxe model
Assembling	2	5
Finishing	1	2
Packaging	1	1

The total times available for assembling, finishing, and packaging are 3000 hours, 1300 hours, and 1000 hours, respectively. What is the optimal production level for each model? What is the optimal profit?

74. Optimal Profit The costs to a merchant for two models of digital camcorders are $25 and $75. The $25 model yields a profit of $5 and the $75 model yields a profit of $25. The merchant estimates that the total monthly demand will not exceed 350 units. There should be no more than $06,250 in digital camcorder inventory. Find the number of units of each model that should be stocked in order to optimize profit. What is the optimal profit?

75. Optimal Profit The costs to a merchant for two models of home theater systems are $70 and $55. The $70 model yields a profit of $0 and the $55 model yields a profit of $5. The merchant estimates that the total monthly demand will not exceed 100 units. There should be no more than $6,250 in home theater system inventory. Find the number of units of each model that should be stocked in order to optimize profit. What is the optimal profit?

76. Optimal Revenue An accounting firm has 800 hours of staff time and 90 hours of review time available each week. The firm charges $500 for an audit and $00 for a tax return. Each audit requires 100 hours of staff time and 10 hours of review time. Each tax return requires 10 hours of staff time and 2 hours of review time. What numbers of audits and tax returns will bring in an optimal revenue? What is the optimal revenue?

Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–6, solve the system of equations using the indicated method.

1. *Substitution*

$$\begin{cases} 5x - 7y = -18 \\ 4x + 3y = 20 \end{cases}$$

2. *Substitution*

$$\begin{cases} x + y = 3 \\ x^2 + y = 9 \end{cases}$$

3. *Graphing*

$$\begin{cases} 5x - y = 6 \\ 2x^2 + y = 8 \end{cases}$$

4. *Graphing*

$$\begin{cases} 1.5x - 2.25y = 8 \\ 2.5x + 2y = 5.75 \end{cases}$$

5. *Elimination*

$$\begin{cases} 2x - 4y + z = 11 \\ x + 2y + 3z = 9 \\ 3y + 5z = 12 \end{cases}$$

6. *Elimination*

$$\begin{cases} 3x - 2y + z = 16 \\ 5x - z = 6 \\ 2x - y - z = 3 \end{cases}$$

7. A total of $80,000 is invested in two funds paying 9% and 9.5% simple interest. The total annual interest is $7300. How much is invested in each fund?

8. Find the point of equilibrium for a system that has a demand equation of $p = 49 - 0.0003x$ and a supply equation of $p = 33 + 0.00002x$.

9. The numbers y of adults (in millions) who participated in baking as a leisure activity in the years 2001 to 2005 are shown in the table at the left. Find the least squares regression parabola $y = at^2 + bt + c$ for the data by solving the following system. *(Source: Mediamark Research, Inc.)*

$$\begin{cases} 5c + 10a = 188.4 \\ 10b = 5.1 \\ 10c + 34a = 383.1 \end{cases}$$

Use the model to predict the number of adults who participated in baking as a leisure activity in 2006.

Year	t	Number, y
2001	-2	37.6
2002	-1	36.6
2003	0	36.9
2004	1	37.7
2005	2	39.6

Table for 9

In Exercises 10–13, sketch the graph of the inequality.

10. $x \geq 0$

11. $y \geq 0$

12. $x + 3y \leq 12$

13. $3x + 2y \leq 15$

14. Sketch the solution set of the system of inequalities composed of the inequalities in Exercises 10–13.

15. Find the minimum and maximum values of the objective function $z = 6x + 7y$, subject to the constraints given in Exercises 10–13.

16. A manufacturer produces two models of stair climbers. The times required for assembling, painting, and packaging each model are as follows.

- Assembling: 3.5 hours for model A; 8 hours for model B
- Painting: 2.5 hours for model A; 2 hours for model B
- Packaging: 1.3 hours for model A; 0.9 hour for model B

The total times available for assembling, painting, and packaging are 5600 hours, 2000 hours, and 900 hours, respectively. The profits per unit are $800 for model A and $875 for model B. What is the optimal production level for each model? What is the optimal profit? Explain your reasoning.

Matrices and Determinants

6



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The Bank of New York, which opened for business on June 9, 1784, is the oldest bank in the United States. In 1789, Alexander Hamilton negotiated the first loan given to the government for $200,000. You can use matrices to find amounts of money borrowed at various interest rates. (See Section 6.1, Exercises 83 and 84.)

Applications

Matrices are used to solve many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Contract Bonuses, Exercise 67, page 495
- Raw Materials, Exercises 75–78, page 505
- Gypsy Moths, Exercise 15, page 525

6.1 Matrices and Linear Systems

6.2 Operations with Matrices

6.3 The Inverse of a Square Matrix

6.4 The Determinant of a Square Matrix

6.5 Applications of Matrices and Determinants

Section 6.1

Matrices and Linear Systems

- Determine the order of a matrix.
- Perform elementary row operations on a matrix in order to write the matrix in row-echelon form or reduced row-echelon form.
- Solve a system of linear equations using Gaussian elimination.
- Solve a system of linear equations using Gauss-Jordan elimination.

Matrices

In this section, you will study a streamlined technique for solving systems of linear equations. This technique involves the use of a rectangular array of real numbers called a **matrix.** The plural of matrix is **matrices.**

Definition of a Matrix

If m and n are positive integers, an $m \times n$ matrix (read "m by n") is a rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

(m rows; n columns)

in which each **entry,** a_{ij}, of the matrix is a number. An $m \times n$ matrix has m **rows** (horizontal lines) and n **columns** (vertical lines).

The entry in the ith row and jth column is denoted by the *double subscript* notation a_{ij}. That is, a_{21} refers to the entry in row 2, column 1. A matrix having m rows and n columns is said to be of **order** $m \times n$. If $m = n$, the matrix is **square** of order n. For a square matrix, the entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **main diagonal** entries.

Example 1 Orders of Matrices

The following matrices have the indicated orders.

a. *Order:* 1×4

$$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$$

b. *Order:* 2×2

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

c. *Order:* 3×2

$$\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$$

A matrix that has only one row [such as the matrix in Example 1(a)] is called a **row matrix,** and a matrix that has only one column is called a **column matrix.**

✓ **CHECKPOINT 1**

Determine the order of the matrix.

$$\begin{bmatrix} 0 & 8 & 5 \\ 3 & 1 & -2 \end{bmatrix}$$

A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the **augmented matrix** of the system. Moreover, the matrix derived from the coefficients of the system (but that does not include the constant terms) is the **coefficient matrix** of the system. Note in the matrices below the use of 0 for the coefficient of the missing y-variable in the third equation. Also note that the fourth column (the column of constant terms) in the augmented matrix is separated from the coefficients of the linear system by vertical dots.

System *Augmented Matrix* *Coefficient Matrix*

$$\begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x - 4z = 6 \end{cases} \quad \left[\begin{array}{ccc:c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right] \quad \left[\begin{array}{ccc} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{array}\right]$$

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by vertically aligning the variables in the equations and using zeros for the coefficients of any missing variables.

Original System *Line Up Variables* *Form Augmented Matrix*

$$\begin{cases} x + 3y = 9 \\ -y + 4z = -2 \\ x - 5z = 0 \end{cases} \quad \begin{cases} x + 3y = 9 \\ -y + 4z = -2 \\ x - 5z = 0 \end{cases} \quad \left[\begin{array}{ccc:c} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & -5 & 0 \end{array}\right]$$

Elementary Row Operations

In Section 5.3, you studied three operations that can be used on a system of linear equations to produce an equivalent system.

1. Interchange two equations.

2. Multiply an equation by a nonzero constant.

3. Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix of a given system of linear equations produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** if one can be obtained from the other by a sequence of elementary row operations.

Elementary Row Operations

1. Interchange two rows.

2. Multiply a row by a nonzero constant.

3. Add a multiple of a row to another row.

Although elementary row operations are simple to perform, they involve a lot of arithmetic. Because it is easy to make a mistake, you should get in the habit of noting in each step, next to the row you are changing, the elementary row operation performed, so that you can go back and check your work.

The next example demonstrates each of the elementary row operations that can be performed on a matrix to produce a row-equivalent matrix.

Example 2 Elementary Row Operations

STUDY TIP

Notice in Example 2 that the notation R_n is used to designate each row in the matrix. For example, Row 1 is represented by R_1.

a. Interchange the first and second rows.

Original Matrix

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

New Row-Equivalent Matrix

$$R_2 \\ R_1 \curvearrowleft \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

b. Multiply the first row by $\frac{1}{2}$.

Original Matrix

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\frac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

c. Add -2 times the first row to the third row.

Original Matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$-2R_1 + R_3 \rightarrow \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

Note that the elementary row operation is written beside *the row that is changing.*

✓CHECKPOINT 2

Identify the elementary row operation being performed.

Original Matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$ ■

TECHNOLOGY

Most graphing utilities can perform elementary row operations on matrices. The screens below show how one graphing utility displays each new row-equivalent matrix from Example 2. For specific instructions on how to use the elementary row operations features of a graphing utility, go to the text website at *college.hmco.com/info/larsonapplied.*

a. Interchange the first and second rows.

```
rowSwap([A],1,2)
   [[-1 2  0 3]
    [0  1  3 4]
    [2  -3 4 1]]
```

b. Multiply the first row by $\frac{1}{2}$.

```
*row(.5,[A],1)
   [[1  -2 3   -1]
    [1  3  -3  0 ]
    [5  -2 1   2 ]]
```

c. Add -2 times the first row to the third row.

```
*row+(-2,[A],1,3
)
   [[1 2   -4 3 ]
    [0 3   -2 -1]
    [0 -3 13 -8]]
```

In Example 3 in Section 5.3, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

Example 3 Comparing Linear Systems and Matrix Operations

Linear System | *Associated Augmented Matrix*

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

Add the first equation to the second equation. | Add the first row to the second row $(R_1 + R_2)$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases} \qquad R_1 + R_2 \rightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

Add -2 times the first equation to the third equation. | Add -2 times the first row to the third row $(-2R_1 + R_3)$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases} \qquad -2R_1 + R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & -1 & -1 & \vdots & -1 \end{bmatrix}$$

Add the second equation to the third equation. | Add the second row to the third row $(R_2 + R_3)$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases} \qquad R_2 + R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{bmatrix}$$

Multiply the third equation by $\frac{1}{2}$. | Multiply the third row by $\frac{1}{2}$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases} \qquad \frac{1}{2}R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

At this point, you can use back-substitution to find that the solution is

$x = 1$, $y = -1$, and $z = 2$

as shown in Example 1 in Section 5.3.

✓CHECKPOINT 3

Write the system of equations represented by the augmented matrix. Use back-substitution to find the solution. (Use the variables x, y, and z.)

$$\begin{bmatrix} 1 & -2 & 5 & \vdots & 3 \\ 0 & 1 & 4 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Remember that you can check a solution by substituting the values of x, y, and z into each equation in the original system. For example, you can check the solution to Example 3 as follows.

Equation 1: $1 - 2(-1) + 3(2) = 9$ ✓

Equation 2: $-1 + 3(-1) = -4$ ✓

Equation 3: $2(1) - 5(-1) + 5(2) = 17$ ✓

The last matrix in Example 3 is said to be in *row-echelon form.* The term *echelon* refers to the stair-step pattern formed by the nonzero entries of the matrix. To be in this form, a matrix must have the following properties.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** if every column that has a leading 1 has zeros in every position above and below the leading 1.

TECHNOLOGY

Some graphing utilities can automatically transform a matrix to row-echelon form and reduced row-echelon form. For specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

Example 4 Row-Echelon Form

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

SOLUTION The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in reduced row-echelon form because every column that has a leading 1 has zeros in every position above and below the leading 1. The matrix in (b) is not in row-echelon form because the row of all zeros does not occur at the bottom of the matrix. The matrix in (e) is not in row-echelon form because the first nonzero entry in row 2 is not 1.

✓CHECKPOINT 4

Determine whether the matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Every matrix can be converted to a row-equivalent matrix that is in row-echelon form. For instance, in Example 4, you can change the matrix in part (e) to row-echelon form by multiplying its second row by $\frac{1}{2}$, as shown below.

Original Matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Row-Echelon Form

$$\tfrac{1}{2}R_2 \rightarrow \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Gaussian Elimination with Back-Substitution

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which the elementary row operations are performed is important. You should operate from *left to right* by columns, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

Example 5 Gaussian Elimination with Back-Substitution

Solve the system.

$$\begin{cases} y + z - 2w = -3 \\ x + 2y - z = 2 \\ 2x + 4y + z - 3w = -2 \\ x - 4y - 7z - w = -19 \end{cases}$$

SOLUTION

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix} \quad R_2 \curvearrowright R_1$$

Interchange R_1 and R_2 so that there is a leading 1 in the upper left corner.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & -6 & -6 & -1 & \vdots & -21 \end{bmatrix} \quad \begin{matrix} -2R_1 + R_3 \to \\ -R_1 + R_4 \to \end{matrix}$$

Perform operations on R_3 and R_4 so that the first column has zeros below the leading 1.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & 0 & 0 & -13 & \vdots & -39 \end{bmatrix} \quad 6R_2 + R_4 \to$$

Perform operations on R_4 so that the second column has zeros below the leading 1.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & -13 & \vdots & -39 \end{bmatrix} \quad \tfrac{1}{3}R_3 \to$$

Multiply R_3 by $\frac{1}{3}$ so that the third row has a leading 1.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \quad -\tfrac{1}{13}R_4 \to$$

Multiply R_4 by $-\frac{1}{13}$ so that the fourth row has a leading 1.

The matrix is now in row-echelon form, and the corresponding system is

$$\begin{cases} x + 2y - z = 2 \\ y + z - 2w = -3 \\ z - w = -2 \\ w = 3 \end{cases}$$

Using back-substitution, you can determine that the solution is $x = -1$, $y = 2$, $z = 1$, and $w = 3$. Check this in the original system of equations.

✓CHECKPOINT 5

Solve the system.

$$\begin{cases} y + 2z - w = -5 \\ x - 3y - z = 0 \\ 2x - 6y + z + 3w = 6 \\ 4x + 4y - 2z + w = 1 \end{cases}$$

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row with zeros except for the last entry, it is unnecessary to continue the elimination process. You can conclude that the system has no solution, or is inconsistent.

Example 6 A System with No Solution

Solve the system.

$$\begin{cases} 3x + 2y - z = 1 \\ x + z = 6 \\ 2x - 3y + 5z = 4 \\ x - y + 2z = 4 \end{cases}$$

SOLUTION

$$\begin{bmatrix} 3 & 2 & -1 & \vdots & 1 \\ 1 & 0 & 1 & \vdots & 6 \\ 2 & -3 & 5 & \vdots & 4 \\ 1 & -1 & 2 & \vdots & 4 \end{bmatrix}$$

$$\begin{matrix} R_4 \\ \\ \\ R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & 6 \\ 2 & -3 & 5 & \vdots & 4 \\ 3 & 2 & -1 & \vdots & 1 \end{bmatrix}$$

$$\begin{matrix} \\ -R_1 + R_2 \to \\ -2R_1 + R_3 \to \\ -3R_1 + R_4 \to \end{matrix} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -4 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix}$$

$$\begin{matrix} \\ \\ R_2 + R_3 \to \\ \end{matrix} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & -2 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix}$$

Note that the third row of this matrix consists of zeros except for the last entry. This means that the original system of linear equations is *inconsistent.* You can see why this is true by converting back to a system of linear equations.

$$\begin{cases} x - y + 2z = 4 \\ y - z = 2 \\ 0 = -2 \\ 5y - 7z = -11 \end{cases}$$

Because $0 = -2$ is a false statement, the system has no solution.

✓CHECKPOINT 6

Solve the system.

$$\begin{cases} x + 2y - z = 3 \\ -x - y + 3z = 2 \\ 2x + 3y - 4z = 0 \\ 3x + 2y + z = -4 \end{cases}$$

Gauss-Jordan Elimination

With Gaussian elimination, elementary row operations are applied to a matrix to obtain a (row-equivalent) row-echelon form of the matrix. A second method of elimination, called **Gauss-Jordan elimination** after Carl Friedrich Gauss and Wilhelm Jordan (18421899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in Example 7.

STUDY TIP

Either Gaussian elimination or Gauss-Jordan elimination can be used to solve a system of equations. The method you use depends on your preference.

Example 7 Gauss-Jordan Elimination

Use Gauss-Jordan elimination to solve the system.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

SOLUTION In Example 3, Gaussian elimination was used to obtain the row-echelon form

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Now, apply elementary row operations until you obtain a matrix in reduced row-echelon form. To do this, you must produce zeros above each of the leading 1's, as follows.

$$2R_2 + R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Perform operations on R_1 so that the second column has a zero above the leading 1.

$$\begin{matrix} -9R_3 + R_1 \rightarrow \\ -3R_3 + R_2 \rightarrow \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Perform operations on R_1 and R_2 so that the third column has zeros above the leading 1.

Now, converting back to a system of linear equations, you have

$$\begin{cases} x = 1 \\ y = -1. \\ z = 2 \end{cases}$$

An advantage of Gauss-Jordan elimination is that you can read the solution from the matrix in reduced row-echelon form.

✓CHECKPOINT 7

Use Gauss-Jordan elimination to solve the system.

$$\begin{cases} x - 3y + 2z = 1 \\ -x - y + 3z = 4 \\ y - 2z = -5 \end{cases}$$

The elimination procedures described in this section sometimes result in fractional coefficients. For instance, in the elimination procedure for the system

$$\begin{cases} 2x - 5y + 5z = 17 \\ 3x - 2y + 3z = 11 \\ -3x + 3y = 6 \end{cases}$$

you may be inclined to multiply the first row by $\frac{1}{2}$ to produce a leading 1, which will result in working with fractional coefficients. You can sometimes avoid fractions by judiciously choosing the order in which you apply elementary row operations.

Example 8 Comparing Row-Echelon Forms

Compare the row-echelon form obtained below with the one found in Example 3. Is it the same? Does it yield the same solution?

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

$$\overset{R_2}{\underset{R_1}{\curvearrowleft}} \begin{bmatrix} -1 & 3 & 0 & \vdots & -4 \\ 1 & -2 & 3 & \vdots & 9 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

$$-R_1 \rightarrow \begin{bmatrix} 1 & -3 & 0 & \vdots & 4 \\ 1 & -2 & 3 & \vdots & 9 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

$$\begin{matrix} -R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & -3 & 0 & \vdots & 4 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 1 & 5 & \vdots & 9 \end{bmatrix}$$

$$-R_2 + R_3 \rightarrow \begin{bmatrix} 1 & -3 & 0 & \vdots & 4 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{bmatrix}$$

$$\tfrac{1}{2}R_3 \rightarrow \begin{bmatrix} 1 & -3 & 0 & \vdots & 4 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

SOLUTION This row-echelon form is different from the one that was obtained in Example 3. The corresponding system of linear equations for this matrix is

$$\begin{cases} x - 3y = 4 \\ y + 3z = 5. \\ z = 2 \end{cases}$$

Using back-substitution on this system, you obtain the solution

$$x = 1, y = -1, \text{ and } z = 2$$

which is the same solution that was obtained in Example 3. This row-echelon form is not the same as the one found in Example 3, but both forms yield the same solution. ■

✓CHECKPOINT 8

Compare the row-echelon form below with the one found in Example 8. Is it the same? Does it yield the same solution?

$$\begin{bmatrix} 1 & -2 & 0 & \vdots & 3 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

In Example 8, you discovered that the row-echelon form of a matrix is *not* unique. Two different sequences of elementary row operations may yield different row-echelon forms. However, the *reduced* row-echelon form of a given matrix *is* unique. Try applying Gauss-Jordan elimination to the row-echelon matrix in Example 8 to see that you obtain the same reduced row-echelon form as in Example 7.

STUDY TIP

Recall from Chapter 5 that when there are fewer equations than variables in a system of equations, then the system has either no solution or infinitely many solutions.

Example 9 A System with an Infinite Number of Solutions

Solve the system.

$$\begin{cases} 2x + 4y - 2z = 0 \\ 3x + 5y \quad\quad = 1 \end{cases}$$

SOLUTION

$$\begin{bmatrix} 2 & 4 & -2 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$\tfrac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$-3R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -1 & 3 & \vdots & 1 \end{bmatrix}$$

$$-R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

$$-2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x \quad + 5z = 2 \\ \quad y - 3z = -1 \end{cases}.$$

Solving for x and y in terms of z, you have $x = -5z + 2$ and $y = 3z - 1$. To write a solution of the system that does not use any of the three variables of the system, let a represent any real number and let $z = a$. Now, substitute a for z in the equations for x and y.

$$x = -5z + 2 = -5a + 2$$

$$y = 3z - 1 = 3a - 1$$

So, the solution set has the form

$$(-5a + 2, 3a - 1, a)$$

where a is a real number. Try substituting values for a to obtain a few solutions. Then check each solution in the original system of equations.

STUDY TIP

Remember that the solution set of a system with an infinite number of solutions can be written in several ways. For example, the solution set in Example 9 can be written as

$$\left(\frac{1 - 5b}{3}, b, \frac{b + 1}{3}\right)$$

where b is a real number.

✓CHECKPOINT 9

Solve the system. $\begin{cases} \quad y + 5z = -2 \\ -x + y - 4z = 8 \end{cases}$ ■

CONCEPT CHECK

1. A matrix has four columns and three rows. Is the order of the matrix 4×3? Explain.
2. Can every matrix be written in row-echelon form? Explain.
3. When solving a system of equations using Gaussian elimination, you obtain the statement $0 = 4$. What can you conclude? Explain.
4. Explain the difference between using Gaussian elimination and using Gauss-Jordan elimination when solving a system of linear equations.

Skills Review 6.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 5.1, and 5.3.

In Exercises 1–4, evaluate the expression.

1. $2(-1) - 3(5) + 7(2)$

2. $-4(-3) + 6(7) + 8(-3)$

3. $11\left(\frac{1}{2}\right) - 7\left(-\frac{3}{2}\right) - 5(2)$

4. $\frac{2}{3}\left(\frac{1}{2}\right) + \frac{4}{3}\left(-\frac{1}{3}\right)$

In Exercises 5 and 6, determine whether $x = 1$, $y = 3$, and $z = -1$ is a solution of the system.

5. $\begin{cases} 4x - 2y + 3z = -5 \\ x + 3y - z = 11 \\ -x + 2y = 5 \end{cases}$

6. $\begin{cases} -x + 2y + z = 4 \\ 2x - 3z = 5 \\ 3x + 5y - 2z = 21 \end{cases}$

In Exercises 7–10, use back-substitution to solve the system of linear equations.

7. $\begin{cases} 2x - 3y = 4 \\ y = 2 \end{cases}$

8. $\begin{cases} 5x + 4y = 0 \\ y = -3 \end{cases}$

9. $\begin{cases} x - 3y + z = 0 \\ y - 3z = 8 \\ z = 2 \end{cases}$

10. $\begin{cases} 2x - 5y + 3z = -2 \\ y - 4z = 0 \\ z = 1 \end{cases}$

Exercises 6.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, determine the order of the matrix.

1. $\begin{bmatrix} 0 & -3 & 0 \\ 9 & 2 & -7 \end{bmatrix}$

2. $\begin{bmatrix} -7 & 21 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 4 & 1 \\ 8 & 3 & 0 \\ -1 & 2 & 1 \\ 1 & 5 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 5 \\ 6 \end{bmatrix}$

5. $\begin{bmatrix} 33 & 45 \\ -9 & 20 \\ 12 & 15 \\ 16 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 12 & -2 & 4 \\ -3 & 4 & 0 \\ -8 & 12 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 7 & 11 & -3 \\ -1 & 10 & -5 & 0 \end{bmatrix}$

8. $\begin{bmatrix} -11 \end{bmatrix}$

In Exercises 9–12, fill in the blank(s) to form a new row-equivalent matrix.

	Original Matrix	*New Row-Reduced Matrix*
9.	$\begin{bmatrix} 1 & 1 & 1 \\ 5 & -2 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \square & -1 \end{bmatrix}$
10.	$\begin{bmatrix} -3 & 3 & 12 \\ 18 & -8 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & \square \\ 18 & -8 & 4 \end{bmatrix}$
11.	$\begin{bmatrix} 1 & 5 & 4 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & \square & \square \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$
12.	$\begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & 1 & 0 & \square \\ 0 & 0 & 1 & \square \end{bmatrix}$

In Exercises 13–16, identify the elementary row operation(s) being performed to obtain the new row-equivalent matrix.

	Original Matrix	*New Row-Equivalent Matrix*
13.	$\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix}$	$\begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$
14.	$\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$	$\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$
15.	$\begin{bmatrix} 0 & -1 & -5 & 5 \\ -1 & 3 & -7 & 6 \\ 4 & -5 & 1 & 3 \end{bmatrix}$	$\begin{bmatrix} -1 & 3 & -7 & 6 \\ 0 & -1 & -5 & 5 \\ 0 & 7 & -27 & 27 \end{bmatrix}$

Original Matrix *New Row-Equivalent Matrix*

16. $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$ $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 0 & -9 & 7 & -11 \\ 0 & -6 & 8 & -4 \end{bmatrix}$

In Exercises 17–22, determine whether the matrix is in row-echelon form. If it is, determine if it is also in *reduced* row-echelon form.

17. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **18.** $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & -1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ **20.** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

23. Use a graphing utility to perform the sequence of row operations in parts (a) through (d) to reduce the matrix to row-echelon form.

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -3 \\ 2 & -1 & 1 \end{bmatrix}$$

(a) Add -3 times R_1 to R_2.

(b) Add -2 times R_1 to R_3.

(c) Add 3 times R_2 to R_3.

(d) Multiply R_3 by $-\frac{1}{30}$.

24. Use a graphing utility to perform the sequence of row operations in parts (a) through (f) to reduce the matrix to *reduced* row-echelon form.

$$\begin{bmatrix} 7 & 1 \\ 0 & 2 \\ -3 & 4 \\ 4 & 1 \end{bmatrix}$$

(a) Add R_3 to R_4.

(b) Interchange R_1 and R_4.

(c) Add 3 times R_1 to R_3.

(d) Add -7 times R_1 to R_4.

(e) Multiply R_2 by $\frac{1}{2}$.

(f) Add the appropriate multiple of R_2 to R_1, R_3, and R_4.

In Exercises 25–28, write the matrix in row-echelon form. (Note: Row-echelon forms are not unique.)

25. $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 3 & 2 & 1 & 11 \\ 4 & 8 & 1 & 10 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & -5 & 14 \\ -2 & -1 & -3 & 8 \end{bmatrix}$

27. $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 5 & -4 & 1 & 8 \\ -6 & 8 & 18 & 0 \end{bmatrix}$

28. $\begin{bmatrix} 1 & -3 & 0 & -7 \\ -3 & 10 & 1 & 23 \\ 1 & 0 & 1 & 12 \\ 4 & -10 & 2 & -24 \end{bmatrix}$

In Exercises 29–34, write the matrix in reduced row-echelon form.

29. $\begin{bmatrix} 4 & 4 & 8 \\ 1 & 2 & 2 \\ -3 & 6 & -9 \end{bmatrix}$ **30.** $\begin{bmatrix} 1 & 3 & 2 \\ 5 & 15 & 9 \\ 2 & 6 & 10 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & -6 & 6 \\ 2 & 0 & 5 & -4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ **32.** $\begin{bmatrix} 1 & 2 & 3 & -5 \\ 1 & 2 & 4 & -9 \\ -2 & -4 & -4 & 3 \\ 4 & 8 & 11 & -14 \end{bmatrix}$

33. $\begin{bmatrix} 2 & 1 \\ 1 & 4 \\ -2 & -1 \end{bmatrix}$ **34.** $\begin{bmatrix} 1 & -3 \\ -1 & 8 \\ 0 & 4 \\ -2 & 10 \end{bmatrix}$

In Exercises 35–38, write the system of linear equations represented by the augmented matrix. (Use the variables *x*, *y*, z, and *w*.)

35. $\left[\begin{array}{cc:c} 2 & 4 & 6 \\ -1 & 3 & -8 \end{array}\right]$

36. $\left[\begin{array}{cc:c} 7 & -2 & 7 \\ -8 & 3 & -3 \end{array}\right]$

37. $\left[\begin{array}{ccc:c} 1 & 0 & 2 & -10 \\ 0 & 3 & -1 & 5 \\ 4 & 2 & 0 & 3 \end{array}\right]$

38. $\left[\begin{array}{cccc:c} 5 & 8 & 2 & 0 & -1 \\ -2 & 15 & 5 & 1 & 9 \\ 1 & 6 & -7 & 0 & -3 \end{array}\right]$

In Exercises 39–44, write the augmented matrix for the system of linear equations.

39. $\begin{cases} 2x - y = 3 \\ 5x + 7y = 12 \end{cases}$ **40.** $\begin{cases} 8x + 3y = 25 \\ 3x - 9y = 12 \end{cases}$

41. $\begin{cases} x + 10y - 3z = 2 \\ 5x - 3y + 4z = 0 \\ 2x + 4y = 6 \end{cases}$

42. $\begin{cases} 2x + 3y - z = 8 \\ y + 2z = -10 \\ x - 2y - 3z = 21 \end{cases}$

43. $\begin{cases} 9w - 3x + 20y + z = 13 \\ 12w - 8y = 5 \\ w + 2x + 3y - 4z = -2 \\ -w - x + y + z = 1 \end{cases}$

44. $\begin{cases} w + 2x - 3y + z = 18 \\ 3w - 5y = 8 \\ w + x + y + 2z = 15 \\ -w - x + 2y + z = -3 \end{cases}$

In Exercises 45–48, write the system of equations represented by the augmented matrix. Use back-substitution to find the solution. (Use *x*, *y*, z, and *w*.)

45. $\left[\begin{array}{cc|c} 1 & -5 & 6 \\ 0 & 1 & -2 \end{array}\right]$

46. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$

47. $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & 1 & 4 & -12 \\ 0 & 0 & 1 & -5 \end{array}\right]$

48. $\left[\begin{array}{cccc|c} 1 & 2 & -2 & 0 & -1 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$

In Exercises 49–54, an augmented matrix that represents a system of linear equations (in variables *x*, *y*, and *z*) has been reduced using Gauss-Jordan elimination. Write the solution represented by the augmented matrix.

49. $\left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 6 \end{array}\right]$

50. $\left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & -3 \end{array}\right]$

51. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right]$

52. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

53. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

54. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 9 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

In Exercises 55–78, use matrices to solve the system of equations (if possible). Use Gaussian elimination with back-substitution or Gauss-Jordan elimination.

55. $\begin{cases} x + 2y = 7 \\ 2x + y = 8 \end{cases}$

56. $\begin{cases} 2x + 6y = 16 \\ 2x + 3y = 7 \end{cases}$

57. $\begin{cases} -3x + 5y = -22 \\ 3x + 4y = 4 \\ 4x - 8y = 32 \end{cases}$

58. $\begin{cases} x + 2y = 0 \\ x + y = 6 \\ 3x - 2y = 8 \end{cases}$

59. $\begin{cases} 8x - 4y = 7 \\ 5x + 2y = 1 \end{cases}$

60. $\begin{cases} x - 3y = 5 \\ -2x + 6y = -10 \end{cases}$

61. $\begin{cases} -x + 2y = 1.5 \\ 2x - 4y = 3 \end{cases}$

62. $\begin{cases} 2x - y = -0.1 \\ 3x + 2y = 1.6 \end{cases}$

63. $\begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = -28 \\ -x + y = 14 \end{cases}$

64. $\begin{cases} -x + y - z = -14 \\ 2x - y + z = 21 \\ 3x + 2y + z = 19 \end{cases}$

65. $\begin{cases} 2x + 3z = 3 \\ 4x - 3y + 7z = 5 \\ 8x - 9y + 15z = 9 \end{cases}$

66. $\begin{cases} 2x - y + 3z = 24 \\ 2y - z = 14 \\ 7x - 5y = 6 \end{cases}$

67. $\begin{cases} x + y - 5z = 3 \\ x - 2z = 1 \\ 2x - y - z = 1 \end{cases}$

68. $\begin{cases} x - 3z = -2 \\ 3x + y - 2z = 5 \\ 2x + 2y + z = 4 \end{cases}$

69. $\begin{cases} x + 2y + z = 8 \\ 3x + 7y + 6z = 26 \end{cases}$

70. $\begin{cases} x + y + 4z = 5 \\ 2x + y - z = 9 \end{cases}$

71. $\begin{cases} 3x + 3y + 12z = 6 \\ x + y + 4z = 2 \\ 2x + 5y + 20z = 10 \\ -x + 2y + 8z = 4 \end{cases}$

72. $\begin{cases} 2x + 10y + 2z = 6 \\ x + 5y + 2z = 6 \\ x + 5y + z = 3 \\ -3x + 15y - 3z = -9 \end{cases}$

73. $\begin{cases} 4x + 12y - 7z - 20w = 22 \\ 3x + 9y - 5z - 28w = 30 \end{cases}$

74. $\begin{cases} x + 2y + 2z + 4w = 11 \\ 3x + 6y + 5z + 12w = 30 \end{cases}$

75. $\begin{cases} x + 2y = 0 \\ -x - y = 0 \end{cases}$

76. $\begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases}$

77. $\begin{cases} x + y + z = 0 \\ 2x + 3y + z = 0 \\ 3x + 5y + z = 0 \end{cases}$

78. $\begin{cases} x - 2y + z + 3w = 0 \\ x - y + w = 0 \\ y - z + 2w = 0 \end{cases}$

In Exercises 79–82, determine whether the two systems of linear equations yield the same solution. If so, find the solution using matrices.

79. (a) $\begin{cases} x - 2y + z = -6 \\ y - 5z = 16 \\ z = -3 \end{cases}$ (b) $\begin{cases} x + y - 2z = 6 \\ y + 3z = -8 \\ z = -3 \end{cases}$

80. (a) $\begin{cases} x - 3y + 4z = -11 \\ y - z = -4 \\ z = 2 \end{cases}$ (b) $\begin{cases} x + 4y = -11 \\ y + 3z = 4 \\ z = 2 \end{cases}$

81. (a) $\begin{cases} x - 4y + 5z = 27 \\ y - 7z = -54 \\ z = 8 \end{cases}$ (b) $\begin{cases} x - 6y + z = 15 \\ y + 5z = 42 \\ z = 8 \end{cases}$

82. (a) $\begin{cases} x + 3y - z = 19 \\ y + 6z = -18 \\ z = -4 \end{cases}$ (b) $\begin{cases} x - y + 3z = -15 \\ y - 2z = 14 \\ z = -4 \end{cases}$

83. Breeding Facility A city zoo borrowed $1,000,000 at simple annual interest to construct a breeding facility. Some of the money was borrowed at 8%, some at 9%, and some at 12%. Use a system of equations to determine how much was borrowed at each rate if the total annual interest was $86,000 and the amount borrowed at 8% was twice the amount borrowed at 12%. Solve the system using matrices.

84. Museum A natural history museum borrowed $1,000,000 at simple annual interest to purchase new exhibits. Some of the money was borrowed at 7%, some at 8.5%, and some at 9.5%. Use a system of equations to determine how much was borrowed at each rate if the total annual interest was $69,750 and the amount borrowed at 8.5% was four times the amount borrowed at 9.5%. Solve the system using matrices.

85. You and a friend solve the following system of equations independently.

$$\begin{cases} 2x - 4y - 3z = 3 \\ x + 3y + z = -1 \\ 5x + y - 2z = 2 \end{cases}$$

You write your solution set as

$(a, -a, 2a - 1)$

where a is any real number. Your friend's solution set is

$\left(\frac{1}{2}b + \frac{1}{2}, -\frac{1}{2}b - \frac{1}{2}, b\right)$

where b is any real number. Are you both correct? Explain. If you let $a = 3$, what value of b must be selected so that you both have the same ordered triple?

86. Describe how you would explain to another student that the augmented matrix below represents a dependent system of equations. Describe a way to write the infinitely many solutions of this system.

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & -6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

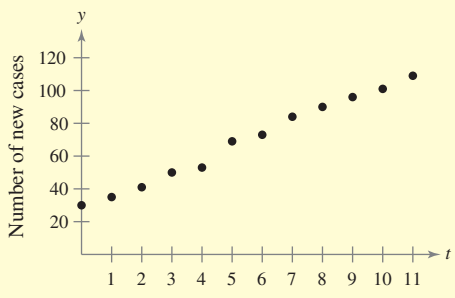
87. Health and Wellness From 1997 to 2008, the number of new cases of a waterborne disease in a small city increased in a pattern that was approximately linear (see figure). Find the least squares regression line

$y = at + b$

for the data shown in the figure by solving the following system using matrices. Let t represent the year, with $t = 0$ corresponding to 1997.

$$\begin{cases} 12b + 66a = 831 \\ 66b + 506a = 5643 \end{cases}$$

Use the result to predict the number of new cases of the waterborne disease in 2011. Is the estimate reasonable? Explain.



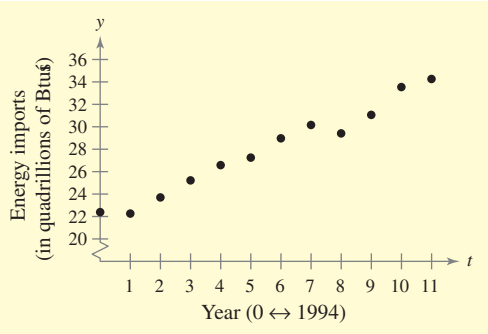
88. Energy Imports From 1994 to 2005, the total energy imports y (in quadrillions of Btu's) to the United States increased in a pattern that was approximately linear (see figure). Find the least squares regression line

$y = at + b$

for the data shown in the figure by solving the following system using matrices. Let t represent the year, with $t = 0$ corresponding to 1994.

$$\begin{cases} 12b + 66a = 334.80 \\ 66b + 506a = 1999.91 \end{cases}$$

Use the result to predict the total energy imports in 2010. Is the estimate reasonable? Explain. *(Source: Energy Information Administration)*



Section 6.2

Operations with Matrices

- Determine whether two matrices are equal.
- Add or subtract two matrices and multiply a matrix by a scalar.
- Find the product of two matrices.
- Solve a matrix equation.
- Use matrix multiplication to solve an application problem.

Equality of Matrices

In Section 6.1, you used matrices to solve systems of linear equations. Matrices, however, can do much more than this. There is a rich mathematical theory of matrices, with numerous applications. This section and the next introduce some fundamentals of matrix theory. It is standard mathematical convention to represent matrices in any of the following three ways.

1. A matrix can be denoted by an uppercase letter such as A, B, or C.
2. A matrix can be denoted by a representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.
3. A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if they have the same order $(m \times n)$ and if $a_{ij} = b_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$. In other words, two matrices are equal if their corresponding entries are equal.

Example 1 Equality of Matrices

Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

SOLUTION Because two matrices are equal only if their corresponding entries are equal, you can conclude that

$$a_{11} = 2, \quad a_{12} = -1, \quad a_{21} = -3, \quad \text{and} \quad a_{22} = 0.$$

✓CHECKPOINT 1

Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$$

Matrix Addition and Scalar Multiplication

You can **add** two matrices (of the same order) by adding their corresponding entries.

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, their **sum** is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different orders is undefined.

Example 2 Addition of Matrices

a. $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

d. The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined because A is of order 3×3 and B is of order 3×2.

✓CHECKPOINT 2

Find $\begin{bmatrix} 2 & -7 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ -1 & -6 \end{bmatrix}$. ■

In operations with matrices, numbers are usually referred to as **scalars.** In this text, scalars will always be real numbers. You can multiply a matrix A by a scalar c by multiplying each entry in A by c, as shown below.

Scalar Matrix

$$3\begin{bmatrix} -1 & 2 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 3(-1) & 3(2) \\ 3(6) & 3(5) \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 18 & 15 \end{bmatrix}$$

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, the **scalar multiple** of A by c is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

The symbol $-A$ represents the negation of A, or the scalar product $(-1)A$. Moreover, if A and B are of the same order, then $A - B$ represents the sum of A and $(-1)B$. That is,

$$A - B = A + (-1)B.$$ Subtraction of matrices

Example 3 **Scalar Multiplication and Matrix Subtraction**

For the following matrices, find (a) $3A$, (b) $-A$, and (c) $3A - B$.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

STUDY TIP

The order of operations for matrix expressions is similar to that for real numbers. In particular, you perform scalar multiplication before matrix addition and subtraction, as shown in Example 3(c).

SOLUTION

a. $3A = 3\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ Scalar multiplication

$= \begin{bmatrix} 3(2) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix}$ Multiply each entry by 3.

$= \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$ Simplify.

b. $-A = (-1)\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ Definition of negation

$= \begin{bmatrix} -2 & -2 & -4 \\ 3 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix}$ Multiply each entry by -1.

c. $3A - B = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$ Matrix subtraction

$= \begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$ Subtract corresponding entries.

✓CHECKPOINT 3

For the following matrices, find (a) $2A$ and (b) $2A - B$.

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 1 & 3 \\ -3 & 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 6 & 3 \\ 7 & -4 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For instance, in the first matrix below, the scalar $\frac{1}{2}$ has been factored out of the matrix, and in the second matrix the scalar -2 has been factored out of the matrix.

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -20 \\ -10 & -2 \end{bmatrix} = -2\begin{bmatrix} 2 & 10 \\ 5 & 1 \end{bmatrix}$$

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers.

Properties of Matrix Addition and Scalar Multiplication

If A, B, and C are $m \times n$ matrices and c and d are scalars, then the following properties are true.

1. $A + B = B + A$	Commutative Property of Matrix Addition
2. $A + (B + C) = (A + B) + C$	Associative Property of Matrix Addition
3. $(cd)A = c(dA)$	Associative Property of Scalar Multiplication
4. $1A = A$	Scalar Identity Property
5. $c(A + B) = cA + cB$	Distributive Property
6. $(c + d)A = cA + dA$	Distributive Property

Note that the Associative Property of Matrix Addition allows you to write expressions such as $A + B + C$ without ambiguity, because you obtain the same sum regardless of how the matrices are grouped. In other words, you obtain the same sum whether you group $A + B + C$ as $(A + B) + C$ or as $A + (B + C)$. This same reasoning applies to sums of four or more matrices.

Example 4 **Addition of More than Two Matrices**

Add the following four matrices.

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

SOLUTION By adding corresponding entries, you obtain the following sum of four matrices.

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

✓CHECKPOINT 4

Add the following three matrices.

$$\begin{bmatrix} 2 & -9 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 7 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 6 \\ 5 & -2 \end{bmatrix}$$

TECHNOLOGY

Most graphing utilities can add and subtract matrices and multiply matrices by scalars. Use your graphing utility to find (a) $A + B$, (b) $A - B$, (c) $4A$, and (d) $4A + B$. For specific keystrokes on how to perform matrix operations using a graphing utility, go to the text website at *college.hmco.com/info/larsonapplied.*

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 4 \\ 2 & -5 \end{bmatrix}$$

One important property of addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number c. For matrices, a similar property holds. That is, if A is an $m \times n$ matrix and O is the $m \times n$ **zero matrix** consisting entirely of zeros, then $A + O = A$.

In other words, O is the **additive identity** for the set of all $m \times n$ matrices. For example, the following matrices are the additive identities for the sets of all 2×3 and 2×2 matrices, respectively.

$$O = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2 \times 3 \text{ zero matrix}} \quad \text{and} \quad O = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{2 \times 2 \text{ zero matrix}}$$

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the following solutions.

Real Numbers (*Solve for x.*)	*m × n Matrices* (*Solve for X.*)
$x + a = b$	$X + A = B$
$x + a + (-a) = b + (-a)$	$X + A + (-A) = B + (-A)$
$x + 0 = b - a$	$X + O = B - A$
$x = b - a$	$X = B - A$

This means that you can apply some of your knowledge of solving real number equations to solving matrix equations. It is often easier to complete the algebraic steps first, and then substitute the matrices into the equation, as illustrated in Example 5.

Example 5 Solving a Matrix Equation

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

SOLUTION Begin by solving the equation for X to obtain

$$3X = B - A$$

$$X = \frac{1}{3}(B - A).$$

Now, using the matrices A and B, you have

$$X = \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right)$$ Substitute the matrices.

$$= \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}$$ Subtract matrix A from matrix B.

$$= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$ Multiply the resulting matrix by $\frac{1}{3}$.

✓CHECKPOINT 5

Solve for X in the equation $2X - A = B$, where

$$A = \begin{bmatrix} 7 & 0 \\ -1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}.$$

TECHNOLOGY

Some graphing utilities can multiply two matrices. Use your graphing utility to find the product AB.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -5 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & -2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

Now use your graphing utility to find the product BA. What is the result of this operation? For specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

Matrix Multiplication

The third basic matrix operation is **matrix multiplication.** At first glance the definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, the **product** AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$.

The definition of matrix multiplication indicates a *row-by-column* multiplication, where the entry in the ith row and jth column of the product AB is obtained by multiplying the entries in the ith row of A by the corresponding entries in the jth column of B and then adding the results. The general pattern for matrix multiplication is as follows.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$$

Example 6 **Finding the Product of Two Matrices**

Find the product AB using $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

SOLUTION First, note that the product AB is defined because the number of *columns* of A is equal to the number of *rows* of B. Moreover, the product AB has order 3×2. To find the entries of the product, multiply each row of A by each column of B.

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

✓CHECKPOINT 6

Find the product AB using

$$A = \begin{bmatrix} -2 & 2 \\ 0 & 4 \\ 3 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 0 \\ 1 & -4 \end{bmatrix}.$$

DISCOVERY

Use a graphing utility to multiply the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}.$$

Do you obtain the same result for the product AB as for the product BA? What does this tell you about matrix multiplication and commutativity?

Be sure you understand that for the product of two matrices to be defined, the number of *columns* of the first matrix must equal the number of *rows* of the second matrix. That is, the middle two indices must be the same and the outside two indices give the order of the product, as shown below.

$$\underset{m \times n}{A} \quad \underset{n \times p}{B} \quad = \quad \underset{m \times p}{AB}$$

Equal

Order of AB

Example 7 Finding the Product of Two Matrices

Find the product AB using $A = \begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ -3 & 5 \end{bmatrix}$.

SOLUTION Note that the order of A is 3×3 and the order of B is 3×2. So, the product AB is defined and is of order 3×2.

$$AB = \begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6(1) + 2(2) + 0(-3) & 6(0) + 2(7) + 0(5) \\ 3(1) + (-1)(2) + 2(-3) & 3(0) + (-1)(7) + 2(5) \\ 1(1) + 4(2) + 6(-3) & 1(0) + 4(7) + 6(5) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 \\ -5 & 3 \\ -9 & 58 \end{bmatrix}$$

✓CHECKPOINT 7

Find the product AB using

$$A = \begin{bmatrix} 0 & 4 & -3 \\ 2 & 1 & 7 \\ 3 & -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 0 \\ 0 & -4 \\ 1 & 2 \end{bmatrix}.$$

Example 8 Patterns in Matrix Multiplication

a. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$

$2 \times 3 \qquad 3 \times 3 \qquad 2 \times 3$

b. $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

c. The product AB for the following matrices is not defined.

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

$3 \times 2 \qquad 3 \times 4$

✓CHECKPOINT 8

Find AB, if possible, using $A = \begin{bmatrix} 2 & 0 & 1 \\ 6 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 3 \\ 3 & 0 \end{bmatrix}$.

Example 9 **Patterns in Matrix Multiplication**

a. $\underset{1\times 2}{\begin{bmatrix}1 & -2\end{bmatrix}}\underset{2\times 1}{\begin{bmatrix}2\\-1\end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix}4\end{bmatrix}}$

b. $\underset{2\times 1}{\begin{bmatrix}2\\-1\end{bmatrix}}\underset{1\times 2}{\begin{bmatrix}1 & -2\end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix}2 & -4\\-1 & 2\end{bmatrix}}$

✓CHECKPOINT 9

Find AB and BA using $A = \begin{bmatrix}3 & -1\end{bmatrix}$ and $B = \begin{bmatrix}1\\-3\end{bmatrix}$. ■

In Example 9, note that the two products are different. Even if AB and BA are defined, matrix multiplication is not, in general, commutative. That is, for most matrices, $AB \neq BA$.

Properties of Matrix Multiplication

Let A, B, and C be matrices and let c be a scalar.

1. $A(BC) = (AB)C$ — Associative Property of Matrix Multiplication
2. $A(B + C) = AB + AC$ — Left Distributive Property
3. $(A + B)C = AC + BC$ — Right Distributive Property
4. $c(AB) = (cA)B = A(cB)$ — Associative Property of Scalar Multiplication

Definition of the Identity Matrix

The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of order *n*** and is denoted by

$$I_n = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\\0 & 1 & 0 & \cdots & 0\\0 & 0 & 1 & \cdots & 0\\\vdots & \vdots & \vdots & & \vdots\\0 & 0 & 0 & \cdots & 1\end{bmatrix}.$$ Identity matrix

Note that an identity matrix must be *square*. When the order is understood to be n, you can denote I_n simply by I.

If A is an $n \times n$ matrix, the identity matrix has the property that $AI_n = A$ and $I_nA = A$. For example,

$$\begin{bmatrix}3 & -2 & 5\\1 & 0 & 4\\-1 & 2 & -3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}3 & -2 & 5\\1 & 0 & 4\\-1 & 2 & -3\end{bmatrix} \quad AI = A$$

and

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & -2 & 5\\1 & 0 & 4\\-1 & 2 & -3\end{bmatrix} = \begin{bmatrix}3 & -2 & 5\\1 & 0 & 4\\-1 & 2 & -3\end{bmatrix}. \quad IA = A$$

Applications

One application of matrix multiplication is the representation of a system of linear equations. Note how the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

can be written as the matrix equation $AX = B$, where A is the *coefficient matrix* of the system, and X and B are column matrices.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A \quad \times X = B$$

STUDY TIP

The column matrix B is also called a *constant matrix*. Its entries are the constant terms in the system of equations.

Example 10 Solving a System of Linear Equations

Consider the system of linear equations.

$$\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$$

a. Write this system as a matrix equation $AX = B$.

b. Use Gauss-Jordan elimination on $[A \vdots B]$ to solve for the matrix X.

SOLUTION

a. In matrix form $AX = B$, the system is written as

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}.$$

Coefficient matrix Constant matrix

b. The augmented matrix is formed by adjoining matrix B to matrix A.

$$[A \vdots B] = \left[\begin{array}{ccc:c} 1 & -2 & 1 & -4 \\ 0 & 1 & 2 & 4 \\ 2 & 3 & -2 & 2 \end{array}\right]$$

Using Gauss-Jordan elimination, you can rewrite this matrix as

$$[I \vdots X] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

So, the solution of the matrix equation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

STUDY TIP

The notation $[A \vdots B]$ represents the augmented matrix formed when matrix B is adjoined to matrix A. The notation $[I \vdots X]$ represents the reduced row-echelon form of the augmented matrix that yields the solution of the system.

✓CHECKPOINT 10

Write the system of linear equations as a matrix equation $AX = B$. Then use Gauss-Jordan elimination on the augmented matrix $[A \vdots B]$ to solve for the matrix X.

$$\begin{cases} -2x_1 - 3x_2 = -4 \\ 6x_1 + x_2 = -36 \end{cases}$$

Example 11 Long-Distance Phone Plans

The charges (in dollars per minute) of two long-distance telephone companies are shown in the table.

	Company A	Company B
In-state	0.07	0.095
State-to-state	0.10	0.08
International	0.28	0.25

You plan to use 120 minutes on in-state long-distance calls, 80 minutes on state-to-state calls, and 20 minutes on international calls. Use matrices to determine which company you should choose to be your long-distance carrier.

SOLUTION The charges C and amounts of time T spent on the phone can be written in matrix form as

$$C = \begin{bmatrix} 0.07 & 0.095 \\ 0.10 & 0.08 \\ 0.28 & 0.25 \end{bmatrix} \quad \text{and} \quad T = [120 \quad 80 \quad 20].$$

The total amount that each company charges is given by the product

$$TC = [120 \quad 80 \quad 20] \begin{bmatrix} 0.07 & 0.095 \\ 0.10 & 0.08 \\ 0.28 & 0.25 \end{bmatrix} = [22 \quad 22.8].$$

Company A charges \$22 for the calls and Company B charges \$22.80. Company A charges less for the calling pattern, so you should choose Company A. Notice that you cannot find the total amount that each company charges using the product CT because CT is not defined. That is, the number of columns of C does not equal the number of rows of T.

✓CHECKPOINT 11

In Example 11, suppose you plan to use 100 minutes on in-state long-distance calls, 70 minutes on state-to-state calls, and 40 minutes on international calls. Use matrices to determine which company you should choose to be your long-distance carrier. ■

CONCEPT CHECK

1. Under what conditions are matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ equal?
2. What is the sum of a matrix A and the negation of A?
3. Discuss the similarities and differences between solving real number equations and solving matrix equations.
4. Explain why AB is not defined and BA is defined when matrix A is of order 1×3 and matrix B is of order 2×1.

Skills Review 6.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2 and 6.1.

In Exercises 1 and 2, evaluate the expression.

1. $-3\left(-\frac{5}{6}\right) + 10\left(-\frac{3}{4}\right)$

2. $-22\left(\frac{5}{2}\right) + 6(8)$

In Exercises 3 and 4, determine whether the matrix is in *reduced* row-echelon form.

3. $\begin{bmatrix} 0 & 1 & 0 & -5 \\ 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}$

In Exercises 5 and 6, write the augmented matrix for the system of linear equations.

5. $\begin{cases} -5x + 10y = 12 \\ 7x - 3y = 0 \end{cases}$

6. $\begin{cases} 10x + 15y - 9z = 42 \\ 6x - 5y = 0 \end{cases}$

In Exercises 7–10, solve the system of linear equations represented by the augmented matrix.

7. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

10. $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Exercises 6.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find x and y.

1. $\begin{bmatrix} 4 & x \\ -1 & y \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} x & -7 \\ 9 & y \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ 9 & -8 \end{bmatrix}$

3. $\begin{bmatrix} -4 & 3 \\ 6 & -1 \\ 8 & 2 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} x-2 & 3 \\ 6 & -1 \\ 8 & -x \\ 5 & 2y-1 \end{bmatrix}$

4. $\begin{bmatrix} x+2 & 8 & -3 \\ 1 & 2y & 2x \\ 7 & -2 & y+2 \end{bmatrix} = \begin{bmatrix} 2x+6 & 8 & -3 \\ 1 & 18 & -8 \\ 7 & -2 & 11 \end{bmatrix}$

In Exercises 5–10, find (a) $A + B$, (b) $A - B$, (c) $3A$, and (d) $3A - 2B$.

5. $A = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$

6. $A = \begin{bmatrix} 7 & 4 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 8 & -4 \end{bmatrix}$

7. $A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$

8. $A = \begin{bmatrix} 6 & 8 & -3 & 2 & 1 \\ -4 & 2 & 1 & 5 & -2 \end{bmatrix}$,
$B = \begin{bmatrix} 6 & 0 & 4 & -1 & 3 \\ 4 & 5 & -2 & 1 & 2 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 1 & -2 \\ 1 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 4 & 9 \\ 0 & -7 & 8 \end{bmatrix}$

10. $A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} -4 \\ 6 \\ 2 \end{bmatrix}$

In Exercises 11–16, evaluate the expression.

11. $\begin{bmatrix} -5 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -10 & -8 \\ 14 & 6 \end{bmatrix}$

12. $\begin{bmatrix} 6 & 8 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} -11 & -7 \\ 2 & -1 \end{bmatrix}$

13. $4\left(\begin{bmatrix} -4 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 3 & -6 & 0 \end{bmatrix}\right)$

14. $\frac{1}{2}([5 \quad -2 \quad 4 \quad 0] + [14 \quad 6 \quad -18 \quad 9])$

15. $-3\left(\begin{bmatrix} 0 & -3 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 8 & 1 \end{bmatrix}\right) - 2\begin{bmatrix} 4 & -4 \\ 7 & -9 \end{bmatrix}$

16. $-1\begin{bmatrix} 4 & 11 \\ -2 & -1 \\ 9 & 3 \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} -5 & -1 \\ 3 & 4 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ -9 & -1 \\ 6 & -1 \end{bmatrix}\right)$

Ⓣ In Exercises 17–20, use the matrix capabilities of a graphing utility to evaluate the expression. Round your results to three decimal places, if necessary.

17. $\frac{3}{7}\begin{bmatrix} 2 & 5 \\ -1 & -4 \end{bmatrix} + 6\begin{bmatrix} -3 & 0 \\ 2 & 2 \end{bmatrix}$

18. $55\left(\begin{bmatrix} 14 & -11 \\ -22 & 19 \end{bmatrix} + \begin{bmatrix} -22 & 20 \\ 13 & 6 \end{bmatrix}\right)$

19. $-\begin{bmatrix} 3.211 & 6.829 \\ -1.004 & 4.914 \\ 0.055 & -3.889 \end{bmatrix} - \begin{bmatrix} -1.630 & -3.090 \\ 5.256 & 8.335 \\ -9.768 & 4.251 \end{bmatrix}$

20. $-12\left(\begin{bmatrix} 6 & 20 \\ 1 & -9 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 14 & -15 \\ -8 & -6 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} -31 & -19 \\ 16 & 10 \\ 24 & -10 \end{bmatrix}\right)$

In Exercises 21–24, solve for X when

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}.$$

21. $X = 3A - 2B$

22. $2X = 2A - B$

23. $2X + 3A = B$

24. $2A + 4B = -2X$

In Exercises 25–32, find AB, if possible.

25. $A = \begin{bmatrix} 3 & -2 \\ 4 & 5 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 4 & -2 & 5 \\ 2 & 1 & 3 & -1 \end{bmatrix}$

26. $A = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$

27. $A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

29. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

30. $A = \begin{bmatrix} 6 \\ -2 \\ 1 \\ 6 \end{bmatrix}, \quad B = [10 \quad 12]$

31. $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ 5 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

32. $A = \begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 6 & 13 & 8 & -17 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$

Ⓣ In Exercises 33–38, use the matrix capabilities of a graphing utility to find AB, if possible.

33. $A = \begin{bmatrix} 5 & 6 & -3 \\ -2 & 5 & 1 \\ 10 & -5 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 2 \\ 8 & 1 & 4 \\ 4 & -2 & 9 \end{bmatrix}$

34. $A = \begin{bmatrix} 11 & -12 & 4 \\ 14 & 10 & 12 \\ 6 & -2 & 9 \end{bmatrix}, B = \begin{bmatrix} 12 & 10 \\ -5 & 12 \\ 15 & 16 \end{bmatrix}$

35. $A = \begin{bmatrix} -3 & 8 & -6 & 8 \\ -12 & 15 & 9 & 6 \\ 5 & -1 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & 6 \\ 24 & 15 & 14 \\ 16 & 10 & 21 \\ 8 & -4 & 10 \end{bmatrix}$

36. $A = \begin{bmatrix} 15 & -18 \\ -4 & 12 \\ -8 & 22 \end{bmatrix}, B = \begin{bmatrix} -7 & 22 & 1 \\ 8 & 16 & 24 \end{bmatrix}$

37. $A = \begin{bmatrix} -2 & 4 & 8 \\ 21 & 5 & 6 \\ 13 & 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ -7 & 15 \\ 32 & 14 \\ 0.5 & 1.6 \end{bmatrix}$

38. $A = \begin{bmatrix} 9 & 10 & -38 & 18 \\ 100 & -50 & 250 & 75 \end{bmatrix}, B = \begin{bmatrix} 52 & -85 & 27 & 45 \\ 40 & -35 & 60 & 82 \end{bmatrix}$

In Exercises 39–44, find (a) AB, (b) BA, and, if possible, (c) A^2. (*Note:* $A^2 = AA$.)

39. $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

40. $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix}$

41. $A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -1 & -2 \\ 2 & 4 \end{bmatrix}$

42. $A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

43. $A = [-4 \quad 2 \quad 3]$, $B = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

44. $A = [3 \quad 2 \quad 1 \quad 0]$, $B = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

In Exercises 45–50, (a) write the system of linear equations as a matrix equation $AX = B$, and (b) use Gauss-Jordan elimination on the augmented matrix $[A \vdots B]$ to solve for the matrix X.

45. $\begin{cases} -x + y = 4 \\ -2x + y = 0 \end{cases}$

46. $\begin{cases} 2x + 3y = 5 \\ x + 4y = 10 \end{cases}$

47. $\begin{cases} x + 2y = 3 \\ 3x - y = 2 \end{cases}$

48. $\begin{cases} 2x - 4y + z = 0 \\ -x + 3y + z = 1 \\ x + y = 3 \end{cases}$

49. $\begin{cases} x - 2y + 3z = 9 \\ -x + 3y - z = -6 \\ 2x - 5y + 5z = 17 \end{cases}$

50. $\begin{cases} x + y - 3z = -1 \\ -x + 2y = 1 \\ x - y + z = 2 \end{cases}$

51. Factory Production A corporation that makes sunglasses has four factories, each of which manufactures two products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 100 & 120 & 60 & 40 \\ 140 & 160 & 200 & 80 \end{bmatrix}.$$

Find the production levels if production is increased by 10%. (*Hint:* Because an increase of 10% corresponds to 100% + 10% multiply the matrix by 1.10.)

52. Factory Production A tire corporation has three factories, each of which manufactures two products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 80 & 120 & 140 \\ 40 & 100 & 80 \end{bmatrix}.$$

Find the production levels if production is decreased by 5%. (*Hint:* Because a decrease of 5% corresponds to 100% − 5% multiply the matrix by 0.95.)

53. Hotel Pricing A convention planning service has identified three suitable hotels for a convention. The quoted room rates are for single, double, triple, and quadruple occupancy. The current rates for the four types of rooms at the three hotels are represented by the matrix A.

$$A = \begin{array}{c} \begin{matrix} \text{Hotel } x & \text{Hotel } y & \text{Hotel } z \end{matrix} \\ \begin{bmatrix} 85 & 92 & 110 \\ 100 & 120 & 130 \\ 110 & 130 & 140 \\ 110 & 140 & 155 \end{bmatrix} \begin{matrix} \text{Single} \\ \text{Double} \\ \text{Triple} \\ \text{Quadruple} \end{matrix} \end{array} \} \text{Occupancy}$$

If room rates are guaranteed not to increase by more than 15% by the time of the convention, what is the maximum rate per room per hotel?

54. Vacation Packages A vacation service has identified four resort hotels with a special all-inclusive package (room and meals included) at a popular travel destination. The quoted room rates are for double and family (maximum of four people) occupancy for 5 days and 4 nights. The current rates for the two types of rooms at the four hotels are represented by the matrix A.

$$A = \begin{array}{c} \begin{matrix} \text{Hotel } w & \text{Hotel } x & \text{Hotel } y & \text{Hotel } z \end{matrix} \\ \begin{bmatrix} 615 & 670 & 740 & 990 \\ 995 & 1030 & 1180 & 1105 \end{bmatrix} \begin{matrix} \text{Double} \\ \text{Family} \end{matrix} \end{array} \} \text{Occupancy}$$

If room rates are guaranteed not to increase by more than 12% by next season, what is the maximum rate per package per hotel?

55. Inventory Levels A company sells five different models of computers through three retail outlets. The inventories of the five models at the three outlets are given by the matrix S.

$$S = \begin{array}{c} \overbrace{\begin{matrix} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \end{matrix}}^{\text{Model}} \\ \begin{bmatrix} 3 & 2 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 & 3 \\ 4 & 2 & 1 & 3 & 2 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array} \} \text{Outlet}$$

The wholesale and retail prices for each model are given by the matrix T.

$$T = \begin{array}{c} \text{Price} \\ \overbrace{\begin{matrix} \text{Wholesale} & \text{Retail} \end{matrix}} \\ \begin{bmatrix} \$900 & \$200 \\ \$200 & \$450 \\ \$400 & \$650 \\ \$650 & \$250 \\ \$050 & \$375 \end{bmatrix} \end{array} \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{matrix} \} \text{Model}$$

(a) What is the total retail price of the inventory at Outlet 1?

(b) What is the total wholesale price of the inventory at Outlet 3?

(c) Compute the product ST and interpret the result in the context of the problem.

56. Labor/Wage Requirements A company that manufactures boats has the following labor-hour and wage requirements.

Labor-Hour Requirements (per boat)

$$S = \begin{array}{c} \text{Department} \\ \overbrace{\begin{matrix} \text{Cutting} & \text{Assembly} & \text{Packaging} \end{matrix}} \\ \begin{bmatrix} 1.0 \text{ hour} & 0.5 \text{ hour} & 0.2 \text{ hour} \\ 1.6 \text{ hours} & 1.0 \text{ hour} & 0.2 \text{ hour} \\ 2.5 \text{ hours} & 2.0 \text{ hours} & 0.4 \text{ hour} \end{bmatrix} \end{array} \begin{matrix} \text{Small} \\ \text{Medium} \\ \text{Large} \end{matrix} \} \begin{matrix} \text{Boat} \\ \text{size} \end{matrix}$$

Wage Requirements (per hour)

$$T = \begin{array}{c} \text{Plant} \\ \overbrace{\begin{matrix} \text{A} & \text{B} \end{matrix}} \\ \begin{bmatrix} \$ & \$5 \\ \$3 & \$2 \\ \$0 & \$1 \end{bmatrix} \end{array} \begin{matrix} \text{Cutting} \\ \text{Assembly} \\ \text{Packaging} \end{matrix} \} \text{Department}$$

(a) What is the labor cost for a medium boat at Plant B?

(b) What is the labor cost for a large boat at Plant A?

(c) Compute ST and interpret the result.

57. Exercise The numbers of calories burned by individuals of different body weights while performing different types of aerobic exercises for a 20-minute time period are shown in the matrix A.

$$A = \begin{array}{c} \text{Calories burned} \\ \overbrace{\begin{matrix} \text{120-lb person} & \text{150-lb person} \end{matrix}} \\ \begin{bmatrix} 109 & 136 \\ 127 & 159 \\ 64 & 79 \end{bmatrix} \end{array} \begin{matrix} \text{Bicycling} \\ \text{Jogging} \\ \text{Walking} \end{matrix}$$

(a) A 120-pound person and a 150-pound person bicycled for 40 minutes, jogged for 10 minutes, and walked for 60 minutes. Organize the times spent exercising in a matrix B.

(b) Compute BA and interpret the result.

58. Agriculture A fruit grower raises apples and peaches, which are shipped to three different outlets. The numbers of units of apples and peaches that are shipped to the three outlets are shown in the matrix A.

$$A = \begin{array}{c} \text{Outlet} \\ \overbrace{\begin{matrix} \text{X} & \text{Y} & \text{Z} \end{matrix}} \\ \begin{bmatrix} 125 & 100 & 75 \\ 100 & 175 & 125 \end{bmatrix} \end{array} \begin{matrix} \text{Apples} \\ \text{Peaches} \end{matrix} \} \text{Units shipped}$$

(a) The profit per unit of apples is \$.50 and the profit per unit of peaches is \$. Organize the profits per unit in a matrix B.

(b) Compute BA and interpret the result.

Think About It In Exercises 59–66, let matrices A, B, C, and D be of orders 2×3, 2×3, 3×2, and 2×2, respectively. Determine whether the matrices are of proper order to perform the operation(s). If so, give the order of the answer.

59. $A + 2C$

60. $B - 3C$

61. AB

62. BC

63. $BC - D$

64. $CB - D$

65. $D(A - 3B)$

66. $(BC - D)A$

67. Contract Bonuses Professional athletes frequently have bonus or incentive clauses in their contracts. For example, a defensive football player might receive bonuses for defensive plays such as sacks, interceptions, and/or key tackles. In one contract, a sack is worth \$000, an interception is worth \$000, and a key tackle is worth \$00. The table shows the numbers of sacks, interceptions, and key tackles for three players.

Player	Sacks	Interceptions	Key tackles
Player X	3	0	4
Player Y	1	2	5
Player Z	2	3	3

(a) Write a matrix D that represents the number of each type of defensive play i made by each player j using the data from the table. State what each entry d_{ij} of the matrix represents.

(b) Write a matrix B that represents the bonus amount received for each type of defensive play. State what each entry b_{ij} of the matrix represents.

(c) Find the product BD of the two matrices and state what each entry of matrix BD represents.

(d) Which player receives the largest bonus?

68. Long-Distance Plans You are choosing between two monthly long-distance phone plans offered by two different companies. Company A charges ¢0.05 per minute for in-state calls, ¢0.12 per minute for state-to-state calls, and ¢0.30 per minute for international calls. Company B charges ¢0.085 per minute for in-state calls, ¢0.10 per minute for state-to-state calls, and ¢0.25 per minute for international calls. In a month, you normally use 20 minutes on in-state calls, 60 minutes on state-to-state calls, and 30 minutes on international calls.

(a) Write a matrix C that represents the charges for each type of call i by each company j. State what each entry c_{ij} of the matrix represents.

(b) Write a matrix T that represents the times spent on the phone for each type of call. State what each entry of the matrix represents.

(c) Find the product TC and state what each entry of the matrix represents.

(d) Which company should you choose? Explain.

69. Voting Preference The matrix

$$P = \begin{array}{c} \\ \text{R} \\ \text{D} \\ \text{I} \end{array} \overset{\overset{\text{From}}{\overbrace{}}}{\begin{bmatrix} \text{R} & \text{D} & \text{I} \\ 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix}} \begin{array}{l} \text{R} \\ \text{D} \\ \text{I} \end{array} \left.\vphantom{\begin{bmatrix}0\\0\\0\end{bmatrix}}\right\} \text{To}$$

is called a *stochastic matrix*. Each entry p_{ij} $(i \neq j)$ represents the proportion of the voting population that changes from Party i to Party j, and p_{ii} represents the proportion that remains loyal to the party from one election to the next. Use a graphing utility to find P^2. (This matrix gives the transition probabilities from the first election to the third.)

70. Voting Preference Use a graphing utility to find P^3, P^4, P^5, P^6, P^7, and P^8 for the matrix given in Exercise 69. Can you detect a pattern as P is raised to higher and higher powers?

In Exercises 71 and 72, find a matrix B such that AB is the identity matrix. Is there more than one correct result?

71. $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

72. $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$

73. If a, b, and c are real numbers such that $c \neq 0$ and $ac = bc$, then $a = b$. However, if A, B, and C are matrices such that $AC = BC$, then A is *not* necessarily equal to B. Illustrate this using the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix},$$

and $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix}$

74. If a and b are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$. However, if A and B are matrices such that $AB = O$, then it is *not* necessarily true that $A = O$ or $B = O$. Illustrate this using the following matrices.

$$A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Find another example of two nonzero matrices whose product is the zero matrix.

In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.

75. $\begin{bmatrix} 3 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -4 \end{bmatrix}$

76. $\begin{bmatrix} -6 & -2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -6 & -2 \\ 2 & -6 \end{bmatrix}$

77. Cable Television Two competing companies offer cable television to a city with 100,000 households. Gold Cable Company has 25,000 subscribers and Galaxy Cable Company has 30,000 subscribers. (The other 45,000 households do not subscribe.) The percent changes in cable subscriptions each year are shown in the matrix below.

$$\begin{array}{c} \\ \text{Percent Changes} \\ \\ \end{array} \overset{\overset{\text{Percent Changes}}{\overbrace{}}}{\begin{array}{r} \\ \text{To Gold} \\ \text{To Galaxy} \\ \text{To Nonsubscriber} \end{array} \begin{bmatrix} \text{From Gold} & \text{From Galaxy} & \text{From Non-subscriber} \\ 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}}$$

(a) Find the number of subscribers each company will have in one year using matrix multiplication. Explain how you obtained your answer.

(b) Find the number of subscribers each company will have in two years using matrix multiplication. Explain how you obtained your answer.

(c) Find the number of subscribers each company will have in three years using matrix multiplication. Explain how you obtained your answer.

(d) What is happening to the number of subscribers to each company? What is happening to the number of nonsubscribers?

78. Extended Application To work an extended application analyzing airline routes with matrices, visit this text's website at *college.hmco.com.*

Section 6.3

The Inverse of a Square Matrix

- Verify that a matrix is the inverse of a given matrix.
- Find the inverse of a matrix.
- Find the inverse of a 2 × 2 matrix using a formula.
- Use an inverse matrix to solve a system of linear equations.

The Inverse of a Matrix

This section further develops the algebra of matrices. To begin, consider the real number equation $ax = b$. To solve this equation for x, multiply each side of the equation by a^{-1} (provided $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number a^{-1} is called the *multiplicative inverse of a* because $a^{-1}a = 1$. The definition of the multiplicative inverse of a matrix is similar.

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is called the **inverse** of A. (The symbol A^{-1} is read "A inverse.")

Example 1 The Inverse of a Matrix

Show that B is the inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

SOLUTION To show that B is the inverse of A, show that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

STUDY TIP

Recall that it is not always true that $AB = BA$, even if both products are defined. However, if A and B are both square matrices and $AB = I_n$, it can be shown that $BA = I_n$. So, in Example 1, you need only check that $AB = I_2$.

✓CHECKPOINT 1

Show that B is the inverse of A, where $A = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$. ■

If a matrix A has an inverse, A is called **invertible** (or **nonsingular**); otherwise, A is called **singular.** A nonsquare matrix cannot have an inverse. To see this, note that if A is of order $m \times n$ and B is of order $n \times m$ (where $m \neq n$), the products AB and BA are of different orders and therefore cannot be equal to each other. Not all square matrices have inverses (see the matrix at the bottom of page 500). If, however, a matrix does have an inverse, that inverse is unique. The following example shows how to use a system of equations to find an inverse.

Example 2 Finding the Inverse of a Matrix

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

SOLUTION To find the inverse of A, try to solve the matrix equation $AX = I$ for X.

$$AX = I$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Write matrix equation.}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Multiply } A \text{ and } X.$$

Equating corresponding entries, you obtain the following two systems of linear equations.

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \qquad \begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases}$$

You can solve these systems using the methods learned in Chapter 5. From the first system you can determine that $x_{11} = -3$ and $x_{21} = 1$, and from the second system you can determine that $x_{12} = -4$ and $x_{22} = 1$. So, the inverse of A is

$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

You can use matrix multiplication to check this result.

CHECK

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1}A = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

✓CHECKPOINT 2

Find the inverse of the matrix

$$A = \begin{bmatrix} -4 & -1 \\ 5 & 1 \end{bmatrix}.$$

Finding Inverse Matrices

In Example 2, note that the two systems of linear equations have the *same coefficient matrix A*. Rather than solve the two systems represented by

$$\left[\begin{array}{cc:c} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc:c} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right]$$

separately, you can solve them simultaneously by adjoining the identity matrix to the coefficient matrix to obtain

$$\begin{array}{cc:cc} A & & I & \\ \left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right]. \end{array}$$

Then, by applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process, as follows.

$$\begin{array}{rl} & \left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \\ R_1 + R_2 \rightarrow & \left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \\ -4R_2 + R_1 \rightarrow & \left[\begin{array}{cc:cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

So, from the "doubly augmented" matrix $[A \vdots I]$, you obtain $[I \vdots A^{-1}]$.

$$\begin{array}{cc:cc} A & & I & \\ \left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] & \Rightarrow & \begin{array}{cc:cc} I & & A^{-1} & \\ \left[\begin{array}{cc:cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array} \end{array}$$

This procedure (or algorithm) works for an arbitrary square matrix that has an inverse.

Finding an Inverse Matrix

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \vdots I]$. Note that the matrices A and I are separated by a dotted line. This process is called **adjoining** the matrices A and I.
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \vdots I]$. The result will be the matrix $[I \vdots A^{-1}]$. If this is not possible, A is not invertible.
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

DISCOVERY

Select two 2×2 matrices A and B that have inverses. Calculate $(AB)^{-1}$ and then calculate $B^{-1}A^{-1}$ and $A^{-1}B^{-1}$. Make a conjecture about the inverse of the product of two invertible matrices.

Example 3 **Finding the Inverse of a Matrix**

Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$.

SOLUTION Begin by adjoining the identity matrix to A to form the matrix

$$[A \vdots I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right].$$

Use elementary row operations to obtain the matrix $[I \vdots A^{-1}]$, as follows.

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -6R_1 + R_3 \rightarrow \end{array} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 \rightarrow \\ \\ -4R_2 + R_3 \rightarrow \end{array} \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$\begin{array}{r} R_3 + R_1 \rightarrow \\ R_3 + R_2 \rightarrow \\ \end{array} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

So, the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}.$$

Confirm this result by multiplying A and A^{-1} to obtain I, as follows.

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

✓CHECKPOINT 3

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & 2 \\ 1 & -2 & 0 \end{bmatrix}.$$

The process shown in Example 3 applies to any $n \times n$ matrix A. If A has an inverse, this process will find it. When using this process, if the matrix A does not reduce to the identity matrix, then A does not have an inverse.

To confirm that matrix A shown below has no inverse, begin by adjoining the identity matrix to A to form the following.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix} \Rightarrow [A \vdots I] = \left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

Then use elementary row operations to obtain

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right].$$

At this point in the elimination process, you can see that it is impossible to obtain the identity matrix I on the left. So, A is not invertible.

The Inverse of a 2 × 2 Matrix (Quick Method)

DISCOVERY

Use a graphing utility with matrix operations to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$$

What message appears on the screen? Why does the graphing utility display an error message?

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of order 3×3 or greater. For 2×2 matrices, however, many people prefer to use a formula for the inverse rather than Gauss-Jordan elimination. This simple formula, which works *only* for 2×2 matrices, is explained as follows. If A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Try verifying this inverse by multiplication.

The denominator $ad - bc$ is called the **determinant** of the 2×2 matrix A. You will study determinants in the next section.

Example 4 Finding the Inverse of a 2 × 2 Matrix

If possible, find the inverse of each matrix.

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

SOLUTION

a. For the matrix A, begin by applying the formula for the determinant of a 2×2 matrix to obtain

$$ad - bc = 3(2) - (-1)(-2) = 4.$$

Because this quantity is not zero, the matrix is invertible. The inverse is formed by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$, as follows.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Formula for inverse of a 2×2 matrix

$$= \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$ Substitute for a, b, c, d, and the determinant.

$$= \begin{bmatrix} \frac{1}{4}(2) & \frac{1}{4}(1) \\ \frac{1}{4}(2) & \frac{1}{4}(3) \end{bmatrix}$$ Multiply by the scalar $\frac{1}{4}$.

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$ Simplify.

b. For the matrix B, you have

$$ad - bc = 3(2) - (-1)(-6)$$
$$= 0.$$

Because $ad - bc = 0$, B is not invertible.

✓CHECKPOINT 4

Find the inverse of the matrix

$$A = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}.$$

Systems of Linear Equations

You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution. If the coefficient matrix A of a *square* system (a system that has the same number of equations as variables) is invertible, then the system has a unique solution, which is defined as follows.

TECHNOLOGY

To solve a system of equations with a graphing utility, enter the matrices A and B in the *matrix editor*. Then, using the inverse key, solve for X.

A [x⁻¹] B [ENTER]

The screen will display the solution, matrix X.

A System of Equations with a Unique Solution

If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$.

Example 5 Solving a System of Equations Using an Inverse Matrix

Use an inverse matrix to solve the system.

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$$

SOLUTION Begin by writing the system in the matrix form $AX = B$.

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

Next, use Gauss-Jordan elimination to find A^{-1}.

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Finally, multiply B by A^{-1} on the left to obtain the solution.

$$X = A^{-1}B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

So, the solution is $x = 2$, $y = -1$, and $z = -2$.

✓CHECKPOINT 5

Use an inverse matrix to solve the system.

$$\begin{cases} -x + y + z = 4 \\ 2x - y - 3z = -7 \\ -2x + 3y + 2z = 10 \end{cases}$$

CONCEPT CHECK

1. What is the product of a square matrix of order n and its inverse?
2. Matrix A is a singular matrix of order n. Does a matrix B exist such that $AB = I$? Explain.
3. Consider the matrix $A = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$, where $x_{11} \cdot x_{22} = x_{12} \cdot x_{21}$. Is A invertible? Explain.
4. Matrix A is nonsingular. Can a system of linear equations represented by $AX = B$ have infinitely many solutions? Explain.

Skills Review 6.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 6.1 and 6.2.

In Exercises 1–8, perform the indicated matrix operations.

1. $4\begin{bmatrix} 1 & 6 \\ 0 & -4 \\ 12 & 2 \end{bmatrix}$

2. $\frac{1}{2}\begin{bmatrix} 11 & 10 & 48 \\ 1 & 0 & 16 \\ 0 & 2 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -10 & 3 \\ 4 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 3 & -4 & 8 \\ 0 & 7 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 5 & 20 \\ -7 & 15 \end{bmatrix} - 3\begin{bmatrix} 6 & 3 \\ 4 & -2 \end{bmatrix}$

5. $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & 5 \\ 3 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & -2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -9 & -8 & 11 \\ 6 & 5 & -7 \end{bmatrix}$

In Exercises 9 and 10, rewrite the matrix in reduced row-echelon form.

9. $\begin{bmatrix} 3 & -2 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 2 & 8 & 0 & 0 & 1 \end{bmatrix}$

Exercises 6.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, show that B is the inverse of A.

1. $A = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$

2. $A = \begin{bmatrix} -4 & 1 \\ -9 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -1 \\ 9 & -4 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & -1 \\ 5 & -4 \end{bmatrix}, \; B = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ \frac{5}{3} & -\frac{2}{3} \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -2 \\ 3 & -10 \end{bmatrix}, \; B = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$

5. $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}, \; B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \; B = \frac{1}{10}\begin{bmatrix} -6 & 0 & 4 \\ -3 & -5 & 2 \\ 2 & 0 & 2 \end{bmatrix}$

7. $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

8. $A = \begin{bmatrix} -1 & 1 & -3 \\ 2 & -1 & 4 \\ -1 & 1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 3 & -1 & 1 & 0 \end{bmatrix},$

$B = \frac{1}{3}\begin{bmatrix} -1 & 3 & -2 & -2 \\ -2 & 9 & -7 & -10 \\ 1 & 0 & -1 & -1 \\ 3 & -6 & 6 & 6 \end{bmatrix}$

10. $A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 2 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix},$

$B = \frac{1}{4}\begin{bmatrix} -4 & 1 & 1 & -4 \\ -4 & -1 & 3 & -4 \\ 0 & 1 & 1 & 0 \\ -4 & -2 & 2 & 0 \end{bmatrix}$

In Exercises 11–30, find the inverse of the matrix (if it exists).

11. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 11 & 1 \\ -1 & 0 \end{bmatrix}$

14. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

15. $\begin{bmatrix} 8 & 4 \\ -2 & -2 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

17. $\begin{bmatrix} 0 & 4 \\ -3 & 6 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 7 & 1 \\ -3 & -9 & 2 \end{bmatrix}$

20. $\begin{bmatrix} -2 & 5 \\ 6 & -15 \\ 0 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

24. $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \\ -4 & 4 & 3 \end{bmatrix}$

25. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

26. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

30. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

Ⓣ In Exercises 31–38, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).

31. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$

32. $\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

33. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

34. $\begin{bmatrix} 0.6 & 0 & -0.3 \\ 0.7 & -1 & 0.2 \\ 1 & 0 & -0.9 \end{bmatrix}$

35. $\begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 4 & -12 & 8 \\ 3 & 0 & 5 & -2 \\ 0 & -3 & 9 & -6 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

37. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

38. $\begin{bmatrix} 4 & 8 & -7 & 14 \\ 2 & 5 & -4 & 6 \\ 0 & 2 & 1 & -7 \\ 3 & 6 & -5 & 10 \end{bmatrix}$

In Exercises 39–44, use the formula on page 501 to find the inverse of the matrix (if it exists).

39. $\begin{bmatrix} 5 & -2 \\ 2 & 3 \end{bmatrix}$

40. $\begin{bmatrix} 7 & 12 \\ -8 & -5 \end{bmatrix}$

41. $\begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$

42. $\begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$

43. $\begin{bmatrix} \frac{7}{2} & -\frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$

44. $\begin{bmatrix} -\frac{1}{4} & \frac{9}{4} \\ \frac{5}{3} & \frac{8}{9} \end{bmatrix}$

In Exercises 45–48, use the inverse matrix found in Exercise 11 to solve the system of linear equations.

45. $\begin{cases} x + 2y = 0 \\ 3x + 7y = 1 \end{cases}$

46. $\begin{cases} x + 2y = -5 \\ 3x + 7y = -16 \end{cases}$

47. $\begin{cases} x + 2y = 8 \\ 3x + 7y = 26 \end{cases}$

48. $\begin{cases} x + 2y = -6 \\ 3x + 7y = -21 \end{cases}$

In Exercises 49–52, use the inverse matrix found in Exercise 16 to solve the system of linear equations.

49. $\begin{cases} 2x + 3y = 5 \\ x + 4y = 10 \end{cases}$

50. $\begin{cases} 2x + 3y = 0 \\ x + 4y = 3 \end{cases}$

51. $\begin{cases} 2x + 3y = 4 \\ x + 4y = 2 \end{cases}$

52. $\begin{cases} 2x + 3y = 1 \\ x + 4y = -2 \end{cases}$

In Exercises 53 and 54, use the inverse matrix found in Exercise 21 to solve the system of linear equations.

53. $\begin{cases} x + y + z = 0 \\ 3x + 5y + 4z = 5 \\ 3x + 6y + 5z = 2 \end{cases}$

54. $\begin{cases} x + y + z = -1 \\ 3x + 5y + 4z = 2 \\ 3x + 6y + 5z = 0 \end{cases}$

In Exercises 55 and 56, use the inverse matrix found in Exercise 37 to solve the system of linear equations.

55. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 0 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = -1 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = 2 \end{cases}$

56. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = -2 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = 0 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = -3 \end{cases}$

In Exercises 57–64, use an inverse matrix to solve (if possible) the system of linear equations.

57. $\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases}$

58. $\begin{cases} 18x + 12y = 13 \\ 30x + 24y = 23 \end{cases}$

59. $\begin{cases} -0.4x + 0.8y = 1.6 \\ 2x - 4y = 5 \end{cases}$

60. $\begin{cases} 0.2x - 0.6y = 2.4 \\ -x + 1.4y = -8.8 \end{cases}$

61. $\begin{cases} -\frac{1}{4}x + \frac{3}{8}y = -2 \\ \frac{3}{2}x + \frac{3}{4}y = -12 \end{cases}$

62. $\begin{cases} \frac{5}{6}x - y = -20 \\ \frac{4}{3}x - \frac{7}{2}y = -51 \end{cases}$

63. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$

64. $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

(T) In Exercises 65 and 66, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.

65. $\begin{cases} 7x - 3y + 2w = 41 \\ -2x + y - w = -13 \\ 4x + z - 2w = 12 \\ -x + y - w = -8 \end{cases}$

66. $\begin{cases} 2x + 5y + w = 11 \\ x + 4y + 2z - 2w = -7 \\ 2x - 2y + 5z + w = 3 \\ x - 3w = -1 \end{cases}$

In Exercises 67 and 68, develop for the given matrix a system of equations that has the given solution. Use an inverse matrix to verify that the system of equations has the desired solution.

67. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & -2 \\ 0 & 3 & 2 \end{bmatrix}$ $x = 2$, $y = -3$, $z = 5$

68. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ $x = 5$, $y = -2$, $z = 1$

Bond Investment In Exercises 69–72, you invest in AAA-rated bonds, A-rated bonds, and B-rated bonds. Your average yield is 9% on AAA bonds, 7% on A bonds, and 8% on B bonds. You invest twice as much in B bonds as in A bonds. The desired system of linear equations (where x, y, and z represent the amounts invested in AAA, A, and B bonds, respectively) is as follows.

$$\begin{cases} x + y + z = \text{(total investment)} \\ 0.09x + 0.07y + 0.08z = \text{(annual return)} \\ 2y - z = 0 \end{cases}$$

Use the inverse of the coefficient matrix of this system to find the amount invested in each type of bond for the given total investment and annual return.

69. Total investment = \$5,000; annual return = \$950

70. Total investment = \$0,000; annual return = \$180

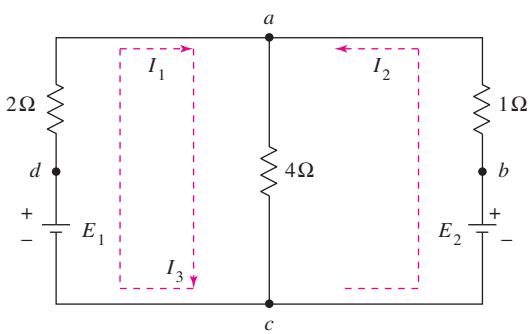
71. Total investment = \$6,000; annual return = \$040

72. Total investment = \$5,000; annual return = \$770

Circuit Analysis In Exercises 73 and 74, consider the circuit shown in the figure. The currents I_1, I_2, and I_3, in amperes, are the solution of the system of linear equations

$$\begin{cases} 2I_1 + 4I_3 = E_1 \\ I_2 + 4I_3 = E_2 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$

where E_1 and E_2 are voltages. Use the inverse of the coefficient matrix of this system to find the unknown currents for the given voltages.



73. $E_1 = 28$ volts, $E_2 = 21$ volts

74. $E_1 = 24$ volts, $E_2 = 23$ volts

Raw Materials In Exercises 75–78, consider a company that specializes in potting soil. Each bag of potting soil for seedlings requires 2 units of sand, 1 unit of loam, and 1 unit of peat moss. Each bag of potting soil for general potting requires 1 unit of sand, 2 units of loam, and 1 unit of peat moss. Each bag of potting soil for hardwood plants requires 2 units of sand, 2 units of loam, and 2 units of peat moss. Find the numbers of bags of the three types of potting soil that the company can produce with the given amounts of raw materials.

75. 500 units of sand
500 units of loam
400 units of peat moss

76. 500 units of sand
750 units of loam
450 units of peat moss

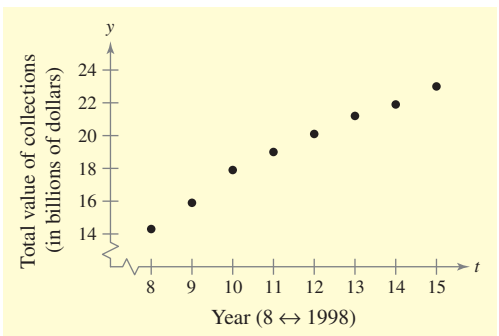
77. 350 units of sand
445 units of loam
345 units of peat moss

78. 975 units of sand
1050 units of loam
725 units of peat moss

79. Child Support The total values y (in billions of dollars) of child support collections from 1998 to 2005 are shown in the figure. The least squares regression parabola $y = at^2 + bt + c$ for these data is found by solving the system

$$\begin{cases} 8c + 92b + 1100a = 153.3 \\ 92c + 1100b + 13{,}616a = 1813.9. \\ 1100c + 13{,}616b + 173{,}636a = 22{,}236.7 \end{cases}$$

Let t represent the year, with $t = 8$ corresponding to 1998. *(Source: U.S. Department of Health and Human Services)*



(a) Use a graphing utility to find an inverse matrix to solve this system, and find the equation of the least squares regression parabola.

(b) Use the result from part (a) to estimate the value of child support collections in 2007.

(c) An analyst predicted that the value of child support collections in 2007 would be $4.0 billion. How does this value compare with your estimate in part (b)? Do both estimates seem reasonable?

80. Alaskan Fishing The total annual profits y (in thousands of dollars) for an Alaskan fishing captain from 2000 to 2008 are shown in the figure. The least squares regression parabola $y = at^2 + bt + c$ for these data is found by solving the system

$$\begin{cases} 9c + 36b + 204a = 1152 \\ 36c + 204b + 1296a = 4399. \\ 204c + 1296b + 8772a = 24{,}597 \end{cases}$$

Let t represent the year, with $t = 0$ corresponding to 2000.

(a) Use a graphing utility to find an inverse matrix to solve this system, and find the equation of the least squares regression parabola.

(b) Use the result from part (a) to predict the captain's profit in 2010.

(c) Due to increased competition, the captain projects profits of $15,000 in 2010. How does this value compare with your prediction in part (b)?

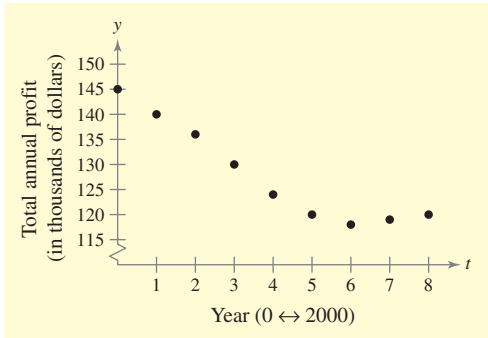


Figure for 80

In Exercises 81 and 82, use the following matrices.

$$A = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 13 & 4 \\ 1 & 8 \end{bmatrix}$$

81. Find AB and BA. What do you observe about the two products?

82. Find C^{-1}, $A^{-1} \cdot B^{-1}$, and $B^{-1} \cdot A^{-1}$. What do you observe about the three resulting matrices?

In Exercises 83 and 84, find a value of k that makes the matrix invertible and then find a value of k that makes the matrix singular. (There are many correct answers.)

83. $\begin{bmatrix} 4 & 3 \\ -2 & k \end{bmatrix}$ **84.** $\begin{bmatrix} 2k + 1 & 3 \\ -7 & 1 \end{bmatrix}$

In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.

85. There exists a matrix A such that $A = A^{-1}$.

86. Multiplication of a nonsingular matrix and its inverse is commutative.

87. If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, verify that the inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

88. Exploration Consider matrices of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

(a) Write a 2×2 matrix and a 3×3 matrix of the form of A. Find the inverse of each.

(b) Use the result from part (a) to make a conjecture about the inverses of matrices of the form of A.

Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, write a matrix of the given order.

1. 4×3 **2.** 3×1

In Exercises 3 and 4, write the augmented matrix for the system of equations.

3. $\begin{cases} 3x + 2y = -2 \\ 5x - y = 19 \end{cases}$

4. $\begin{cases} x + 3z = -5 \\ x + 2y - z = 3 \\ 3x + 4z = 0 \end{cases}$

5. Use Gaussian elimination with back-substitution to solve the augmented matrix found in Exercise 3.

6. Use Gauss-Jordan elimination to solve the augmented matrix found in Exercise 4.

In Exercises 7–12, use the following matrices to find the indicated matrix (if possible).

$$A = \begin{bmatrix} -1 & 4 \\ -2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -2 \\ 3 & 1 \end{bmatrix}$$

7. $2A + 3C$ **8.** AB

9. $A - 3C$ **10.** C^2

11. A^{-1} **12.** B^{-1}

In Exercises 13 and 14, solve for X using matrices A and C from Exercises 7–12.

13. $X = 3A - 2C$ **14.** $2X + 4A = 2C$

In Exercises 15–18, a hang glider manufacturer has the labor-hour and wage requirements indicated at the left.

15. What is the labor cost for model A at Plant 1?

16. What is the labor cost for model B at Plant 2?

17. What is the labor cost for model C at Plant 2?

18. Compute LW and interpret the result.

In Exercises 19 and 20, use an inverse matrix to solve the system of linear equations.

19. $\begin{cases} x - 3y = 10 \\ -2x + y = -10 \end{cases}$

20. $\begin{cases} 2x - y + z = 3 \\ 3x - z = 15 \\ 4y + 3z = -1 \end{cases}$

$$L = \begin{array}{c} \begin{matrix} \text{Assembly} & \text{Finishing} & \text{Packaging} \end{matrix} \\ \begin{bmatrix} 1.0 & 0.6 & 0.2 \\ 2.4 & 1.0 & 0.2 \\ 2.8 & 2.0 & 0.5 \end{bmatrix} \end{array} \begin{matrix} \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{matrix}$$

Labor-Hour Requirements
(in hours per hang glider)

$$W = \begin{array}{c} \begin{matrix} \text{Plant 1} & \text{Plant 2} \end{matrix} \\ \begin{bmatrix} 15 & 12 \\ 10 & 11 \\ 9 & 8 \end{bmatrix} \end{array} \begin{matrix} \text{Assembly} \\ \text{Finishing} \\ \text{Packaging} \end{matrix}$$

Wage Requirements
(in dollars per hour)

Matrices for 15–18

Section 6.4

The Determinant of a Square Matrix

- Evaluate the determinant of a 2 × 2 matrix.
- Find the minors and cofactors of a matrix.
- Find the determinant of a square matrix.
- Find the determinant of a triangular matrix.

The Determinant of a 2 × 2 Matrix

Every *square* matrix can be associated with a real number called its **determinant.** Determinants have many uses, and several will be discussed in this and the next section. The use of determinants is derived from special number patterns that occur when systems of linear equations are solved. For instance, the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution given by

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Note that the denominator of each fraction is the same. This denominator is called the *determinant* of the coefficient matrix of the system.

Coefficient Matrix	*Determinant*
$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\det(A) = a_1b_2 - a_2b_1$

The determinant of the matrix A can also be denoted by vertical bars on both sides of the matrix, as indicated in the following definition.

Definition of the Determinant of a 2 × 2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is given by

$$\det(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of A. Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended.

A convenient method for remembering the formula for the determinant of a 2×2 matrix is shown in the following diagram.

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Note that the determinant is the difference of the products of the two diagonals of the matrix. In Example 1 you will see that the determinant of a matrix can be positive, zero, or negative.

Example 1 The Determinant of a 2 × 2 Matrix

Find the determinant of each matrix.

a. $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ **c.** $C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$

SOLUTION Use the formula $\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

a. $\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$

b. $\det(B) = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$

c. $\det(C) = \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$

STUDY TIP

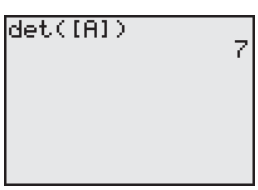
The determinant of a matrix of order 1×1 is defined simply as the entry of the matrix. For instance, if $A = [-2]$, then $\det(A) = -2$.

✓CHECKPOINT 1

Find the determinant of $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$. ■

TECHNOLOGY

Most graphing utilities can evaluate the determinant of a matrix. Use a graphing utility to find the determinant of matrix A from Example 1. The result should be 7, as shown below. For specific keystrokes on how to use a graphing utility to evaluate the determinant of a matrix, go to the text website at *college.hmco.com/info/larsonapplied.*



Try evaluating the determinant of B with your graphing utility.

$$B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

What happens when you try to evaluate the determinant of a nonsquare matrix?

Minors and Cofactors

To define the determinant of a square matrix of order 3×3 or higher, it is convenient to introduce the concepts of **minors** and **cofactors.**

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

Minors and Cofactors of a Square Matrix

If A is a square matrix, the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is given by $C_{ij} = (-1)^{i+j}M_{ij}$.

In the sign pattern for cofactors at the left, notice that *odd* positions (where $i + j$ is odd) have negative signs and *even* positions (where $i + j$ is even) have positive signs.

Example 2 Finding the Minors and Cofactors of a Matrix

Find all the minors and cofactors of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

SOLUTION To find the minor M_{11}, delete the first row and first column of A and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find M_{12}, delete the first row and second column.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain the following minors.

$$\begin{aligned} M_{11} &= -1 & M_{12} &= -5 & M_{13} &= 4 \\ M_{21} &= 2 & M_{22} &= -4 & M_{23} &= -8 \\ M_{31} &= 5 & M_{32} &= -3 & M_{33} &= -6 \end{aligned}$$

Now, to find the cofactors, combine the minors above with the checkerboard pattern of signs for a 3×3 matrix shown at the upper left.

$$\begin{aligned} C_{11} &= -1 & C_{12} &= 5 & C_{13} &= 4 \\ C_{21} &= -2 & C_{22} &= -4 & C_{23} &= 8 \\ C_{31} &= 5 & C_{32} &= 3 & C_{33} &= -6 \end{aligned}$$

✓CHECKPOINT 2

Find all the minors and cofactors of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$. ■

The Determinant of a Square Matrix

The definition below is called **inductive** because it uses determinants of matrices of order $n - 1$ to define determinants of matrices of order n.

Determinant of a Square Matrix

If A is a square matrix (of order 2×2 or greater), then the determinant of A is the sum of the entries in any row (or column) of A multiplied by their respective cofactors. For instance, expanding along the first row yields

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Applying this definition to find a determinant is called **expanding by cofactors.**

Try checking that for a 2×2 matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

this definition of the determinant yields

$$|A| = a_1b_2 - a_2b_1$$

as previously defined.

Example 3 The Determinant of a Matrix of Order 3×3

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

SOLUTION Note that this is the same matrix that was given in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \quad C_{12} = 5, \quad \text{and} \quad C_{13} = 4.$$

So, by the definition of a determinant, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First-row expansion} \\ &= 0(-1) + 2(5) + 1(4) \\ &= 14. \end{aligned}$$

In Example 3, the determinant was found by expanding by the cofactors in the first row. You could have used any row or column. For instance, you could have expanded along the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second-row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) \\ &= 14. \end{aligned}$$

✓CHECKPOINT 3

Find the determinant of

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 2 & -1 \\ 2 & 0 & -4 \end{bmatrix}.$$

When expanding by cofactors, you do not need to find cofactors of zero entries, because zero times its cofactor is zero.

$$a_{ij}C_{ij} = (0)C_{ij} = 0$$

So, the row (or column) containing the most zeros is usually the best choice for expansion by cofactors. This is demonstrated in the next example.

Example 4 The Determinant of a Matrix of Order 4 × 4

Find the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & 2 \end{bmatrix}.$$

SOLUTION After inspecting this matrix, you can see that three of the entries in the third column are zeros. So, you can eliminate some of the work in the expansion by using the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

Because C_{23}, C_{33}, and C_{43} have zero coefficients, you only need to find the cofactor C_{13}. To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix}$$ Delete 1st row and 3rd column.

$$= \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix}$$ Simplify.

Expanding by minors in the second row yields

$$C_{13} = 0(-1)^3\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} + 2(-1)^4\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} + 3(-1)^5\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= 0 + 2(1)(-8) + 3(-1)(-7)$$

$$= 5.$$

So, you obtain

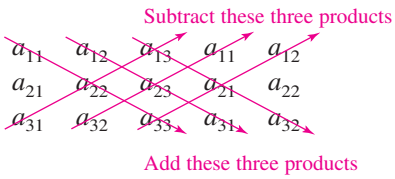
$$|A| = 3C_{13} = 3(5) = 15.$$

✓CHECKPOINT 4

Find the determinant of $A = \begin{bmatrix} 3 & 0 & 7 & 0 \\ 2 & 6 & 0 & 11 \\ 4 & 1 & 0 & 2 \\ 1 & 5 & 0 & 10 \end{bmatrix}$. ■

Try using a graphing utility to confirm the result of Example 4.

There is an alternative method that is commonly used to evaluate the determinant of a 3×3 matrix A. This method works *only* for 3×3 matrices. To apply this method, copy the first and second columns of A to form fourth and fifth columns. The determinant of A is then obtained by adding the products of the three "downward diagonals" and subtracting the products of the three "upward diagonals," as shown in the following diagram.



So, the determinant of the 3×3 matrix A is given by

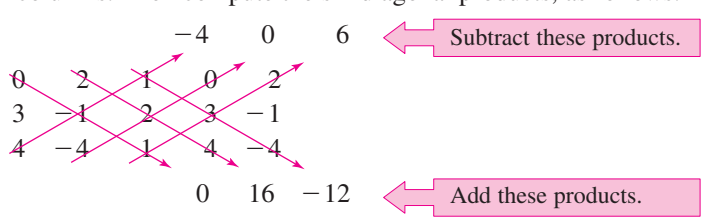
$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Example 5 The Determinant of a 3×3 Matrix

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}.$$

SOLUTION Because A is a 3×3 matrix, you can use the alternative procedure for finding $|A|$. Begin by copying the first and second columns to form fourth and fifth columns. Then compute the six diagonal products, as follows.



Now, by adding the lower three products and subtracting the upper three products, you find the determinant of A to be

$$|A| = 0 + 16 + (-12) - (-4) - 0 - 6$$
$$= 2.$$

✓CHECKPOINT 5

Find the determinant of $A = \begin{bmatrix} -3 & -4 & 0 \\ 4 & 1 & -2 \\ 0 & 2 & 3 \end{bmatrix}$. ■

Be sure you understand that the diagonal process illustrated in Example 5 is valid *only* for matrices of order 3×3. For matrices of higher orders, another method must be used, such as expansion by cofactors or a graphing utility.

Triangular Matrices

Evaluating determinants of matrices of order 4 or higher can be tedious. There is, however, an important exception: the determinant of a **triangular** matrix. A triangular matrix is a square matrix with all *zero* entries either below or above its main diagonal. A square matrix is **upper triangular** if it has all zero entries below its main diagonal and **lower triangular** if it has all zero entries above its main diagonal. A matrix that is both upper and lower triangular is called **diagonal.** That is, a diagonal matrix is one in which all entries above and below the main diagonal are zero.

Upper Triangular Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

To find the determinant of a triangular matrix of any order, simply find the product of the entries on the main diagonal.

Example 6 **The Determinant of a Triangular Matrix**

a. $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{vmatrix} = 2(-2)(1)(3) = -12$

b. $\begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} = -1(3)(2)(4)(-2) = 48$

✓CHECKPOINT 6

Find the determinant of

$$A = \begin{bmatrix} -1 & -2 & -1 & 6 \\ 0 & 5 & 4 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

CONCEPT CHECK

1. Explain the difference between $\begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}$ and $\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix}$.
2. Explain the difference between the minors and cofactors of a square matrix.
3. Consider the matrix $A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 0 & 2 \\ 3 & 0 & 5 \end{bmatrix}$. When expanding by cofactors, which column reduces the amount of work in finding the determinant? Explain your reasoning.
4. What is the determinant of any identity matrix? Explain.

Skills Review 6.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.3, and 6.2.

In Exercises 1–4, perform the indicated matrix operations.

1. $\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 4 & -3 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 5 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix}$

3. $3\begin{bmatrix} 3 & -4 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$

4. $4\begin{bmatrix} 0 & 2 & 3 \\ -1 & 2 & 3 \\ -2 & 1 & -2 \end{bmatrix}$

In Exercises 5–10, perform the indicated arithmetic operations.

5. $[(1)(3) + (-3)(2)] - [(1)(4) + (3)(5)]$

6. $[(4)(4) + (-1)(-3)] - [(-1)(2) + (-2)(7)]$

7. $\dfrac{4(7) - 1(-2)}{(-5)(-2) - 3(4)}$

8. $\dfrac{3(6) - 2(7)}{6(-5) - 2(1)}$

9. $-5(-1)^2[6(-2) - 7(-3)]$

10. $4(-1)^3[3(6) - 2(7)]$

Exercises 6.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, find the determinant of the matrix.

1. $[-5]$

2. $[6]$

3. $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$

6. $\begin{bmatrix} -7 & -4 \\ 8 & 7 \end{bmatrix}$

7. $\begin{bmatrix} 9 & 3 \\ 12 & 4 \end{bmatrix}$

8. $\begin{bmatrix} -5 & -2 \\ 10 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

10. $\begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$

11. $\begin{bmatrix} \frac{2}{3} & 0 \\ -1 & 6 \end{bmatrix}$

12. $\begin{bmatrix} 9 & -\frac{1}{4} \\ 8 & 0 \end{bmatrix}$

13. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \\ -6 & \frac{1}{3} \end{bmatrix}$

14. $\begin{bmatrix} \frac{2}{3} & \frac{4}{3} \\ -1 & -\frac{1}{3} \end{bmatrix}$

Ⓣ In Exercises 15–20, use the matrix capabilities of a graphing utility to find the determinant of the matrix.

15. $\begin{bmatrix} 0.1 & 0.3 & 0.2 \\ -0.3 & -0.2 & 0.1 \\ 1 & 2 & 3 \end{bmatrix}$

16. $\begin{bmatrix} 0.2 & -0.1 & -0.3 \\ 0.1 & -0.1 & 0.4 \\ -0.5 & -0.2 & -0.1 \end{bmatrix}$

17. $\begin{bmatrix} 0.9 & 0.7 & 0 \\ -0.1 & 0.3 & 1.3 \\ 2.2 & 4.2 & 6.1 \end{bmatrix}$

18. $\begin{bmatrix} 0.1 & 0.1 & -4.3 \\ 7.5 & 6.2 & 0.7 \\ 0.3 & 0.6 & -1.2 \end{bmatrix}$

19. $\begin{bmatrix} 5 & -3 & 2 \\ 7 & 5 & -7 \\ 0 & 6 & -1 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & -2 \end{bmatrix}$

In Exercises 21–28, find all (a) minors and (b) cofactors of the matrix.

21. $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$

22. $\begin{bmatrix} 11 & 0 \\ -3 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 3 & 1 \\ -2 & -4 \end{bmatrix}$

24. $\begin{bmatrix} -6 & 5 \\ 7 & -2 \end{bmatrix}$

25. $\begin{bmatrix} 4 & 0 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

26. $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 5 \\ 4 & -6 & 4 \end{bmatrix}$

27. $\begin{bmatrix} 3 & -2 & 8 \\ 3 & 2 & -6 \\ -1 & 3 & 6 \end{bmatrix}$

28. $\begin{bmatrix} -2 & 9 & 4 \\ 7 & -6 & 0 \\ 6 & 7 & -6 \end{bmatrix}$

In Exercises 29–34, find the determinant of the matrix by the method of expansion by cofactors. Expand using the indicated row or column.

29. $\begin{bmatrix} 4 & 1 & -3 \\ 6 & 5 & -2 \\ -1 & 3 & -4 \end{bmatrix}$

(a) Row 3

(b) Column 2

30. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$

(a) Row 2

(b) Column 3

31. $\begin{bmatrix} 7 & 0 & -4 \\ 2 & -3 & 0 \\ 5 & 8 & 1 \end{bmatrix}$

(a) Row 1

(b) Column 3

32. $\begin{bmatrix} 10 & -5 & 5 \\ 30 & 0 & 10 \\ 0 & 10 & 1 \end{bmatrix}$

(a) Row 3

(b) Column 1

33. $\begin{bmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{bmatrix}$

(a) Row 2 (b) Column 2

34. $\begin{bmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 1 & 0 & -3 & 2 \end{bmatrix}$

(a) Row 3 (b) Column 1

In Exercises 35–52, find the determinant of the matrix. Expand by cofactors on the row or column that appears to make the computations easiest. Use a graphing utility to confirm your result.

35. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

36. $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

37. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

38. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

39. $\begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

40. $\begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

41. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 6 & -6 \\ -2 & 1 & 4 \end{bmatrix}$

42. $\begin{bmatrix} -1 & 3 & 1 \\ 4 & 2 & 5 \\ -2 & 1 & 6 \end{bmatrix}$

43. $\begin{bmatrix} 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ -0.4 & 0.4 & 0.3 \end{bmatrix}$

44. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

45. $\begin{bmatrix} 6 & 3 & -7 \\ 0 & 0 & 0 \\ 4 & -6 & 3 \end{bmatrix}$

46. $\begin{bmatrix} 5 & 0 & 3 \\ -4 & 0 & 8 \\ 3 & 0 & -6 \end{bmatrix}$

47. $\begin{bmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{bmatrix}$

48. $\begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{bmatrix}$

49. $\begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix}$

50. $\begin{bmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{bmatrix}$

51. $\begin{bmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{bmatrix}$

52. $\begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 53–60, use the matrix capabilities of a graphing utility to evaluate the determinant.

53. $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{vmatrix}$

54. $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix}$

55. $\begin{vmatrix} 7 & 0 & -14 \\ -2 & 5 & 4 \\ -6 & 2 & 12 \end{vmatrix}$

56. $\begin{vmatrix} 3 & 0 & 0 \\ -2 & 5 & 0 \\ 12 & 5 & 7 \end{vmatrix}$

57. $\begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix}$

58. $\begin{vmatrix} 0 & -3 & 8 & 3 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$

59. $\begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix}$

60. $\begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}$

In Exercises 61–70, evaluate the determinant of the matrix. Do not use a graphing utility.

61. $\begin{bmatrix} 2 & 0 & 0 \\ 4 & -3 & 0 \\ 6 & 5 & 1 \end{bmatrix}$

62. $\begin{bmatrix} 1 & 0 & 0 \\ -4 & -1 & 0 \\ 5 & 1 & 5 \end{bmatrix}$

63. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -1 & -3 & 5 \\ 0 & 0 & -2 & 7 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

64. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 6 & -2 & 3 & -1 \end{bmatrix}$

65. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

66. $\begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$

67. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 6 & -5 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 1 & 2 & 7 & -1 \end{bmatrix}$

68. $\begin{bmatrix} 5 & 3 & 6 & 1 \\ 0 & -10 & 4 & 3 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$

69. $\begin{bmatrix} -6 & 7 & 2 & 0 & 5 \\ 0 & -1 & 3 & 4 & -3 \\ 0 & 0 & -7 & 0 & 4 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$

70. $\begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 7 & -8 & 7 & 0 & 0 \\ 6 & 4 & 0 & -2 & 0 \\ 1 & 5 & 1 & -10 & 6 \end{bmatrix}$

In Exercises 71–74, find the determinant of the matrix. Tell which method you used.

71. $\begin{bmatrix} 2 & 1 & 3 \\ 7 & 3 & -2 \\ 4 & 1 & 1 \end{bmatrix}$

72. $\begin{bmatrix} 6 & -5 & 2 \\ 0 & 5 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

73. $\begin{bmatrix} 3 & 0 & 0 \\ 4 & -2 & 0 \\ 5 & 4 & 3 \end{bmatrix}$

74. $\begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & -3 \\ 0 & 1 & 0 \end{bmatrix}$

In Exercises 75–82, find (a) $|A|$, (b) $|B|$, (c) AB, and (d) $|AB|$.

75. $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

76. $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

77. $A = \begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

78. $A = \begin{bmatrix} 5 & 4 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 6 \\ 1 & -2 \end{bmatrix}$

79. $A = \begin{bmatrix} 0 & 1 & 2 \\ -3 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 0 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

80. $A = \begin{bmatrix} 3 & 2 & 0 \\ -1 & -3 & 4 \\ -2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

81. $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

82. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

In Exercises 83–86, find a 4 × 4 *upper* triangular matrix whose determinant is equal to the given value and a 4 × 4 *lower* triangular matrix whose determinant is equal to the given value. Use a graphing utility to confirm your results.

83. -18

84. -40

85. 28

86. 36

In Exercises 87–90, explain why the determinant of the matrix is equal to zero.

87. $\begin{bmatrix} 3 & 4 & -2 & 7 \\ 1 & 3 & -1 & 2 \\ 0 & 5 & 7 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix}$

88. $\begin{bmatrix} 3 & 2 & -1 \\ -6 & -4 & 2 \\ 5 & -7 & 9 \end{bmatrix}$

89. $\begin{bmatrix} 3 & 0 & 1 & 7 \\ 2 & -1 & 4 & 3 \\ 11 & 5 & -7 & 8 \\ -6 & 3 & -12 & -9 \end{bmatrix}$

90. $\begin{bmatrix} -1 & 3 & 2 \\ 5 & 7 & 0 \\ -1 & 3 & 2 \end{bmatrix}$

In Exercises 91 and 92, determine whether the statement is true or false. Justify your answer.

91. If a square matrix has an entire row of zeros, the determinant will always be zero.

92. If two columns of a square matrix are the same, the determinant of the matrix will be zero.

In Exercises 93–96, evaluate the determinant(s) to verify the equation.

93. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = -\begin{vmatrix} y & z \\ w & x \end{vmatrix}$

94. $\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c\begin{vmatrix} w & x \\ y & z \end{vmatrix}$

95. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix}$

96. $\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$

Section 6.5

Applications of Matrices and Determinants

- Find the area of a triangle using a determinant.
- Determine whether three points are collinear using a determinant.
- Use a determinant to find an equation of a line.
- Encode and decode a cryptogram using a matrix.

Area of a Triangle

In this section, you will study some additional applications of matrices and determinants. The first involves a formula for finding the area of a triangle whose vertices are given by three points on a rectangular coordinate system.

Area of a Triangle

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol $(\pm)$ indicates that the appropriate sign should be chosen to yield a positive area.

Example 1 Finding the Area of a Triangle

Find the area of the triangle whose vertices are (1, 0), (2, 2), and (4, 3), as shown in Figure 6.1.

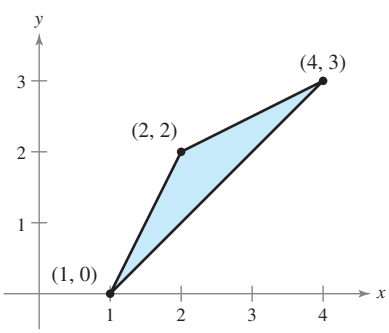


FIGURE 6.1

SOLUTION Let $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 2)$, and $(x_3, y_3) = (4, 3)$. Then, to find the area of the triangle, evaluate the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= 1(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 0(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= 1(-1) + 0 + 1(-2) = -3.$$

Using this value, you can conclude that the area of the triangle is

$$\text{Area} = -\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\frac{1}{2}(-3) = \frac{3}{2}.$$

Choose $(-)$ so that the area is positive.

✓CHECKPOINT 1

Find the area of the triangle whose vertices are (2, 1), (3, 5), and (10, 5). ■

Lines in the Plane

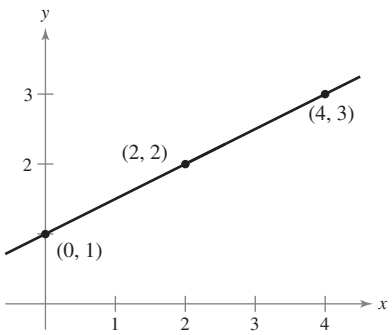


FIGURE 6.2

Suppose the three points in Example 1 had been on the same line. What would have happened had the area formula been applied to three such points? The answer is that the determinant would have been zero. Consider, for instance, the three collinear points (0, 1), (2, 2), and (4, 3), as shown in Figure 6.2. The area of the "triangle" that has these three points as vertices is

$$\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left[0(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 1(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}\right]$$

$$= \frac{1}{2}[0(-1) - 1(-2) + 1(-2)] = 0.$$

This result is generalized as follows.

Test for Collinear Points

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear (lie on the same line) if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Example 2 Testing for Collinear Points

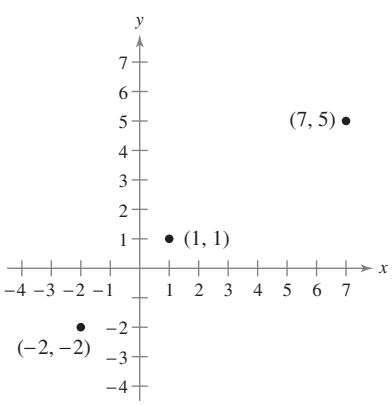


FIGURE 6.3

Determine whether the points $(-2, -2)$, $(1, 1)$, and $(7, 5)$ are collinear. (See Figure 6.3.)

SOLUTION Letting $(x_1, y_1) = (-2, -2)$, $(x_2, y_2) = (1, 1)$, and $(x_3, y_3) = (7, 5)$, you have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -2 & 1 \\ 1 & 1 & 1 \\ 7 & 5 & 1 \end{vmatrix}$$

$$= -2(-1)^2\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} + (-2)(-1)^3\begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix}$$

$$= -2(-4) + 2(-6) + 1(-2)$$

$$= -6.$$

Because the value of this determinant is *not* zero, you can conclude that the three points *do not* lie on the same line and are not collinear.

✓CHECKPOINT 2

Determine whether the points $(-2, 4)$, $(3, 0)$, and $(6, -4)$ are collinear. ■

Another way to test for collinear points in Example 2 is to find the slope of the line between $(-2, -2)$ and $(1, 1)$ and the slope of the line between $(-2, -2)$ and $(7, 5)$. Try doing this. If the slopes are equal, then the points are collinear. If the slopes are not equal, then the points are not collinear.

The test for collinear points can be adapted to another use. That is, if you are given two points on a rectangular coordinate system, you can find an equation of the line passing through the two points, as follows.

Two-Point Form of the Equation of a Line

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

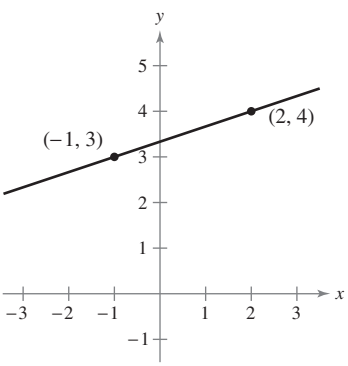


FIGURE 6.4

Example 3 **Finding an Equation of a Line**

Find an equation of the line passing through the two points $(2, 4)$ and $(-1, 3)$, as shown in Figure 6.4.

SOLUTION Let $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (-1, 3)$. Applying the determinant formula for the equation of a line produces

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, you can expand by cofactors along the first row to obtain the following.

$$x(-1)^2\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} + y(-1)^3\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 0$$

$$x(1) - y(3) + (1)(10) = 0$$

$$x - 3y + 10 = 0$$

So, an equation of the line is

$$x - 3y + 10 = 0.$$

✓CHECKPOINT 3

Find an equation of the line passing through the two points $(-3, -1)$ and $(3, 5)$. ■

Note that this method of finding the equation of a line works for all lines, including horizontal and vertical lines. For instance, the equation of the vertical line through $(2, 0)$ and $(2, 2)$ is

$$\begin{vmatrix} x & y & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$$4 - 2x = 0$$

$$x = 2.$$



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During World War II, Navajo soldiers created a code using their native language to send messages between batallions. Native words were assigned to represent characters in the English alphabet, and they created a number of expressions for important military terms, like iron-fish to mean submarine. Without the Navajo Code Talkers, the Second World War might have had a very different outcome.

Cryptography

A **cryptogram** is a message written according to a secret code. (The Greek word *kryptos* means "hidden.") Matrix multiplication can be used to **encode** and **decode** messages. To begin, you need to assign a number to each letter in the alphabet (with 0 assigned to a blank space), as follows.

0 = __	9 = I	18 = R
1 = A	10 = J	19 = S
2 = B	11 = K	20 = T
3 = C	12 = L	21 = U
4 = D	13 = M	22 = V
5 = E	14 = N	23 = W
6 = F	15 = O	24 = X
7 = G	16 = P	25 = Y
8 = H	17 = Q	26 = Z

The message is then converted to numbers and partitioned into **uncoded row matrices,** each having n entries, as demonstrated in Example 4.

Example 4 Forming Uncoded Row Matrices

Write the uncoded row matrices of order 1×3 for the message

MEET ME MONDAY.

SOLUTION Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

[13	5	5]	[20	0	13]	[5	0	13]	[15	14	4]	[1	25	0]
M	E	E	T		M	E		M	O	N	D	A	Y	

Note that a blank space is used to fill out the last uncoded row matrix.

✓CHECKPOINT 4

Write the uncoded row matrices of order 1×3 for the message

OWLS ARE NOCTURNAL. ■

To encode a message, choose an $n \times n$ invertible matrix A and multiply the uncoded row matrices by A to obtain **coded row matrices.** The uncoded matrix should be on the left, whereas the encoding matrix A should be on the right. Here is an example.

Uncoded Matrix	*Encoding Matrix A*		*Coded Matrix*
$[13 \quad 5 \quad 5]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[13 \quad -26 \quad 21]$

This technique is further illustrated in Example 5.

Example 5 Encoding a Message

Use the following matrix to encode the message MEET ME MONDAY.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

SOLUTION The coded row matrices are obtained by multiplying each of the uncoded row matrices found in Example 4 by the matrix A, as follows.

Uncoded Matrix	*Encoding Matrix A*		*Coded Matrix*
$[13 \quad 5 \quad 5]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[13 \quad -26 \quad 21]$
$[20 \quad 0 \quad 13]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[33 \quad -53 \quad -12]$
$[5 \quad 0 \quad 13]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[18 \quad -23 \quad -42]$
$[15 \quad 14 \quad 4]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[5 \quad -20 \quad 56]$
$[1 \quad 25 \quad 0]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$=$	$[-24 \quad 23 \quad 77]$

So, the sequence of coded row matrices is

$$[13 \ -26 \quad 21]\,[33 \ -53 \ -12]\,[18 \ -23 \ -42]\,[5 \ -20 \quad 56]\,[-24 \quad 23 \quad 77].$$

Finally, removing the matrix notation produces the following cryptogram.

$$13 \ -26 \ 21 \ 33 \ -53 \ -12 \ 18 \ -23 \ -42 \ 5 \ -20 \ 56 \ -24 \ 23 \ 77$$

For those who do not know the encoding matrix A, decoding the cryptogram found in Example 5 is difficult. But for an authorized receiver who knows the encoding matrix A, decoding is simple. The receiver only needs to multiply the coded row matrices by A^{-1} (on the right) to retrieve the uncoded row matrices. Here is an example.

$$\underbrace{[13 \quad -26 \quad 21]}_{\text{Coded}} \underbrace{\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}}_{A^{-1}} = \underbrace{[13 \quad 5 \quad 5]}_{\text{Uncoded}}$$

The receiver could then easily refer to the number code chart on page 521 and translate $[13 \quad 5 \quad 5]$ into the letters M E E.

✓ CHECKPOINT 5

Use the following matrix to encode the message OWLS ARE NOCTURNAL.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$$

Example 6 Decoding a Message

Use the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$ to decode the cryptogram.

$$13 \quad -26 \quad 21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77$$

SOLUTION First find A^{-1} by using the techniques demonstrated in Section 6.3. A^{-1} is the decoding matrix. Then partition the message into groups of three to form the coded row matrices. Multiply each coded row matrix on the right by A^{-1} to obtain the decoded row matrices.

Coded Matrix	*Decoding Matrix* A^{-1}		*Decoded Matrix*
$[13 \quad -26 \quad 21]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[13 \quad 5 \quad 5]$
$[33 \quad -53 \quad -12]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[20 \quad 0 \quad 13]$
$[18 \quad -23 \quad -42]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[5 \quad 0 \quad 13]$
$[5 \quad -20 \quad 56]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[15 \quad 14 \quad 4]$
$[-24 \quad 23 \quad 77]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$=$	$[1 \quad 25 \quad 0]$

So, the message is as follows.

$$\begin{array}{cccccccccccccccc} [13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \\ \text{M} & \text{E} & \text{E} & \text{T} & & \text{M} & \text{E} & & \text{M} & \text{O} & \text{N} & \text{D} & \text{A} & \text{Y} & & \end{array}$$

✓CHECKPOINT 6

Use the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$$

to decode the cryptogram.

110, −39, −59, 25, −21, −3, 23, −18, −5, 47, −20, −24, 149, −56, −75, 87, −38, −37 ■

CONCEPT CHECK

1. The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is 4. What are the possible values of the determinant in the area formula? Explain.
2. Suppose the matrix formed by three points is not invertible. What does this tell you about the points?
3. You are finding the equation of a line given two points using the determinant formula. By expanding by cofactors along the first row and second column, you find that the determinant is 0. What does this tell you about the line?
4. Can you use the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$ to encode a message? Explain.

Skills Review 6.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 6.2, 6.3, and 6.4.

In Exercises 1–6, evaluate the determinant.

1. $\begin{vmatrix} 4 & 3 \\ -3 & -2 \end{vmatrix}$

2. $\begin{vmatrix} 10 & -20 \\ -1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 4 & 0 \\ -3 & -2 \end{vmatrix}$

4. $\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$

5. $\begin{vmatrix} 4 & 0 & -2 \\ 3 & 1 & 2 \\ -8 & 0 & 6 \end{vmatrix}$

6. $\begin{vmatrix} 3 & 2 & 5 \\ 0 & 0 & -4 \\ -6 & 1 & 1 \end{vmatrix}$

In Exercises 7 and 8, find the inverse of the matrix.

7. $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

8. $A = \begin{bmatrix} 10 & 5 & -2 \\ -4 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

In Exercises 9 and 10, perform the indicated matrix multiplication.

9. $\begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}$

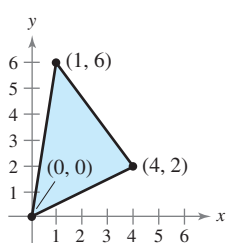
10. $\begin{bmatrix} 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

Exercises 6.5

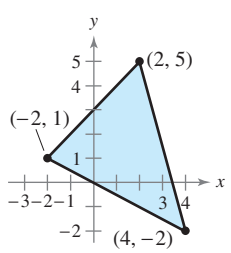
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, use a determinant to find the area of the triangle with the given vertices.

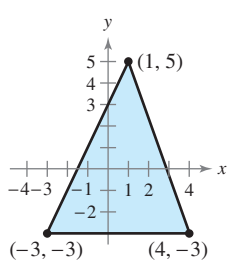
1.



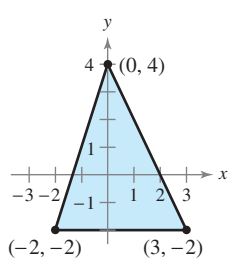
2.



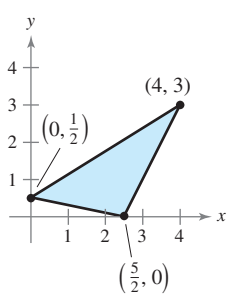
3.



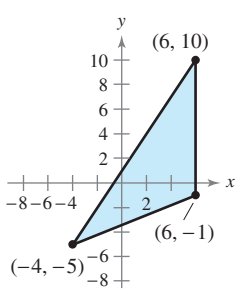
4.



5.



6.



7. $(-2, 4), (2, 3), (-1, 5)$

8. $(0, -2), (-1, 4), (3, 5)$

9. $(-3, 5), (2, 6), (3, -5)$

10. $(-2, 4), (1, 5), (3, -2)$

In Exercises 11 and 12, find a value of y such that the triangle with the given vertices has an area of 4 square units.

11. $(-5, 1), (0, 2), (-2, y)$

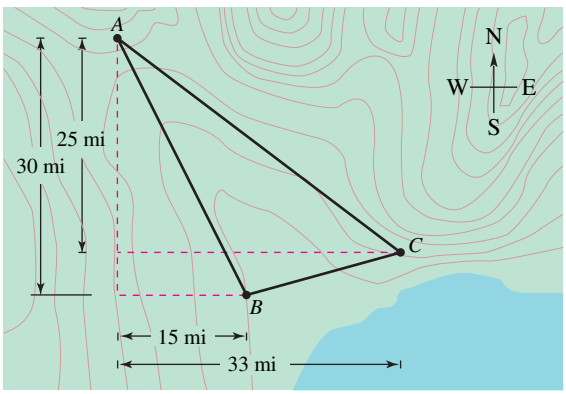
12. $(-4, 2), (-3, 5), (-1, y)$

In Exercises 13 and 14, find a value of y such that the triangle with the given vertices has an area of 6 square units.

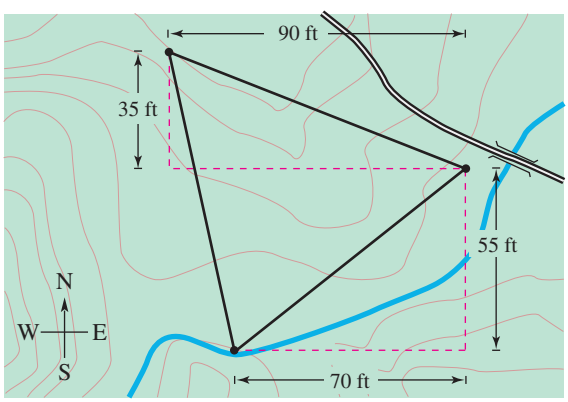
13. $(-2, -3), (1, -1), (-8, y)$

14. $(1, 0), (5, -3), (-3, y)$

15. Gypsy Moths A large region of forest has been infested with gypsy moths. The region is roughly triangular, as shown in the figure. From the northernmost vertex A of the region, the distances to the other vertices are 30 miles south and 15 miles east (for vertex B), and 25 miles south and 33 miles east (for vertex C). Use a graphing utility to approximate the number of square miles in this region.



16. Botany A botanist is studying the plants growing in a triangular tract of land, as shown in the figure. To estimate the number of square feet in the tract, the botanist starts at one vertex, walks 70 feet east and 55 feet north to the second vertex, and then walks 90 feet west and 35 feet north to the third vertex. Use a graphing utility to determine how many square feet there are in the tract of land.



In Exercises 17–24, use a determinant to determine whether the points are collinear.

17. $(-4, -7), (0, -4), (4, -1)$

18. $(2, 4), (4, 5), (-2, 2)$

19. $(-1, -7), (0, -3), (1, 2)$

20. $(1, 7), (0, 4), (-1, 2)$

21. $(-2, -11), (4, 13), (2, 5)$

22. $(4, 3), (3, 1), (2, -1)$

23. $(-2, 3), (2, -1), (7, -4)$

24. $(-3, -4), (-1, -1), (5, 5)$

In Exercises 25 and 26, find y such that the points are collinear.

25. $(2, -5), (3, y), (5, -2)$

26. $(-6, 2), (-4, y), (-3, 5)$

In Exercises 27 and 28, find x such that the points are collinear.

27. $(-4, -1), (-1, 2), (x, 6)$

28. $(1, 5), (5, -1), (x, 3)$

In Exercises 29–36, use a determinant to find an equation of the line passing through the points.

29. $(-1, 2), (5, 3)$

30. $(3, 1), (-2, -5)$

31. $(-4, 3), (2, 1)$

32. $(10, 7), (-2, -7)$

33. $(-4, 5), (-4, -2)$

34. $(3, 3), (6, 3)$

35. $\left(-\frac{1}{2}, 3\right), \left(\frac{5}{2}, 1\right)$

36. $\left(\frac{2}{3}, 4\right), (6, 12)$

In the remaining exercises for this section, use the number code chart on page 521.

In Exercises 37 and 38, find the uncoded 1×2 row matrices for the message. Then encode the message using the encoding matrix.

Message	*Encoding Matrix*
37. COME HOME SOON	$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$
38. HELP IS ON THE WAY	$\begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$

In Exercises 39 and 40, find the uncoded 1×3 row matrices for the message. Then encode the message using the encoding matrix.

Message	*Encoding Matrix*
39. CALL ME TOMORROW	$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$
40. PLEASE SEND MONEY	$\begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

In Exercises 41–46, write a cryptogram for the message using the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}.$$

41. LANDING SUCCESSFUL

42. BEAM ME UP SCOTTY

43. HAPPY BIRTHDAY

44. OPERATION OVERLORD

45. CONTACT AT DAWN

46. HEAD DUE WEST

In Exercises 47–50, use A^{-1} to decode the cryptogram.

47. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

11, 21, 64, 112, 25, 50, 29, 53, 23, 46, 40, 75, 55, 92

48. $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

19, 26, 41, 57, 28, 42, 78, 109, 64, 87, 62, 83, 63, 87, 28, 42, 73, 102, 46, 69

49. $A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

94, 35, 25, 44, 16, 10, 4, −10, 1, 27, 15, 9, 71, 43, 22

50. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

9, −1, −9, 38, −19, −19, 28, −9, −19, −80, 25, 41, −64, 21, 31, −7, −4, 7

In Exercises 51 and 52, decode the cryptogram by using the inverse of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

51. 20, 17, −15, −9, −44, −83, 64, 136, 157, 24, 31, 12, 4, −37, −102

52. −10, −57, −111, 74, 168, 209, 35, 75, 85, 16, 35, 42, 34, 55, 43

53. The following cryptogram was encoded with a 2×2 matrix.

8, 21, −15, −10, −13, −13, 5, 10, 5, 25, 5, 19, −1, 6, 20, 40, −18, −18, 1, 16

The last word of the message is RON. What is the message?

54. The following cryptogram was encoded with a 2×2 matrix.

5, 2, 25, 11, −2, −7, −15, −15, 32, 14, −8, −13, 38, 19, −19, −19, 37, 16

The last word of the message is SUE. What is the message?

55. Cryptography A code breaker intercepted the encoded message below.

45, −35, 38, −30, 18, −18, 35, −30, 81, −60, 42, −28, 75, −55, 2, −2, 22, −21, 15, −10

Let $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$. You know that $[45 \ -35]A^{-1} = [10 \ 15]$ and that $[38 \ -30]A^{-1} = [8 \ 14]$, where A^{-1} is the inverse of the encoding matrix A. Explain how you can find the values of w, x, y, and z. Decode the message.

56. Cryptography Your biology professor gives you the encoded message below.

−204, 47, −231, 53, −265, 61, −223, 51, −9, 2, −117, 28, −117, 26, −166, 37, −265, 61, −145, 34, −112, 25, −76, 19

Let $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$. You know that $[-204 \ \ 47]A^{-1} = [15 \ 16]$ and that $[-231 \ \ 53]A^{-1} = [15 \ \ 19]$, where A^{-1} is the inverse of the encoding matrix A. Explain how you can find the values of w, x, y, and z. Decode the message.

Business Capsule



Kei Uesugi/Getty Images

Voltage Security, Inc. is a leader in secure business communications and data protection. The company provides the most scalable enterprise key management and encryption capabilities for securing data. Invented by Dr. Dan Boneh and Dr. Matt Franklin in 2001, Identity-Based Encryption or IBE is a breakthrough in cryptography. IBE enables users to simply use an identity, such as an e-mail address, to secure business communications.

57. Research Project Use your campus library, the Internet, or some other reference source to find information about a company that generates software which uses cryptography to secure data. Write a brief paper about such a company or small business.

Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 529.
Answers to odd-numbered Review Exercises are given in the back of the book.*

Section 6.1

Review Exercises

- Determine the order of a matrix. *1, 2*

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \Bigg\} m \text{ rows}$$

 A matrix having m rows and n columns is of order $m \times n$.

- Perform elementary row operations on a matrix in order to write the matrix in row-echelon form or reduced row-echelon form. *3–6*
- Solve a system of linear equations using Gaussian elimination or Gauss-Jordan elimination. *7–16*

Section 6.2

- Add or subtract two matrices and multiply a matrix by a scalar. *17–20, 33, 34*

 If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices and c is a scalar, then

 $A + B = [a_{ij} + b_{ij}]$ and $cA = [ca_{ij}]$.

- Find the product of two matrices. *25–32*

 If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then AB is an $m \times p$ matrix

 $AB = [c_{ij}]$

 where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$.

- Solve a matrix equation. *21–24*
- Use matrix multiplication to solve an application problem. *35, 36*

Section 6.3

- Verify that a matrix is the inverse of a given matrix. *37, 38*
- Find the inverse of a matrix. *39, 40*
- Find the inverse of a 2×2 matrix using a formula. *41, 42*

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Use an inverse matrix to solve a system of linear equations. *43–54*

*Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

Section 6.4

	Review Exercises
■ Evaluate the determinant of a 2×2 matrix. $\det(A) = \lvert A \rvert = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$	*55–58*
■ Find the minors of a matrix.	*59–62*
■ Find the cofactors of a matrix.	*59–62*
■ Find the determinant of a square matrix. $\lvert A \rvert = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$	*63–68*
■ Find the determinant of a triangular matrix.	*65, 66*

Section 6.5

■ Find the area of a triangle using a determinant. $\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$	*69–72*
■ Determine whether three points are collinear using a determinant. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$	*73–76*
■ Use a determinant to find an equation of a line. $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$	*77–80*
■ Encode and decode a cryptogram using a matrix.	*81–85*

Study Strategies

■ **Variety of Approaches** You can use a variety of approaches when finding the determinant of a square matrix.

1. For a 2×2 matrix, you can use the definition $\lvert A \rvert = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.
2. For a 3×3 matrix, you can use the diagonal process shown in Example 5 on page 513.
3. For any square matrix (of order 2×2 or greater), you can use expansion by cofactors. Be sure you choose the row or column that makes the computations the easiest.
4. You can always use the matrix capabilities of a graphing utility.

■ **Using Technology** Performing operations with matrices can be tedious. You can use a graphing utility to accomplish the following.

- Perform elementary row operations on matrices.
- Reduce matrices to row-echelon form and reduced row-echelon form.
- Add and subtract matrices.
- Multiply matrices.
- Multiply matrices by scalars.
- Find inverses of matrices.
- Solve systems of equations using matrices.
- Evaluate determinants of matrices.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, determine the order of the matrix.

1. $\begin{bmatrix} 3 & 7 & 4 & -2 \\ 1 & 8 & 6 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 5 \\ -1 \\ 2 \\ 4 \end{bmatrix}$

In Exercises 3 and 4, write the matrix in row-echelon form.

3. $\begin{bmatrix} 1 & 3 & 0 & 2 \\ 3 & 10 & 1 & 8 \\ 2 & 3 & 3 & 10 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & -1 & 0 \\ -2 & -3 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{bmatrix}$

In Exercises 5 and 6, write the matrix in *reduced* row-echelon form.

5. $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 3 & 1 & -5 \\ 1 & 0 & 5 & 2 \\ -1 & 4 & 3 & 6 \\ 0 & -2 & 6 & -8 \end{bmatrix}$

In Exercises 7–14, use matrices to solve the system of equations (if possible). Use Gaussian elimination with back-substitution or Gauss-Jordan elimination.

7. $\begin{cases} 4x - 3y = 18 \\ x + y = 1 \end{cases}$

8. $\begin{cases} 2x + 4y = 16 \\ -x + 3y = 17 \end{cases}$

9. $\begin{cases} 2x + 3y - z = 13 \\ 3x + z = 8 \\ x - 2y + 3z = -4 \end{cases}$

10. $\begin{cases} 3x + 4y + 2z = 5 \\ 2x + 3y = 7 \\ 2y - 3z = 12 \end{cases}$

11. $\begin{cases} x + 2y + 2z = 10 \\ 2x + 3y + 5z = 20 \end{cases}$

12. $\begin{cases} 3x + 10y + 4z = 20 \\ x + 3y - 2z = 8 \end{cases}$

13. $\begin{cases} 2x + y - 3z = 4 \\ x + 2y + 2z = 10 \\ x - 2z = 12 \\ x + y + z = 6 \end{cases}$

14. $\begin{cases} 2x + 4y + 2z = 10 \\ x + 3z = 9 \\ 3x - 2y = 4 \\ x + y + z = 8 \end{cases}$

15. Biology A school district borrowed $300,000 at simple annual interest to upgrade microbiology equipment. Some of the money was borrowed at 8%, some at 10%, and some at 12%. Use a system of equations to determine how much was borrowed at each rate if the total annual interest was $30,000 and the amount borrowed at 10% was three times the amount borrowed at 8%. Solve the system using matrices.

16. Amusement Park An amusement park borrowed $650,000 at simple annual interest to renovate a roller coaster. Some of the money was borrowed at 8.5%, some at 9.5%, and some at 10%. Use a system of equations to determine how much was borrowed at each rate if the total annual interest was $58,250 and the amount borrowed at 8.5% was four times the amount borrowed at 10%. Solve the system using matrices.

In Exercises 17–20, find (a) $A + B$, (b) $A - B$, (c) $4A$, and (d) $4A - 3B$.

17. $A = \begin{bmatrix} -1 & 5 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ -6 & 3 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 5 \\ 2 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -4 & 6 \\ 1 & 2 & -3 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 3 & -2 & 6 \\ 0 & 1 & 3 & 2 \end{bmatrix}$,

$B = \begin{bmatrix} 2 & 1 & 4 & -5 \\ 3 & -6 & 3 & -2 \end{bmatrix}$

20. $A = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$

In Exercises 21–24, solve for X when

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}.$$

21. $X = 4A - 3B$

22. $X = 5B + 2A$

23. $2X - 3A = B$

24. $4X - 8B = 4A$

In Exercises 25–30, find AB, if possible.

25. $A = \begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

26. $A = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}$

27. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$

28. $A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 6 \\ 0 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

29. $A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

30. $A = \begin{bmatrix} 1 & 2 & 3 & 6 & -1 \\ 2 & 8 & 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$

In Exercises 31 and 32, find (a) AB, (b) BA, and, if possible, (c) A^2. (*Note:* $A^2 = AA$.)

31. $A = \begin{bmatrix} 1 & -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

32. $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -2 & 1 \\ 5 & 4 & -2 \end{bmatrix}$

33. Factory Production A window corporation has four factories, each of which manufactures three products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 80 & 120 & 20 & 40 \\ 40 & 60 & 80 & 20 \\ 140 & 60 & 100 & 80 \end{bmatrix}.$$

Find the production levels if production is increased by 20%

34. Factory Production An electronics manufacturer has three factories, each of which manufactures four products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 120 & 140 & 60 \\ 80 & 100 & 40 \\ 40 & 160 & 80 \\ 20 & 120 & 100 \end{bmatrix}.$$

Find the production levels if production is decreased by 10%

35. Inventory Levels A company sells four different models of car sound systems through three retail outlets. The inventories of the four models at the three outlets are given by matrix S.

$$S = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \overbrace{\begin{array}{c} \\ \\ \\ \\ \end{array}}^{\text{Model}} \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{bmatrix} 3 & 2 & 1 & 4 \\ 1 & 3 & 4 & 3 \\ 5 & 3 & 2 & 2 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \} \text{Outlet}$$

The wholesale and retail prices of the four models are given by matrix T.

$$T = \begin{array}{c} \\ \text{Wholesale} \\ \text{\$50} \\ \text{\$25} \\ \text{\$00} \\ \text{\$50} \end{array} \overbrace{\begin{array}{c} \\ \text{Retail} \\ \text{\$00} \\ \text{\$05} \\ \text{\$55} \\ \text{\$150} \end{array}}^{\text{Price}} \begin{array}{l} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \} \text{Model}$$

(a) What is the total retail price of the inventory at Outlet 3?

(b) What is the total wholesale price of the inventory at Outlet 1?

(c) Compute ST and interpret the result in the context of the problem.

36. Labor/Wage Requirements A company that manufactures racing bicycles has the following labor-hour and wage requirements.

Labor-Hour Requirements (per bicycle)

$$S = \begin{array}{c} \\ \text{Cutting} \\ \text{0.9 hour} \\ \text{1.5 hours} \\ \text{3.5 hours} \end{array} \overbrace{\begin{array}{c} \\ \text{Assembly} \\ \text{0.8 hour} \\ \text{1.0 hour} \\ \text{3.0 hours} \end{array}}^{\text{Department}} \begin{array}{c} \\ \text{Packaging} \\ \text{0.2 hour} \\ \text{0.4 hour} \\ \text{0.5 hour} \end{array} \begin{array}{l} \\ \text{Basic} \\ \text{Light} \\ \text{Ultra-light} \end{array} \} \text{Models}$$

Wage Requirements (per hour)

$$T = \begin{array}{c} \\ \text{\$2.00} \\ \text{\$.00} \\ \text{\$.50} \end{array} \overbrace{\begin{array}{c} \text{A} \\ \text{\$3.00} \\ \text{\$.50} \\ \text{\$.00} \end{array}}^{\text{Plant}} \begin{array}{l} \text{B} \\ \\ \\ \\ \end{array} \begin{array}{l} \text{Cutting} \\ \text{Assembly} \\ \text{Packaging} \end{array} \} \text{Department}$$

(a) What is the labor cost for a light racing bicycle at Plant A?

(b) What is the labor cost for an ultra-light racing bicycle at Plant B?

(c) Compute ST and interpret the result.

In Exercises 37 and 38, show that B is the inverse of A.

37. $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 4 \\ 0 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -14 & 5 & -2 \\ 9 & -3 & 1 \\ -3 & 1 & 0 \end{bmatrix}$

38. $A = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 2 & 2 \end{bmatrix}$,

$B = \frac{1}{9}\begin{bmatrix} -4 & 6 & 1 & 1 \\ 10 & -6 & 2 & -7 \\ -7 & 6 & 4 & 4 \\ 12 & -9 & -3 & -3 \end{bmatrix}$

In Exercises 39 and 40, find the inverse of the matrix.

39. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ **40.** $\begin{bmatrix} 3 & 2 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

In Exercises 41 and 42, use the formula on page 501 to find the inverse of the matrix.

41. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ **42.** $\begin{bmatrix} -2 & 1 \\ 4 & 3 \end{bmatrix}$

In Exercises 43 and 44, use the inverse matrix found in Exercise 41 to solve the system of linear equations.

43. $\begin{cases} x + 3y = 15 \\ 2x + 5y = 26 \end{cases}$ **44.** $\begin{cases} x + 3y = 7 \\ 2x + 5y = 11 \end{cases}$

In Exercises 45 and 46, use the inverse matrix found in Exercise 40 to solve the system of linear equations.

45. $\begin{cases} 3x + 2y + 2z = 13 \\ 2y + z = 4 \\ x + z = 5 \end{cases}$ **46.** $\begin{cases} 3x + 2y + 2z = 12 \\ 2y + z = 13 \\ x + z = 3 \end{cases}$

In Exercises 47–50, use an inverse matrix to solve the system of linear equations.

47. $\begin{cases} -3x + 10y = 8 \\ 5x - 17y = -13 \end{cases}$

48. $\begin{cases} 5x - y = 13 \\ -9x + 2y = -24 \end{cases}$

49. $\begin{cases} 3x + 2y - z = 6 \\ x - y + 2z = -1 \\ 5x + y + z = 7 \end{cases}$

50. $\begin{cases} -x + 4y - 2z = 12 \\ 2x - 9y + 5z = -25 \\ -x + 5y - 4z = 10 \end{cases}$

Raw Materials In Exercises 51 and 52, you are making three types of windshield washer fluid in chemistry class. Fluid X requires 9 cups of water, 1 cup of isopropyl alcohol, and 1 tablespoon of detergent. Fluid Y requires 10 cups of water, 3 cups of isopropyl alcohol, and 1 tablespoon of detergent. Fluid Z requires 14 cups of water, 2 cups of isopropyl alcohol, and 2 tablespoons of detergent. A system of linear equations (where x, y, and z represent fluids X, Y, and Z, respectively) is as follows.

$$\begin{cases} 9x + 10y + 14z = (\textit{cups of water}) \\ x + 3y + 2z = (\textit{cups of isopropyl alcohol}) \\ x + y + 2z = (\textit{tablespoons of detergent}) \end{cases}$$

Use the inverse of the coefficient matrix of this system to find the numbers of units of fluids X, Y, and Z that you can produce with the given amounts of ingredients.

51. 240 cups of water
44 cups of isopropyl alcohol
28 tablespoons of detergent

52. 235 cups of water
41 cups of isopropyl alcohol
29 tablespoons of detergent

53. Field of Study The percent y of U.S. college freshmen who identified computer science as their probable field of study from 2001 to 2005 decreased in a pattern that was approximately parabolic. The least squares regression parabola $y = at^2 + bt + c$ for the data is found by solving the system

$$\begin{cases} 5c + 15b + 55a = 9.7 \\ 15c + 55b + 225a = 23.9. \\ 55c + 225b + 979a = 77.3 \end{cases}$$

Let t represent the year, with $t = 1$ corresponding to 2001. *(Source: The Higher Education Research Institute)*

(T) (a) Use a graphing utility to find an inverse matrix with which to solve the system, and find the equation of the least squares regression parabola.

(b) Use the result of part (a) to estimate the percent in 2000.

(c) The actual percent in 2000 was 3.7. How does this value compare with your estimate in part (b)?

54. Carnivorous Plants A Venus flytrap is grown in a greenhouse, and the size y (in millimeters) of its traps is measured at the end of each year for 5 years. The least squares regression parabola $y = at^2 + bt + c$ for the data is found by solving the system

$$\begin{cases} 5c + 15b + 55a = 53.3 \\ 15c + 55b + 225a = 190.6. \\ 55c + 225b + 979a = 755.8 \end{cases}$$

Let t represent the year, with $t = 1$ corresponding to the first year.

(T) (a) Use a graphing utility to find an inverse matrix with which to solve the system, and find the equation of the least squares regression parabola.

(b) Use the result of part (a) to estimate the sizes of the traps after the first and third years.

(c) The actual sizes of the traps were 2.5 millimeters after the first year and 12.8 millimeters after the third year. How do these values compare with your estimates in part (b)?

In Exercises 55–58, find the determinant of the matrix.

55. $\begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$ **56.** $\begin{bmatrix} 7 & 2 \\ 9 & -3 \end{bmatrix}$

57. $\begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix}$ **58.** $\begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$

In Exercises 59–62, find all (a) minors and (b) cofactors of the matrix.

59. $\begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix}$

60. $\begin{bmatrix} 3 & 6 \\ 5 & -4 \end{bmatrix}$

61. $\begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 0 \\ 1 & 8 & 6 \end{bmatrix}$

62. $\begin{bmatrix} 8 & 3 & 4 \\ 6 & 5 & -9 \\ -4 & 1 & 2 \end{bmatrix}$

In Exercises 63–68, find the determinant of the matrix. Tell whether you used expansion by cofactors, the product of the entries on the main diagonal, or upward and downward diagonals.

63. $\begin{bmatrix} 1 & 2 & 3 \\ 8 & 6 & 7 \\ 0 & 2 & -1 \end{bmatrix}$

64. $\begin{bmatrix} -2 & 3 & 3 \\ -1 & 0 & 5 \\ 1 & 2 & -1 \end{bmatrix}$

65. $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

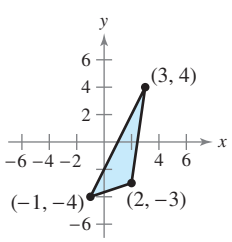
66. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 1 & -2 & 0 \\ 6 & 3 & 1 & 1 \end{bmatrix}$

67. $\begin{bmatrix} -2 & 4 & 1 \\ 6 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$

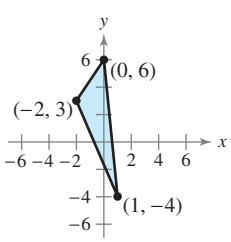
68. $\begin{bmatrix} 4 & 7 & -1 \\ 2 & -3 & 4 \\ -5 & 1 & -1 \end{bmatrix}$

In Exercises 69–72, use a determinant to find the area of the triangle with the given vertices.

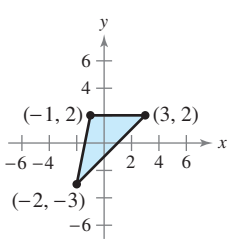
69.



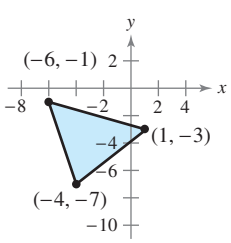
70.



71.



72.



In Exercises 73–76, use a determinant to determine whether the points are collinear.

73. (0, 3), (1, 5), (2, 8)

74. (2, 6), (−2, 3), (0, 5)

75. (−4, 1), (6, 6), (0, 3)

76. (−3, −1), (0, 5), (−4, −3)

In Exercises 77–80, use a determinant to find an equation of the line passing through the points.

77. (−7, 3), (8, 2)

78. (5, −4), (−3, 2)

79. (2, 4), (2, −7)

80. (−5, −1), (1, −1)

In the remaining exercises for this review, use the number code chart on page 521.

In Exercises 81 and 82, find the uncoded row matrices for the message. Then encode the message using the encoding matrix.

Message	*Encoding Matrix*
81. TRANSMIT NOW	$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$
82. CALL AT MIDNIGHT	$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

In Exercises 83 and 84, use A^{-1} to decode the cryptogram.

83. $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

14, 53, −17, 96, 5, 10, 12, 64, 5, 10, 3, 11, 25, 50

84. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

−14, −1, 10, −38, 2, 27, −94, 18, 57, 7, −11, −1, −96, 20, 57, −74, 23, 35, 17, −12, −5

85. Cryptography A family sends the encoded message below to a relative overseas.

−57, −13, 91, 26, 97, 29, −76, −19, 5, 5, −84, −21, 55, 16, −28, −7, 97, 28, −8, −2

Let $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

(a) You know that $[-57 \quad -13]A^{-1} = [23 \quad 5]$ and that $[91 \quad 26]A^{-1} = [0 \quad 13]$, where A^{-1} is the inverse of the encoding matrix A. Explain how you can find the values of w, x, y, and z.

(b) Decode the message.

Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, write the augmented matrix for the system of linear equations.

1. $\begin{cases} 2x + y + 4z = 2 \\ x + 4y - z = 0 \\ -x + 3y + 3z = -1 \end{cases}$

2. $\begin{cases} 3x + 4y + 2z = 4 \\ 2x + 3y = -2 \\ 2y - 3z = -13 \end{cases}$

In Exercises 3–5, use matrices to solve the system of equations.

3. $\begin{cases} x + 2y + 3z = 16 \\ 5x + 4y - z = 22 \end{cases}$

4. $\begin{cases} x - 2y + z = 14 \\ y - 3z = 2 \\ z = -6 \end{cases}$

5. $\begin{cases} 2x - 3y + z = 14 \\ x + 2y = -4 \\ y - z = -4 \end{cases}$

In Exercises 6–9, use the matrices to find the indicated matrix.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -2 \\ 3 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

6. $2A + C$ **7.** CA **8.** BD **9.** A^2

In Exercises 10–12, find the inverse of the matrix.

10. $A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ **11.** $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **12.** $A = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$

In Exercises 13–15, find the determinant of the matrix.

13. $\begin{bmatrix} -5 & 2 \\ 1 & 3 \end{bmatrix}$ **14.** $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 2 \\ 4 & 5 & 2 \end{bmatrix}$ **15.** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

16. Use the inverse matrix found in Exercise 12 to solve the system in Exercise 2.

17. Find two nonzero matrices whose product is a zero matrix.

18. Find the area of the triangle whose vertices are $(-3, 1)$, $(0, 4)$, and $(5, 2)$.

19. Use a determinant to decide whether $(2, 1)$, $(-3, -14)$, and $(4, 7)$ are collinear.

20. Use a determinant to find an equation of the line passing through the points $(1, -2)$ and $(5, 2)$.

21. A manufacturer produces three models of a product, which are shipped to two warehouses. The number of units i that are shipped to warehouse j is represented by a_{ij} in matrix A below. The prices per unit are represented by matrix B. Find the product BA and interpret the result.

$$A = \begin{bmatrix} 1500 & 4000 \\ 3000 & 4500 \\ 5500 & 7000 \end{bmatrix}$$

$$B = [\$5 \quad \$0 \quad \$3]$$

7 Sequences, Series, and Probability



John Mitchell/Photo Researchers, Inc.

Burmese pythons, which are not native to the United States, were introduced in the Florida Everglades by their former owners. As a result, there have been several documented alligator and python encounters. You can use the terms of a sequence to find a trend in the ratio of alligators to pythons in a marshland. (See Section 7.1, Exercise 87.)

7.1 Sequences and Summation Notation
7.2 Arithmetic Sequences and Partial Sums
7.3 Geometric Sequences and Series
7.4 The Binomial Theorem
7.5 Counting Principles
7.6 Probability
7.7 Mathematical Induction

Applications

Sequences, series, and probability are used to solve many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Annual Payroll, Exercise 90, page 544
- Counting Kidney Donors, Example 6, page 574
- Blogging, Exercises 69 and 70, page 592

Section 7.1

Sequences and Summation Notation

- Use sequence notation to find the terms of a sequence.
- Evaluate a factorial expression.
- Find the sum of a finite sequence and use summation notation to write the sum of a sequence.
- Use a sequence to solve an application problem.

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English. Saying that a collection is listed in *sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers. Rather than using function notation, however, sequences are usually written using subscript notation, as indicated in the definition below.

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

are the **terms** of the sequence. If the domain of the function consists of the first n positive integers only, the sequence is called a **finite sequence.**

On occasion it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become $a_0, a_1, a_2, a_3, \ldots$.

Example 1 Finding Terms of a Sequence

a. The first four terms of the sequence given by $a_n = 3n - 2$ are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10.$ 4th term

b. The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$$a_1 = 3 + (-1)^1 = 3 - 1 = 2 \qquad a_2 = 3 + (-1)^2 = 3 + 1 = 4$$

$$a_3 = 3 + (-1)^3 = 3 - 1 = 2 \qquad a_4 = 3 + (-1)^4 = 3 + 1 = 4.$$

✓CHECKPOINT 1

Write the first four terms of the sequence given by $a_n = 2n + 1$. ■

When working with sequences, note that any variable can be used as a subscript. The most commonly used variable subscripts in sequence and series notation are i, j, k, and n. Also note that the terms of a sequence need not all be positive, as shown in Example 2.

Example 2 Finding Terms of a Sequence

The first four terms of the sequence given by $a_n = \frac{(-1)^n}{2n - 1}$ are

$$a_1 = \frac{(-1)^1}{2(1) - 1} = \frac{-1}{2 - 1} = -1 \qquad \text{1st term}$$

$$a_2 = \frac{(-1)^2}{2(2) - 1} = \frac{1}{4 - 1} = \frac{1}{3} \qquad \text{2nd term}$$

$$a_3 = \frac{(-1)^3}{2(3) - 1} = \frac{-1}{6 - 1} = -\frac{1}{5} \qquad \text{3rd term}$$

$$a_4 = \frac{(-1)^4}{2(4) - 1} = \frac{1}{8 - 1} = \frac{1}{7}. \qquad \text{4th term}$$

✓CHECKPOINT 2

Write the first four terms of the sequence given by $a_n = \frac{(-1)^n}{n + 1}$.

Try finding the first four terms of the sequence whose nth term is

$$a_n = \frac{(-1)^{n+1}}{2n - 1}.$$

How do they differ from the first four terms of the sequence in Example 2?

It is important to realize that simply listing the first few terms is not sufficient to define a unique sequence~~the~~ the nth term *must be given.* To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n + 1)(n^2 - n + 6)}, \ldots$$

TECHNOLOGY

Use the *table* feature of a graphing utility to find the first four terms of each sequence. Be sure to start the table with $n = 1$ and set the table to increment n by 1.

a. $a_n = 2^n + 1$ **b.** $a_n = (-1)^n(2^n + 1)$ **c.** $a_n = (-1)^{n+1}(2^n + 1)$

Describe the differences in the terms of the sequences.

For instructions on how to use the *table* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com/info/larsonapplied.*

Factorial Notation

Some important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

Definition of Factorial

If n is a positive integer, ***n* factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots (n-1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

Here are some values of $n!$ for the first several nonnegative integers. Notice that 0! is 1 by definition.

$$0! = 1$$
$$1! = 1$$
$$2! = 1 \cdot 2 = 2$$
$$3! = 1 \cdot 2 \cdot 3 = 6$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$
$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

The value of n does not have to be large before the value of $n!$ becomes very large. For instance, $10! = 3{,}628{,}800$.

TECHNOLOGY

Most scientific and graphing calculators have a *factorial* feature built in. Use the *factorial* feature to verify the factorials at the right. Try finding such values as 8!, 15!, and 20!. How does the calculator's display change for increasingly large values of n? For specific keystrokes and for information on how to use the *factorial* feature, go to the text website at *college.hmco.com/info/larsonapplied*.

Example 3 Finding Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by

$$a_n = \frac{2}{n!}.$$

Begin with $n = 0$.

SOLUTION

$$a_0 = \frac{2}{0!} = \frac{2}{1} = 2 \qquad \text{0th term}$$

$$a_1 = \frac{2}{1!} = \frac{2}{1} = 2 \qquad \text{1st term}$$

$$a_2 = \frac{2}{2!} = \frac{2}{2} = 1 \qquad \text{2nd term}$$

$$a_3 = \frac{2}{3!} = \frac{2}{6} = \frac{1}{3} \qquad \text{3rd term}$$

$$a_4 = \frac{2}{4!} = \frac{2}{24} = \frac{1}{12} \qquad \text{4th term}$$

✓CHECKPOINT 3

Write the first five terms of the sequence given by $a_n = \frac{6}{n!}$. Begin with $n = 1$. ■

Factorials follow the same conventions for order of operations as do exponents. For instance,

$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n)$, whereas

$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot 2n.$

Example 4 Finding Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by

$$a_n = \frac{2^n}{n!}.$$

Begin with $n = 0$.

SOLUTION

$$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1 \qquad \text{0th term}$$

$$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2 \qquad \text{1st term}$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2 \qquad \text{2nd term}$$

$$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3} \qquad \text{3rd term}$$

$$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3} \qquad \text{4th term}$$

✓CHECKPOINT 4

Write the first five terms of the sequence given by

$$a_n = \frac{(n+1)!}{4n}.$$

Begin with $n = 1$. ■

When working with fractions involving factorials, you will often find that the fractions can be reduced.

Example 5 Simplifying Factorial Expressions

Simplify each factorial expression.

a. $\dfrac{8!}{2! \cdot 6!}$ **b.** $\dfrac{2! \cdot 6!}{3! \cdot 5!}$ **c.** $\dfrac{n!}{(n-1)!}$

SOLUTION

a. $$\frac{8!}{2! \cdot 6!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{7 \cdot 8}{2} = 28$$

b. $$\frac{2! \cdot 6!}{3! \cdot 5!} = \frac{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{6}{3} = 2$$

c. $$\frac{n!}{(n-1)!} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)} = n$$

✓CHECKPOINT 5

Simplify the factorial expression given by $\dfrac{4!(n+1)!}{3!n!}$. ■

Series and Summation Notation

The sum of the terms of an infinite sequence is called an **infinite series.** The sum of the first n terms of a sequence is called a **finite series** or the ***n*th partial sum.** There is a convenient notation for the sum of the terms of a finite sequence. It is called **summation notation** or **sigma notation** because it involves the use of the uppercase Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where i is the **index of summation,** n is the **upper limit of summation,** and 1 is the **lower limit of summation.**

Summation notation is an instruction to add the terms of a sequence. From the definition above, the lower limit of the summation tells you where to begin the sum and the upper limit of the summation tells you where to end the sum.

Example 6 Using Summation Notation

a. $\displaystyle\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$

$= 3(1 + 2 + 3 + 4 + 5)$

$= 3(15) = 45$

b. $\displaystyle\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$

$= 10 + 17 + 26 + 37 = 90$

c. $\displaystyle\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}$

≈ 2.71828

Note that the sum in part (c) is very close to the irrational number $e \approx 2.718281828$. It can be shown that as more terms of the sequence whose nth term is $1/n!$ are added, the sum approaches e.

DISCOVERY

Consider the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Use a graphing utility to find a_{100}, a_{1000}, $a_{10,000}$, and $a_{100,000}$. Compare your results with the approximation of e given by a graphing utility. What do you observe?

✓CHECKPOINT 6

Find the sum $\displaystyle\sum_{i=1}^{4} (4i + 1)$. ■

In Example 6, note that the lower index of a summation does not have to be 1. Also note that the index does not have to be the letter i. For instance, in part (b), the letter k is the index.

STUDY TIP

The sum of the terms of any *finite* sequence must be a finite number. Variations in the upper and lower limits of summation can produce quite different-looking summation notations for *the same sum,* as you can see in Example 7. For example, consider the following two sums.

$$\sum_{i=1}^{3} 3(2^i) = 3\sum_{i=1}^{3} 2^i = 3(2^1 + 2^2 + 2^3)$$

$$\sum_{i=0}^{2} 3(2^{i+1}) = 3\sum_{i=0}^{2} 2^{i+1} = 3(2^1 + 2^2 + 2^3)$$

✓CHECKPOINT 7

Write the partial sum in summation notation. Decide whether the notation is unique.

$$\frac{1-1}{2} + \frac{2-1}{2} + \frac{3-1}{2} + \frac{4-1}{2}$$ ■

Example 7 Writing a Sum in Summation Notation

Write each partial sum in summation notation. Decide whether the notation is unique.

a. $\frac{3}{1+2} + \frac{3}{2+2} + \frac{3}{3+2} + \frac{3}{4+2} + \frac{3}{5+2}$ **b.** $2 - 4 + 8 - 16$

SOLUTION

a. Begin by looking for similarities and differences in each of the terms. Each term has 3 in the numerator. In the denominator, 2 is added to a number that is increasing by 1 for each term. Let i be the index of summation, let 1 be the lower limit of summation, and let 5 be the upper limit of summation. Then, using summation notation, you can write

$$\frac{3}{1+2} + \frac{3}{2+2} + \frac{3}{3+2} + \frac{3}{4+2} + \frac{3}{5+2} = \sum_{i=1}^{5} \frac{3}{i+2}.$$

The sum can also be written in other ways, such as

$$\sum_{i=3}^{7} \frac{3}{i}, \quad \sum_{i=0}^{4} \frac{3}{i+3}, \quad \text{and} \quad \sum_{i=4}^{8} \frac{3}{i-1}.$$

The summation notation for the sum is not unique.

b. This series has terms with alternating signs, and each term can be written as a power of 2. Let n be the index of summation, let 1 be the lower limit of summation, and let 4 be the upper limit of summation. Then, using summation notation, you can write

$$2 - 4 + 8 - 16 = (-1)^{1+1}(2)^1 + (-1)^{2+1}(2)^2 + (-1)^{3+1}(2)^3 + (-1)^{4+1}(2)^4$$

$$= \sum_{n=1}^{4} (-1)^{n+1}(2)^n.$$

The sum can also be written in other ways, such as

$$\sum_{n=0}^{3} (-1)^n(2)^{n+1} \quad \text{and} \quad \sum_{n=2}^{5} (-1)^n(2)^{n-1}.$$

The summation notation for the sum is not unique.

The following properties of sums are helpful when adding the terms of a sequence.

Properties of Sums

1. $\sum_{i=1}^{n} c = cn$, c is a constant.
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, c is a constant.
3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Application

Sequences have many applications in business and science. One such application is illustrated in Example 8.

Example 8
MAKE A DECISION **Population of the United States**

Figure 7.1 shows the population of the United States for the years 1990 to 2005. These data can be approximated by the model

$$a_n = 0.001n^3 - 0.05n^2 + 3.6n + 250, \quad n = 0, 1, \ldots, 15$$

where a_n is the population (in millions) and n represents the year, with $n = 0$ corresponding to 1990. Find the last three terms of this finite sequence. The projected population of the United States in the year 2050 is 420 million. Use the model to predict the population in 2050. Do you think this model should be used to predict the United States population in future years? *(Source: U.S. Census Bureau)*

SOLUTION The last three terms of this finite sequence are as follows.

$$a_{13} = 0.001(13)^3 - 0.05(13)^2 + 3.6(13) + 250 \approx 291 \quad \text{2003 population}$$

$$a_{14} = 0.001(14)^3 - 0.05(14)^2 + 3.6(14) + 250 \approx 293 \quad \text{2004 population}$$

$$a_{15} = 0.001(15)^3 - 0.05(15)^2 + 3.6(15) + 250 \approx 296 \quad \text{2005 population}$$

To predict the population in 2050, find the 60th term of the sequence, as follows.

$$a_{60} = 0.001(60)^3 - 0.05(60)^2 + 3.6(60) + 250 \approx 502$$

This estimate is too high when compared with the Census Bureau's estimate. This model should not be used to predict the U.S. population in future years. From Figure 7.1, you can see that the population grew in a linear or quadratic pattern from 1990 to 2005. So, a linear or quadratic model would be better for predicting the population in future years.

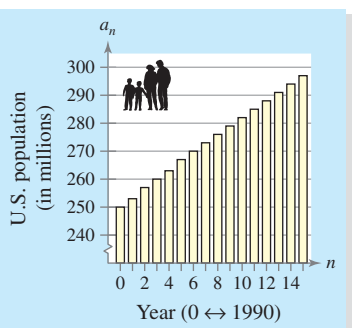


FIGURE 7.1

✓CHECKPOINT 8

The projected population of the United States in the year 2020 is 336 million. Use the model in Example 8 to predict the U.S. population in 2020. How close is the prediction to the projection of the population? ■

CONCEPT CHECK

1. Explain how to find the term a_3 of the sequence given by
$$a_n = n + \frac{1}{n-1}.$$
2. Explain how to evaluate $\frac{88!}{87!}$ without using multiplication.
3. Determine whether the following statement is true or false. Explain your reasoning.
There are three terms in the sum $\sum_{i=3}^{6} 3i$.
4. Describe and correct the error in the partial sum.
$$\sum_{k=1}^{4} c = c + 2c + 3c + 4c = 10c$$

Skills Review 7.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3, 0.6, and 2.4.

In Exercises 1 and 2, find the indicated function value.

1. Find $f(2)$ for $f(n) = \dfrac{2n}{n^2 + 1}$.

2. Find $f(3)$ for $f(n) = \dfrac{4}{3(n + 1)}$.

In Exercises 3–6, factor the expression.

3. $4n^2 - 1$

4. $4n^2 - 8n + 3$

5. $n^2 - 3n + 2$

6. $n^2 + 3n + 2$

In Exercises 7–10, perform the indicated operations and simplify.

7. $\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\left(\frac{5}{6}\right)$

8. $\dfrac{2 \cdot 4 \cdot 6 \cdot 8}{2^4}$

9. $\dfrac{1}{2 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{2 \cdot 4}$

10. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4}$

Exercises 7.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–22, write the first five terms of the sequence. (Assume that n begins with 1.)

1. $a_n = 2n - 7$

2. $a_n = 4 - 2n$

3. $a_n = 2^n$

4. $a_n = (-2)^n$

5. $a_n = \left(\frac{1}{2}\right)^n$

6. $a_n = \left(-\frac{1}{2}\right)^n$

7. $a_n = \dfrac{3}{n + 1}$

8. $a_n = \dfrac{n + 1}{2}$

9. $a_n = \dfrac{(-1)^n}{n}$

10. $a_n = \dfrac{(-1)^{n+1}}{n}$

11. $a_n = 2 - \dfrac{1}{3^n}$

12. $a_n = \dfrac{2^n}{3^n}$

13. $a_n = \dfrac{3^n}{n!}$

14. $a_n = \dfrac{n!}{n}$

15. $a_n = \dfrac{n^2 - 1}{n + 1}$

16. $a_n = \dfrac{n^2 - 1}{n^2 + 2}$

17. $a_n = \dfrac{(-1)^n}{n^2}$

18. $a_n = \dfrac{(-1)^n n}{n + 1}$

19. $a_n = \dfrac{3n!}{(n - 1)!}$

20. $a_n = \dfrac{(n + 1)!}{n!}$

21. $a_n = \dfrac{5n^2 - n + 1}{n^2}$

22. $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$

In Exercises 23–28, find the indicated term of the sequence.

23. $a_n = (-1)^n(3n - 2)$

$a_{25} =$ ____

24. $a_n = (-1)^{n-1}[n(n - 1)]$

$a_{16} =$ ____

25. $a_n = \dfrac{2^n}{n!}$

$a_{10} =$ ____

26. $a_n = \dfrac{n!}{2n}$

$a_8 =$ ____

27. $a_n = \dfrac{4n}{2n^2 - 3}$

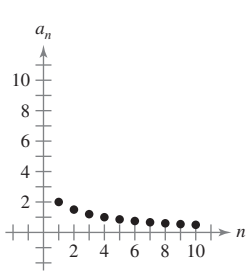
$a_{11} =$ ____

28. $a_n = \dfrac{4n^2 - n + 3}{n(n - 1)(n + 2)}$

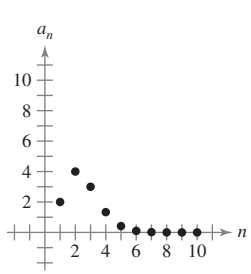
$a_{13} =$ ____

In Exercises 29–32, match the sequence with the graph of its first 10 terms. [The graphs are labeled (a), (b), (c), and (d).]

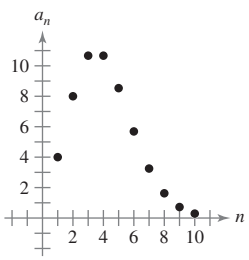
(a)



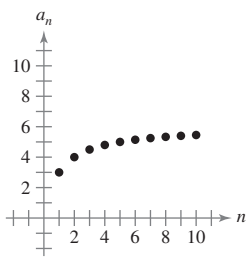
(b)



(c)



(d)



29. $a_n = \dfrac{6}{n+2}$

30. $a_n = \dfrac{6n}{n+1}$

31. $a_n = \dfrac{4^n}{n!}$

32. $a_n = \dfrac{2n}{(n-1)!}$

In Exercises 33–36, evaluate the expression.

33. $\dfrac{8!}{6!}$

34. $\dfrac{10!}{9!}$

35. $\dfrac{28!}{25!}$

36. $\dfrac{49!}{50!}$

In Exercises 37–40, simplify the factorial expression. Then evaluate the expression when $n = 0$ and $n = 1$.

37. $\dfrac{(n+1)!}{n!}$

38. $\dfrac{(n+2)!}{n!}$

39. $\dfrac{(2n-1)!}{(2n+1)!}$

40. $\dfrac{(2n+2)!}{(2n)!}$

In Exercises 41–54, write an expression for the *most apparent* nth term of the sequence. (Assume that n begins with 1.)

41. $1, 4, 7, 10, 13, \ldots$

42. $3, 7, 11, 15, 19, \ldots$

43. $2, -4, 6, -8, 10, \ldots$

44. $1, -1, 1, -1, 1, \ldots$

45. $2, \frac{1}{2}, \frac{2}{9}, \frac{1}{8}, \frac{2}{25}, \ldots$

46. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

47. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

48. $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

49. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, \ldots$

50. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \frac{16}{243}, \ldots$

51. $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

52. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

53. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

54. $1, 2, \dfrac{2^2}{2}, \dfrac{2^3}{6}, \dfrac{2^4}{24}, \dfrac{2^5}{120}, \ldots$

In Exercises 55–72, find the sum.

55. $\displaystyle\sum_{i=1}^{4} (6i + 3)$

56. $\displaystyle\sum_{i=1}^{6} (4i - 2)$

57. $\displaystyle\sum_{i=1}^{7} -3i$

58. $\displaystyle\sum_{i=1}^{5} -3(i - 2)$

59. $\displaystyle\sum_{k=1}^{4} 12$

60. $\displaystyle\sum_{k=1}^{5} 5$

61. $\displaystyle\sum_{i=0}^{4} i^2$

62. $\displaystyle\sum_{i=0}^{5} 3i^2$

63. $\displaystyle\sum_{k=2}^{6} \frac{1}{2k}$

64. $\displaystyle\sum_{j=3}^{5} \frac{1}{j}$

65. $\displaystyle\sum_{i=1}^{4} (i - 1)^2$

66. $\displaystyle\sum_{k=2}^{5} (k + 1)(k - 3)$

67. $\displaystyle\sum_{i=1}^{4} (-1)^i(2i + 4)$

68. $\displaystyle\sum_{i=1}^{4} (-2)^i$

69. $\displaystyle\sum_{k=0}^{4} \frac{1}{k!}$

70. $\displaystyle\sum_{j=0}^{5} \frac{2}{(j+1)!}$

71. $\displaystyle\sum_{j=0}^{6} \frac{(j+2)!}{(2j)!}$

72. $\displaystyle\sum_{i=1}^{4} \frac{i}{(i-1)!}$

In Exercises 73–82, use summation notation to write the sum.

73. $\dfrac{1}{3(1)} + \dfrac{1}{3(2)} + \dfrac{1}{3(3)} + \cdots + \dfrac{1}{3(9)}$

74. $\dfrac{5}{1+1} + \dfrac{5}{1+2} + \dfrac{5}{1+3} + \cdots + \dfrac{5}{1+15}$

75. $\left[2\left(\frac{1}{8}\right) + 3\right] + \left[2\left(\frac{2}{8}\right) + 3\right] + \cdots + \left[2\left(\frac{8}{8}\right) + 3\right]$

76. $\left[1 - \left(\frac{1}{6}\right)^2\right] + \left[1 - \left(\frac{2}{6}\right)^2\right] + \cdots + \left[1 - \left(\frac{6}{6}\right)^2\right]$

77. $3 - 9 + 27 - 81 + 243 - 729$

78. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{128}$

79. $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots - \dfrac{1}{20^2}$

80. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{2 \cdot 4} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{10 \cdot 12}$

81. $\frac{1}{4} + \frac{3}{8} + \frac{7}{16} + \frac{15}{32} + \frac{31}{64}$

82. $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \frac{720}{64}$

83. Evaluate

$$\sum_{j=0}^{4} j^2 \quad \text{and} \quad \sum_{k=2}^{6} (k-2)^2.$$

What do you observe about the two sums? Is the summation notation for a sum unique? Explain.

84. Evaluate

$$\sum_{j=0}^{6} \frac{(j+1)!}{j!} \quad \text{and} \quad \sum_{k=1}^{7} \frac{k!}{(k-1)!}.$$

What do you observe about the two sums? Is the summation notation for a sum unique? Explain.

85. Compound Interest You deposit $5,000 in an account that earns 7% interest compounded monthly. The balance in the account after n months is given by

$$A_n = 25{,}000\left(1 + \frac{0.07}{12}\right)^n, \quad n = 1, 2, 3, \ldots$$

(a) Compute the first six terms of the sequence.

(b) Find the balance in the account after 5 years by computing the 60th term of the sequence.

(c) Is the balance after 10 years twice the balance after 5 years? Explain.

86. Compound Interest A deposit of $10,000 is made in an account that earns 8.5% interest compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 10{,}000\left(1 + \frac{0.085}{4}\right)^n, \quad n = 1, 2, 3, \ldots$$

(a) Compute the first eight terms of the sequence.

(b) Find the balance in the account after 10 years by computing the 40th term of the sequence.

(c) Is the balance after 20 years twice the balance after 10 years? Explain.

87. Ratio of Reptiles The ratio a_n of alligators to pythons in a marshland from 2001 to 2008 can be approximated by the model

$$a_n = 24 - 4.4n + 0.34n^2, \quad n = 1, 2, \ldots, 8$$

where n is the year, with $n = 1, 2, \ldots, 8$ corresponding to 2001, 2002, . . . , 2008.

(a) Use a graphing utility to find the terms of the finite sequence. Interpret the meaning of the terms in the context of the real-life situation.

(b) Construct a bar graph that represents the sequence. Describe any trends you see in your graph.

(c) In 2008, the total number of alligators and pythons in the marsh was about 800. In that year, how many were alligators? How many were pythons?

88. Ratio of Males to Females The ratio a_n of males to females in the United States from 1940 to 2010 can be approximated by the model

$$a_n = 1.05 - 0.038n + 0.0036n^2, \quad n = 1, 2, \ldots, 8$$

where n is the year, with $n = 1, 2, 3, \ldots, 8$ corresponding to 1940, 1950, 1960, . . . , 2010. *(Source: U.S. Census Bureau)*

(a) Use a graphing utility to find the terms of this finite sequence. Interpret the meaning of the terms in the context of the real-life situation.

(b) Construct a bar graph that represents the sequence. Describe any trends you see in your graph.

(c) In 2000, the population of the United States was approximately 281 million. In that year, how many people were females? How many were males?

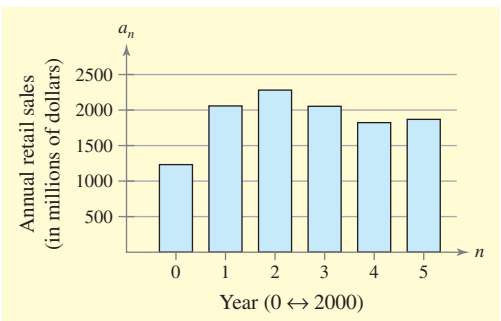
89. Insect Control Sales The annual retail sales a_n (in millions of dollars) of insect control products in the United States from 2000 to 2005 (see bar graph) can be approximated by the model

$$a_n = 53.167n^3 - 500.75n^2 + 1302.7n + 1226,$$
$$n = 0, 1, 2, 3, 4, 5$$

where n represents the year, with $n = 0$ corresponding to 2000. Find the total sales from 2000 to 2005 by evaluating the sum

$$\sum_{n=0}^{5} (53.167n^3 - 500.75n^2 + 1302.7n + 1226).$$

(Source: The National Gardening Association)



90. Annual Payroll The annual payroll a_n (in billions of dollars) of new car dealerships in the United States from 2000 to 2005 can be approximated by the model

$$a_n = 46.3 + 1.53n - 0.102n^2, \quad n = 0, 1, 2, 3, 4, 5$$

where n represents the year, with $n = 0$ corresponding to 2000. Find the total payroll from 2000 to 2005 by evaluating the sum

$$\sum_{n=0}^{5} (46.3 + 1.53n - 0.102n^2).$$

(Source: National Automobile Dealers Association)

In Exercises 91 and 92, determine whether the statement is true or false. Justify your answer.

91. $\sum_{i=1}^{4} (i^2 + 2i) = \sum_{i=1}^{4} i^2 + 2\sum_{i=1}^{4}$

92. $\sum_{j=1}^{4} 2^j = \sum_{j=3}^{6} 2^{j-2}$

In Exercises 93–96, find the first five terms of the sequence.

93. $a_n = \dfrac{x^n}{n!}$

94. $a_n = \dfrac{(-1)^n x^{2n+1}}{2n + 1}$

95. $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$

96. $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n + 1)!}$

Section 7.2

Arithmetic Sequences and Partial Sums

- Recognize and find the nth term of an arithmetic sequence.
- Find a formula for an arithmetic sequence.
- Find the nth partial sum of an arithmetic sequence.
- Use an arithmetic sequence to solve an application problem.

Arithmetic Sequences

A sequence whose consecutive terms have a common difference is called an **arithmetic sequence.**

DISCOVERY

Determine the arithmetic rule used to generate the terms of each sequence.

a. 1, 4, 7, 10, 13, . . .

b. 6, 12, 18, 24, . . .

c. 100, 90, 80, 70, . . .

Definition of Arithmetic Sequence

A sequence is **arithmetic** if the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is arithmetic if there is a number d such that

$$a_2 - a_1 = d, \quad a_3 - a_2 = d, \quad a_4 - a_3 = d, \text{ and so on.}$$

The number d is the **common difference** of the arithmetic sequence.

Example 1 Examples of Arithmetic Sequences

a. The sequence whose nth term is $4n + 3$ is arithmetic. For this sequence, the common difference between consecutive terms is 4.

$$\underbrace{7, 11}_{11 - 7 = 4}, 15, 19, \ldots, 4n + 3, \ldots \qquad \text{Begin with } n = 1.$$

b. The sequence whose nth term is $7 - 5n$ is arithmetic. For this sequence, the common difference between consecutive terms is -5.

$$\underbrace{2, -3}_{-3 - 2 = -5}, -8, -13, \ldots, 7 - 5n, \ldots \qquad \text{Begin with } n = 1.$$

c. The sequence whose nth term is $\frac{1}{4}(n + 3)$ is arithmetic. For this sequence, the common difference between consecutive terms is $\frac{1}{4}$.

$$\underbrace{1, \frac{5}{4}}_{\frac{5}{4} - 1 = \frac{1}{4}}, \frac{3}{2}, \frac{7}{4}, \ldots, \frac{n + 3}{4}, \ldots \qquad \text{Begin with } n = 1.$$

✓CHECKPOINT 1

Determine whether the sequence is arithmetic. If it is, find the common difference.

a. 5, 7, 9, 11, . . . **b.** $-2, 1, 6, 13, \ldots$ **c.** $\frac{7}{2}, 3, \frac{5}{2}, 2, \ldots$ ■

In Example 1, notice that each of the arithmetic sequences has an nth term that is of the form $dn + c$, where the common difference of the sequence is d. An arithmetic sequence may be thought of as a linear function whose domain is the set of natural numbers.

The nth Term of an Arithmetic Sequence

The nth term of an arithmetic sequence has the form

$$a_n = dn + c$$

where d is the common difference between consecutive terms of the sequence and $c = a_1 - d$.

Example 2 Finding the nth Term of an Arithmetic Sequence

Find a formula for the nth term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

SOLUTION Because the sequence is arithmetic, you know that the formula for the nth term is of the form $a_n = dn + c$. Moreover, because the common difference is $d = 3$, the formula must have the form

$$a_n = 3n + c.$$

Because $a_1 = 2$, it follows that

$$c = a_1 - d = 2 - 3 = -1.$$

So, the formula for the nth term is

$$a_n = 3n - 1.$$

The sequence has the form $2, 5, 8, 11, 14, \ldots, 3n - 1, \ldots$.

✓CHECKPOINT 2

Find a formula for the nth term of the arithmetic sequence whose common difference is 2 and whose first term is 5. ■

Example 3 Finding the nth Term of an Arithmetic Sequence

Find the ninth term of the arithmetic sequence whose first two terms are 2 and 9.

SOLUTION For this sequence, the common difference is $d = 9 - 2 = 7$. There are two ways to find the ninth term. One way is to write out the first nine terms by repeatedly adding 7.

$$2, 9, 16, 23, 30, 37, 44, 51, 58$$

Another way to find the ninth term is first to find a formula for the nth term. Because the first term is 2, it follows that

$$c = a_1 - d = 2 - 7 = -5.$$

So, a formula for the nth term is $a_n = 7n - 5$, which implies that the ninth term is

$$a_9 = 7(9) - 5 = 58.$$

✓CHECKPOINT 3

Find the 10th term of the arithmetic sequence whose first two terms are 1 and 4. ■

If you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$th term by using the **recursion formula**

$$a_{n+1} = a_n + d. \qquad \text{Recursion formula}$$

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the previous term. For instance, if you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

If you substitute $a_1 - d$ for c in the formula $a_n = dn + c$, then the nth term of an arithmetic sequence has the alternative recursion formula

$$a_n = a_1 + (n - 1)d. \qquad \text{Alternative recursion formula}$$

Example 4 Using a Recursion Formula

Find the 15th term of the arithmetic sequence whose common difference is 11 and whose 14th term is 136.

SOLUTION Because you know the 14th term and want to find the 15th term, use the recursion formula.

$$\begin{aligned} a_{n+1} &= a_n + d \\ a_{14+1} &= a_{14} + d \\ a_{15} &= 136 + 11 \\ &= 147 \end{aligned}$$

So, the 15th term of the sequence is $a_{15} = 147$.

✓CHECKPOINT 4

Find the 12th term of the arithmetic sequence whose common difference is 6 and whose 11th term is 37. ■

Example 5 Finding the First Several Terms of an Arithmetic Sequence

The fourth term of an arithmetic sequence is 20, and the 13th term is 65. Write the first several terms of this sequence.

SOLUTION You know that $a_4 = 20$ and $a_{13} = 65$. So, you must add the common difference d to the fourth term nine times to obtain the 13th term. Therefore, the fourth and 13th terms of the sequence are related by

$$a_{13} = a_4 + 9d. \qquad a_4 \text{ and } a_{13} \text{ are nine terms apart.}$$

Using $a_4 = 20$ and $a_{13} = 65$, you can conclude that $d = 5$, which implies that the sequence is as follows.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	
5,	10,	15,	20,	25,	30,	35,	40,	45,	50,	55,	. . .

✓CHECKPOINT 5

The third term of an arithmetic sequence is 12, and the 14th term is 45. Write the first several terms of this sequence. ■

The Sum of a Finite Arithmetic Sequence

There is a simple formula for the sum of a finite arithmetic sequence. Be sure you see that this formula works only for *arithmetic* sequences.

The Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

DISCOVERY

To *develop* the formula for the sum of a finite arithmetic sequence, consider writing the sum in two different ways. One way is to write the sum as

$$\begin{aligned} S_n &= a_1 + (a_1 + d) \\ &\quad + (a_1 + 2d) + \cdots \\ &\quad + [a_1 + (n - 1)d]. \end{aligned}$$

In the second way, you repeatedly subtract d from the nth term to obtain

$$\begin{aligned} S_n &= a_n + (a_n - d) \\ &\quad + (a_n - 2d) + \cdots \\ &\quad + [a_n - (n - 1)d]. \end{aligned}$$

Can you discover a way to combine these two versions of S to obtain the following formula?

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Example 6 Finding the Sum of a Finite Arithmetic Sequence

Find the sum: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$.

SOLUTION To begin, notice that the sequence is arithmetic (with a common difference of 2). Moreover, the sequence has 10 terms. So, the sum of the sequence is

$$\begin{aligned} S_{10} &= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 \\ &= \frac{n}{2}(a_1 + a_n) && \text{Formula for the sum of a finite arithmetic sequence} \\ &= \frac{10}{2}(1 + 19) && \text{Substitute 10 for } n, 1 \text{ for } a_1, \text{ and 19 for } a_n. \\ &= 5(20) && \text{Simplify.} \\ &= 100. && \text{Multiply.} \end{aligned}$$

✓CHECKPOINT 6

Find the sum: $4 + 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 + 31$. ■

Example 7 Finding the Sum of a Finite Arithmetic Sequence

Find the sum of the integers from 1 to 100.

SOLUTION The integers from 1 to 100 form an arithmetic sequence that has 100 terms. So, you can use the formula for the sum of an arithmetic sequence, as follows.

$$\begin{aligned} S_n &= 1 + 2 + 3 + 4 + 5 + 6 + \cdots + 99 + 100 \\ &= \frac{n}{2}(a_1 + a_n) && \text{Formula for the sum of a finite arithmetic sequence} \\ &= \frac{100}{2}(1 + 100) && \text{Substitute 100 for } n, 1 \text{ for } a_1, \text{ and 100 for } a_n. \\ &= 50(101) && \text{Simplify.} \\ &= 5050 && \text{Multiply.} \end{aligned}$$

✓CHECKPOINT 7

Find the sum of the integers from 101 to 150. ■

The sum of the first n terms of an infinite sequence is the ***n*th partial sum.** The nth partial sum can be found by using the formula for the sum of a finite arithmetic sequence.

Example 8 Finding a Partial Sum of an Arithmetic Sequence

Find the 150th partial sum of the arithmetic sequence

$$5, 16, 27, 38, 49, \ldots$$

SOLUTION For this arithmetic sequence, you have $a_1 = 5$ and $d = 16 - 5 = 11$. So, $c = a_1 - d = 5 - 11 = -6$, and the nth term is

$$a_n = 11n - 6.$$

So, $a_{150} = 11(150) - 6 = 1644$, and the sum of the first 150 terms is as follows.

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad n\text{th partial sum formula}$$

$$= \frac{150}{2}(5 + 1644) \qquad \text{Substitute 150 for } n, 5 \text{ for } a_1, \text{ and } 1644 \text{ for } a_n.$$

$$= 75(1649) \qquad \text{Simplify.}$$

$$= 123{,}675 \qquad \text{Simplify.}$$

✓CHECKPOINT 8

Find the 100th partial sum of the arithmetic sequence

$$4, 10, 16, 22, 28, \ldots$$ ■

Applications

Example 9 Seating Capacity

An auditorium has 20 rows of seats. There are 20 seats in the first row, 22 seats in the second row, 24 seats in the third row, and so on (see Figure 7.2). How many seats are there in all 20 rows?

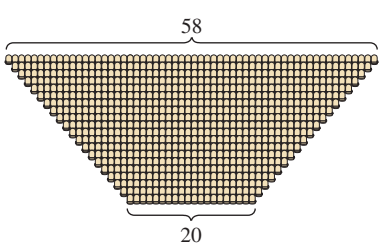


FIGURE 7.2

SOLUTION The numbers of seats in the rows form an arithmetic sequence in which the common difference is $d = 2$. Because $c = a_1 - d = 20 - 2 = 18$, you can determine that the formula for the nth term in the sequence is $a_n = 2n + 18$. So, the 20th term in the sequence is $a_{20} = 2(20) + 18 = 58$, and the total number of seats is

$$S_{20} = 20 + 22 + 24 + \cdots + 58$$

$$= \frac{n}{2}(a_1 + a_n) \qquad n\text{th partial sum formula}$$

$$= \frac{20}{2}(20 + 58) \qquad \text{Substitute 20 for } n, 20 \text{ for } a_1, \text{ and } 58 \text{ for } a_n.$$

$$= 10(78) \qquad \text{Simplify.}$$

$$= 780. \qquad \text{Simplify.}$$

✓CHECKPOINT 9

In Example 9, suppose the auditorium has 30 rows of seats. How many seats are there in all 30 rows? ■



AP/Wide World Photos

There were fewer than 1700 coffee shops in the United States in 1991. By 2006, the number had increased to more than 19,000. *(Source: Specialty Coffee Association of America)*

Example 10 **Total Sales** ®

A small coffee shop sells $0,000 worth of beverages during its first year. The owner of the business has set a goal of increasing annual sales by $500 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years this business is in operation. What kinds of economic factors could prevent the shop owner from meeting the sales goal?

SOLUTION The annual sales form an arithmetic sequence in which $a_1 = 10{,}000$ and $d = 7500$. So,

$$c = a_1 - d = 10{,}000 - 7500 = 2500$$

and the nth term of the sequence is

$$a_n = 7500n + 2500.$$

This implies that the 10th term of the sequence is

$$a_{10} = (7500)(10) + 2500 = 77{,}500.$$

So, the total sales for the first 10 years are as follows.

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad n\text{th partial sum formula}$$

$$= \frac{10}{2}(10{,}000 + 77{,}500) \qquad \text{Substitute 10 for } n\text{, 10,000 for } a_1\text{, and 77,500 for } a_n.$$

$$= 5(87{,}500) \qquad \text{Simplify.}$$

$$= \$437{,}500 \qquad \text{Simplify.}$$

Many economic factors, such as the location of the shop, demand by consumers, and marketing techniques, could prevent the shop owner from meeting the sales goal.

✓CHECKPOINT 10

A company sells $000 worth of printing paper during its first year. The sales manager has set a goal of increasing annual sales of printing paper by $500 each year for 9 years. Assuming that this goal is met, find the total sales of printing paper during the first 10 years this business is in operation. ■

CONCEPT CHECK

1. Explain how to determine whether a sequence of terms is an arithmetic sequence.
2. Each term of a sequence is twice the previous term. Explain why this is not an arithmetic sequence.
3. You know the third term and the common difference of an arithmetic sequence. Explain how to find the fifth term.
4. Determine whether the following statement is true or false. Explain your reasoning.

 To find the sum of a finite arithmetic sequence with n terms, you *must* add all n terms.

Skills Review 7.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.1, 0.2, 2.4, and 7.1.

In Exercises 1 and 2, find the sum.

1. $\sum_{i=1}^{6} (2i - 1)$

2. $\sum_{i=1}^{10} (4i + 2)$

In Exercises 3 and 4, find the distance between the two real numbers.

3. $\frac{5}{2}, 8$

4. $\frac{4}{3}, \frac{14}{3}$

In Exercises 5 and 6, evaluate the function as indicated.

5. Find $f(3)$ for $f(n) = 10 + (n - 1)4$.

6. Find $f(10)$ for $f(n) = 1 + (n - 1)\frac{1}{3}$.

In Exercises 7–10, evaluate the expression.

7. $\frac{11}{2}(1 + 25)$

8. $\frac{16}{2}(4 + 16)$

9. $\frac{20}{2}[2(5) + (12 - 1)3]$

10. $\frac{8}{2}[2(-3) + (15 - 1)5]$

Exercises 7.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, determine whether the sequence is arithmetic. If it is, find the common difference.

1. $3, 7, 11, 15, 19, \ldots$

2. $20, 17, 14, 11, 8, \ldots$

3. $-3, -2, 0, 3, 7, \ldots$

4. $-12, -8, -4, 0, 4, 8, \ldots$

5. $3, \frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}, \ldots$

6. $\frac{9}{4}, 2, \frac{7}{4}, \frac{3}{2}, \frac{5}{4}, 1, \ldots$

7. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \ldots$

8. $\frac{7}{5}, \frac{4}{5}, \frac{1}{5}, \frac{-2}{5}, \ldots$

9. $\log_{10} 10, \log_{10} 100, \log_{10} 1000, \log_{10} 10{,}000, \ldots$

10. $\ln 1, \ln 2, \ln 3, \ln 4, \ln 5, \ldots$

11. $5.3, 5.7, 6.1, 6.5, 6.9, \ldots$

12. $1.2, 2.4, 3.6, 4.8, 6.0, \ldots$

In Exercises 13–22, write the first five terms of the sequence. Determine whether the sequence is arithmetic. If it is, find the common difference.

13. $a_n = 5 + 3n$

14. $a_n = 100 - 3n$

15. $a_n = (2^n)n$

16. $a_n = 2^{n-1}$

17. $a_n = (2 + n) - (1 + n)$

18. $a_n = 1 + (n - 1)4$

19. $a_n = \frac{2}{n - 1}$

20. $a_n = \frac{2n + 1}{4n}$

21. $a_n = (-1)^n$

22. $a_n = (-1)^{2n+1}$

In Exercises 23–34, find a formula for a_n for the arithmetic sequence.

23. $a_1 = 4, d = 5$

24. $a_1 = 3, d = 8$

25. $a_1 = 50, d = -12$

26. $a_1 = 102, d = -6$

27. $a_1 = 7, a_5 = 27$

28. $a_1 = -5, a_7 = -17$

29. $a_3 = 8, a_9 = 32$

30. $a_6 = 29, a_{18} = 95$

31. $2, 8, 14, 20, 26, \ldots$

32. $3, -7, -17, -27, -37, \ldots$

33. $2, \frac{11}{4}, \frac{7}{2}, \frac{17}{4}, 5, \ldots$

34. $\frac{16}{3}, \frac{14}{3}, 4, \frac{10}{3}, \frac{8}{3}, \ldots$

In Exercises 35–42, write the first five terms of the arithmetic sequence.

35. $a_1 = 10, d = -4$

36. $a_1 = 2, d = \frac{1}{3}$

37. $a_1 = 5, a_{n+1} = a_n + 9$

38. $a_1 = 7, a_{n+1} = a_n - 2$

39. $a_1 = 2, a_{12} = 46$

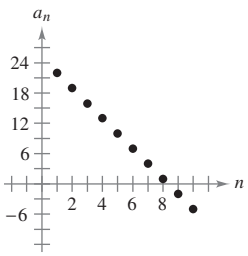
40. $a_5 = 28, a_{10} = 3$

41. $a_8 = 26, a_{12} = 10$

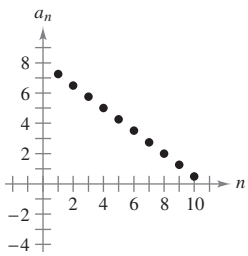
42. $a_4 = 16, a_{10} = 46$

In Exercises 43–46, match the arithmetic sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

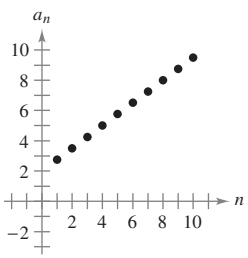
(a)



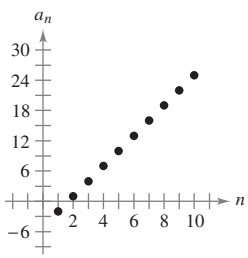
(b)



(c)



(d)



43. $a_n = 3n - 5$

44. $a_n = 25 - 3n$

45. $a_n = -\frac{3}{4}n + 8$

46. $a_n = 2 + \frac{3}{4}n$

Ⓣ In Exercises 47–50, use a graphing utility to graph the first 10 terms of the sequence.

47. $a_n = 15 - \frac{3}{2}n$

48. $a_n = -5 + 2n$

49. $a_n = 0.4n - 2$

50. $a_n = -1.3n + 7$

In Exercises 51–58, find the indicated nth partial sum of the arithmetic sequence.

51. $6, 12, 18, 24, \ldots, n = 8$

52. $4, 11, 18, 25, \ldots, n = 10$

53. $-10, -8, -6, -4, \ldots, n = 12$

54. $-2, -9, -16, -23, \ldots, n = 10$

55. $0.5, 0.9, 1.3, 1.7, \ldots, n = 10$

56. $1.50, 1.45, 1.40, 1.35, \ldots, n = 20$

57. $a_1 = 100,\ a_{25} = 220,\ n = 25$

58. $a_1 = 15,\ a_{100} = 307,\ n = 100$

59. Find the sum of the first 100 odd integers.

60. Find the sum of the integers from -20 to 80.

In Exercises 61–70, find the partial sum.

61. $\sum_{n=1}^{50} n$

62. $\sum_{n=1}^{100} 2n$

63. $\sum_{n=32}^{100} 5n$

64. $\sum_{n=51}^{100} 7n$

65. $\sum_{n=0}^{50} (1000 - 5n)$

66. $\sum_{n=1}^{250} (1000 - n)$

67. $\sum_{n=1}^{20} (3n + 2)$

68. $\sum_{n=1}^{500} (n + 6)$

69. $\sum_{n=1}^{100} \frac{n + 4}{2}$

70. $\sum_{n=0}^{100} \frac{8 - 3n}{16}$

In Exercises 71–74, use summation notation to write the sum.

71. $3 + 6 + 9 + 12 + 15 + \cdots + 300$

72. $5 + 7 + 9 + 11 + 13 + \cdots + 43$

73. $\frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2} + \cdots + \frac{11}{4}$

74. $4 + \frac{13}{3} + \frac{14}{3} + 5 + \frac{16}{3} + \cdots + 12$

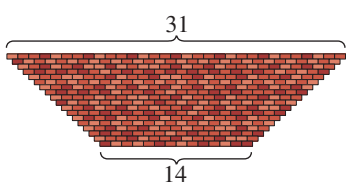
Job Offer In Exercises 75 and 76, consider a job offer with the given starting salary and the given annual raise. (a) Determine the salary during the sixth year of employment. (b) Determine the total compensation from the company through six full years of employment.

	Starting Salary	*Annual Raise*
75.	\$5,000	\$750
76.	\$9,400	\$125

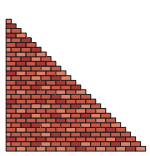
77. Seating Capacity Determine the seating capacity of an auditorium with 30 rows of seats if there are 15 seats in the first row, 20 seats in the second row, 25 seats in the third row, and so on.

78. Seating Capacity Determine the seating capacity of an auditorium with 36 rows of seats if there are 15 seats in the first row, 18 seats in the second row, 21 seats in the third row, and so on.

79. Brick Pattern A brick patio has the approximate shape of a trapezoid (see figure). The patio has 18 rows of bricks. The first row has 14 bricks and the 18th row has 31 bricks. How many bricks are in the patio?



80. Brick Pattern A triangular brick wall is made by cutting some bricks in half to use in the first column of every other row. The wall has 28 rows. The top row is one-half brick wide and the bottom row is 14 bricks wide. How many bricks are used in the finished wall?



81. Total Sales The annual sales a_n (in millions of dollars) for The Cheesecake Factory, Inc., from 2001 to 2006 can be approximated by the model

$a_n = 500.7 + 161.52n, \quad n = 0, 1, \ldots, 5$

where n represents the year, with $n = 0$ corresponding to 2001. *(Source: The Cheesecake Factory, Inc.)*

(a) Construct a bar graph showing the annual sales for The Cheesecake Factory, Inc., from 2001 to 2006.

(b) Find the total sales from 2001 to 2006.

82. Total Sales The annual sales a_n (in millions of dollars) for Starbucks Corporation from 2001 to 2006 can be modeled by

$a_n = 2328.5 + 1032.84n, \quad n = 0, 1, \ldots, 5$

where n represents the year, with $n = 0$ corresponding to 2001. *(Source: Starbucks Corporation)*

(a) Construct a bar graph showing the annual sales for Starbucks Corporation from 2001 to 2006.

(b) Find the total sales from 2001 to 2006.

83. Falling Object A heavy object (with negligible air resistance) is dropped from a plane. During the first second of fall, the object falls 16 feet; during the second second, it falls 48 feet; during the third second, it falls 80 feet; and during the fourth second, it falls 112 feet. Assume this pattern continues. How many feet will the object fall in 8 seconds?

84. Falling Object A heavy object (with negligible air resistance) is dropped from a plane. During the first second of fall, the object falls 4.9 meters; during the second second, it falls 14.7 meters; during the third second, it falls 24.5 meters; and during the fourth second, it falls 34.3 meters. Assume this pattern continues. How many meters will the object fall in 10 seconds?

85. Total Profit A small landscaping company makes a profit of $0,000 during its first year. The company president sets a goal of increasing profit by $0,000 each year for 4 years. Assuming that this goal is met, find the total profit during the first 5 years of this business. What kinds of economic factors could prevent the company from meeting the profit goal?

86. Total Sales An entrepreneur sells $0,000 worth of gumballs in the first year of operation. The entrepreneur has set a goal of increasing annual sales by $000 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years of this gumball business. What kinds of economic factors could prevent the entrepreneur from meeting the sales goal?

87. Reasoning Given an arithmetic sequence for which only the first few terms are known, is it possible to find the nth term? Explain your reasoning.

88. Reasoning If the only known information about a finite arithmetic sequence is its first term and its last term, is it possible to find the sum of the sequence? Explain your reasoning.

89. Total Revenue The annual revenues a_n (in billions of dollars) for McDonald's Corporation from 2000 to 2005 are shown in the table, where n represents the year, with $n = 0$ corresponding to 2000. *(Source: McDonald's Corporation)*

Year, n	Revenue, a_n
0	14.243
1	14.870
2	15.406
3	17.141
4	19.065
5	20.460

(a) Construct a bar graph showing the annual revenues from 2000 to 2005.

(b) Use the *linear regression* feature of a graphing utility to find an arithmetic sequence that approximates the annual revenues from 2000 to 2005.

(c) Use summation notation to represent the *total* revenue from 2000 to 2005. Find the total revenue.

90. Total Revenue The annual revenues a_n (in millions of dollars) for Outback Steakhouse, Inc. (OSI Restaurant Partners, Inc.) from 2001 to 2006 are shown in the table, where n represents the year, with $n = 0$ corresponding to 2001. *(Source: OSI Restaurant Partners, Inc.)*

Year, n	Revenue, a_n
0	2127.1
1	2362.1
2	2744.4
3	3201.8
4	3601.7
5	3941.0

(a) Construct a bar graph showing the annual revenues from 2001 to 2006.

(b) Use the *linear regression* feature of a graphing utility to find an arithmetic sequence that approximates the annual revenues from 2001 to 2006.

(c) Use summation notation to represent the *total* revenue from 2001 to 2006. Find the total revenue.

Section 7.3

Geometric Sequences and Series

- Recognize and find the nth term of a geometric sequence.
- Find the sum of a finite geometric sequence.
- Find the sum of an infinite geometric series.
- Use a geometric sequence to solve an application problem.

Geometric Sequences

In Section 7.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence.** Consecutive terms of a geometric sequence have a common *ratio,* as indicated in the following definition.

DISCOVERY

Determine the rule used to generate the terms of each sequence.

a. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

b. $5, 20, 80, 320, \ldots$

c. $\frac{1}{2}, \frac{3}{4}, \frac{9}{8}, \frac{27}{16}, \ldots$

Definition of Geometric Sequence

A sequence is **geometric** if the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n \ldots$ is geometric if there is a number r such that

$$\frac{a_2}{a_1} = r, \quad \frac{a_3}{a_2} = r, \quad \frac{a_4}{a_3} = r, \quad r \neq 0$$

and so on. The number r is the **common ratio** of the geometric sequence.

Example 1 Examples of Geometric Sequences

a. The sequence whose nth term is 2^n is geometric. For this sequence, the common ratio between consecutive terms is 2.

$$\underbrace{2, 4}_{\frac{4}{2} = 2}, 8, 16, \ldots, 2^n, \ldots$$ Begin with $n = 1$.

b. The sequence whose nth term is $4(3^n)$ is geometric. For this sequence, the common ratio between consecutive terms is 3.

$$\underbrace{12, 36}_{\frac{36}{12} = 3}, 108, 324, \ldots, 4(3^n), \ldots$$ Begin with $n = 1$.

c. The sequence whose nth term is $\left(-\frac{1}{3}\right)^n$ is geometric. For this sequence, the common ratio between consecutive terms is $-\frac{1}{3}$.

$$\underbrace{-\frac{1}{3}, \frac{1}{9}}_{\frac{1/9}{-1/3} = -\frac{1}{3}}, -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots$$ Begin with $n = 1$.

✓CHECKPOINT 1

Determine whether the sequence $2, 8, 32, 128, \ldots$ is geometric. If it is, find its common ratio and write a formula for a_n. ■

In Example 1, notice that each of the geometric sequences has an nth term that is of the form ar^n, where the common ratio of the sequence is r. A geometric sequence may be thought of as an exponential function whose domain is the set of natural numbers.

The nth Term of a Geometric Sequence

The nth term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$\begin{array}{cccccccc} a_1, & a_2, & a_3, & a_4, & a_5, & \ldots, & a_n, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ a_1, & a_1 r, & a_1 r^2, & a_1 r^3, & a_1 r^4, & \ldots, & a_1 r^{n-1}, & \ldots \end{array}$$

If you know the nth term of a geometric sequence, you can find the $(n + 1)$th term by multiplying by r. That is, $a_{n+1} = ra_n$.

TECHNOLOGY

You can generate the geometric sequence in Example 2 with a graphing utility using the following steps.

3 ENTER

2 × 2nd [ANS]

Now press ENTER repeatedly to generate the terms of the sequence.

Example 2 Finding the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$.

SOLUTION Starting with 3, repeatedly multiply by 2 to obtain the following.

$a_1 = 3$ 1st term

$a_2 = 3(2^1) = 6$ 2nd term

$a_3 = 3(2^2) = 12$ 3rd term

$a_4 = 3(2^3) = 24$ 4th term

$a_5 = 3(2^4) = 48$ 5th term

✓CHECKPOINT 2

Write the first five terms of the geometric sequence whose first term is $a_1 = 2$ and whose common ratio is $r = 3$. ■

Example 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

SOLUTION

$a_n = a_1 r^{n-1}$ Formula for the nth term of a geometric sequence

$a_{15} = 20(1.05)^{15-1}$ Substitute 20 for a_1, 1.05 for r, and 15 for n.

≈ 39.599 Use a calculator.

✓CHECKPOINT 3

Find the 12th term of the geometric sequence whose first term is 14 and whose common ratio is 1.2. ■

Example 4 Finding a Term of a Geometric Sequence

Find the 12th term of the geometric sequence

$$5, 15, 45, \ldots$$

SOLUTION

The common ratio of this sequence is $r = \frac{15}{5} = 3$. Because the first term is $a_1 = 5$, you can determine the 12th term $(n = 12)$ to be

$$\begin{aligned} a_n &= a_1 r^{n-1} && \text{Formula for the } n\text{th term of a geometric sequence} \\ a_{12} &= 5(3)^{12-1} && \text{Substitute 5 for } a_1\text{, 3 for } r\text{, and 12 for } n. \\ &= 5(177{,}147) && \text{Use a calculator.} \\ &= 885{,}735. && \text{Simplify.} \end{aligned}$$

✓CHECKPOINT 4

Find the 10th term of the geometric sequence

$$4, 20, 100, \ldots$$ ■

If you know any two terms of a geometric sequence, you can use that information to find a formula for the nth term of the sequence.

Example 5 Finding a Term of a Geometric Sequence

The fourth term of a geometric sequence is 125, and the 10th term is $\frac{125}{64}$. Find the 14th term. (Assume that the terms of the sequence are positive.)

SOLUTION Because r is the common ratio of *consecutive* terms of a sequence, you have to find the relationship between the 10th term and the fourth term.

$$\begin{aligned} a_{10} &= a_1 r^9 \\ &= a_1 \cdot r \cdot r \cdot r \cdot r^6 \\ &= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6 \\ a_{10} &= a_4 r^6. \end{aligned}$$

Because $a_{10} = 125/64$ and $a_4 = 125$, you can solve for r as follows.

$$\begin{aligned} \frac{125}{64} &= 125r^6 \\ \frac{1}{64} &= r^6 \\ \pm\frac{1}{2} &= r \end{aligned}$$

Because the terms of the sequence are assumed to be positive, choose the positive solution, $r = \frac{1}{2}$. You can obtain the 14th term by multiplying the 10th term by r^4.

$$a_{14} = a_{10} r^4 = \frac{125}{64}\left(\frac{1}{2}\right)^4 = \frac{125}{1024}$$

✓CHECKPOINT 5

The second term of a geometric sequence is 6, and the fifth term is $\frac{81}{4}$. Find the eighth term. ■

The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

The Sum of a Finite Geometric Sequence

The sum of the finite geometric sequence

$$a_1, a_1r, a_1r^2, a_1r^3, a_1r^4, \ldots, a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by

$$S_n = \sum_{i=1}^{n} a_1r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

DISCOVERY

To *develop* the formula for the sum of a finite geometric sequence, consider the following two equations.

$$S = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}$$

$$rS = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + a_1r^n$$

Can you discover a way to simplify the difference between these two equations to obtain the formula for the sum of a finite geometric sequence?

Example 6 **Finding the Sum of a Finite Geometric Sequence**

Find the sum $\sum_{n=1}^{12} 4(0.3)^{n-1}$.

SOLUTION By writing out a few terms, you have

$$\sum_{n=1}^{12} 4(0.3)^{n-1} = 4(0.3)^0 + 4(0.3)^1 + 4(0.3)^2 + 4(0.3)^3 + \cdots + 4(0.3)^{11}.$$

Now, because $a_1 = 4$, $r = 0.3$, and $n = 12$, you can apply the formula for the sum of a finite geometric sequence to obtain

$$\sum_{n=1}^{12} 4(0.3)^{n-1} = a_1\left(\frac{1 - r^n}{1 - r}\right) = 4\left[\frac{1 - (0.3)^{12}}{1 - 0.3}\right] \approx 5.714.$$

✓CHECKPOINT 6

Find the sum $\sum_{n=1}^{10} 2(3^{n-1})$. ■

When using the formula for the sum of a finite geometric sequence, be careful to check that the sum is of the form

$$\sum_{i=1}^{n} a_1r^{i-1}.$$

If the sum is not of this form, you must adjust the formula for the nth partial sum. For instance, if the sum in Example 6 were $\sum_{n=1}^{12} 4(0.3)^n$, then you would evaluate the sum as follows.

$$\sum_{n=1}^{12} 4(0.3)^n = 4(0.3) + 4(0.3)^2 + 4(0.3)^3 + \cdots + 4(0.3)^{12}$$

$$= 4(0.3) + [4(0.3)](0.3) + [4(0.3)](0.3)^2 + \cdots + [4(0.3)](0.3)^{11}.$$

So, $a_1 = 4(0.3)$, $r = 0.3$, and $n = 12$, and the sum is

$$S_{12} = 4(0.3)\left(\frac{1 - (0.3)^{12}}{1 - 0.3}\right) \approx 1.714.$$

DISCOVERY

Use a calculator to evaluate r^n for $r = \frac{1}{2}$ and $n = 1, 2, 5, 10, 50$, and 100. What happens to the value of r^n as n increases? Make a conjecture about the value of r^n as $n \to \infty$. Use a graphing utility to graph $y = \left(\frac{1}{2}\right)^x$ to verify your conjecture.

Geometric Series

The summation of the terms of an infinite geometric *sequence* is called an **infinite geometric series,** or simply a **geometric series.**

The formula for the sum of a *finite* geometric *sequence* can, depending on the value of r, be extended to produce a formula for the sum of an *infinite* geometric series. Specifically, if the common ratio r has the property that $|r| < 1$, it can be shown that r^n becomes arbitrarily close to zero as n increases without bound. Consequently,

$$a_1\left(\frac{1 - r^n}{1 - r}\right) \to a_1\left(\frac{1 - 0}{1 - r}\right) \quad \text{as} \quad n \to \infty.$$

This result is summarized as follows.

The Sum of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + \cdots$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1 - r}.$$

Example 7 **Finding the Sum of an Infinite Geometric Series**

Find each sum.

a. $\sum_{n=0}^{\infty} 4(0.6)^n$ **b.** $\sum_{n=1}^{\infty} 3(0.1)^{n-1}$

SOLUTION

a. $\sum_{n=0}^{\infty} 4(0.6)^n = 4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots + 4(0.6)^n + \cdots$

$$= \frac{4}{1 - 0.6} \qquad \frac{a_1}{1 - r}$$

$$= 10$$

b. $\sum_{n=1}^{\infty} 3(0.1)^{n-1} = 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots + 3(0.1)^{n-1} + \cdots$

$$= \frac{3}{1 - 0.1} \qquad \frac{a_1}{1 - r}$$

$$= \frac{10}{3}$$

$$\approx 3.33$$

✓CHECKPOINT 7

Find the sum $\sum_{n=1}^{\infty} 5(0.4)^{n-1}$. ■



Tim Boyle/Getty Images

Digital televisions will eventually replace analog TVs because they deliver better sound and picture quality to viewers, and they are a more efficient way for broadcasters to communicate information.

Application

Example 8 Compound Interest

You make a deposit of $0 on the first day of each month in a savings account that pays 6% compounded monthly. Will there be enough money in the account in 2 years to buy a $500 digital TV?

SOLUTION

The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1 + \frac{0.06}{12}\right)^{24} = 50(1.005)^{24}.$$

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1 + \frac{0.06}{12}\right)^{23} = 50(1.005)^{23}.$$

The last deposit will gain interest for 1 month, and its balance will be

$$A_1 = 50\left(1 + \frac{0.06}{12}\right)^{1} = 50(1.005).$$

The total balance in the account will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.005)$, $r = 1.005$, and $n = 24$, you have

$$S_{24} = 50(1.005)\left[\frac{1 - (1.005)^{24}}{1 - 1.005}\right]$$

$$= \$277.96.$$

You will not have enough money in your account in 2 years to pay for the TV.

✓ CHECKPOINT 8

You make a deposit of $0 on the first day of each month in a savings account that pays 3% compounded monthly. Will there be enough money in the account in 2 years to buy a $725 digital TV? ■

CONCEPT CHECK

1. Explain how to determine whether a sequence of terms is a geometric sequence.
2. You know the fourth term and the common ratio of a geometric sequence. Explain how to find the sixth term.
3. You know the common ratio of a finite geometric sequence. What else do you need to know to apply the formula for finding its sum, S_n?
4. The first term of an infinite geometric series is 4. The second term is 5. Can you find the sum of this infinite geometric series? Explain.

Skills Review 7.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 7.1.

In Exercises 1–4, evaluate the expression.

1. $\left(\frac{4}{5}\right)^3$ **2.** $\left(\frac{3}{4}\right)^2$ **3.** 2^{-4} **4.** $4(3^4)$

In Exercises 5–8, simplify the expression.

5. $(2n)(3n^2)$ **6.** $n(3n)^3$ **7.** $\frac{4n^5}{n^2}$ **8.** $\frac{(2n)^3}{8n}$

In Exercises 9 and 10, use summation notation to write the sum.

9. $2 + 2(3) + 2(3^2) + 2(3^3)$ **10.** $3 + 3(2) + 3(2^2) + 3(2^3)$

Exercises 7.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the common ratio of the geometric sequence.

1. $3, 15, 75, 375, 1875, \ldots$

2. $-2, 8, -32, 128, -512, \ldots$

3. $216, 36, 6, 1, \frac{1}{6}, \ldots$

4. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \ldots$

In Exercises 5–14, determine whether the sequence is geometric. If it is, find its common ratio and write a formula for a_n.

5. $1, 4, 16, 64, \ldots$ **6.** $2, 10, 50, 250, \ldots$

7. $4, -12, 36, -108, \ldots$ **8.** $1, -2, 4, -8, \ldots$

9. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$ **10.** $\frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9}, \ldots$

11. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **12.** $-8, \frac{2}{3}, 4, -2, \ldots$

13. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ **14.** $\frac{1}{5}, \frac{2}{3}, \frac{9}{3}, \frac{4}{11}, \ldots$

In Exercises 15–22, write the first five terms of the geometric sequence.

15. $a_1 = 4, r = 2$ **16.** $a_1 = 6, r = 3$

17. $a_1 = 1, r = \frac{1}{3}$ **18.** $a_1 = 1, r = \frac{2}{3}$

19. $a_1 = \frac{1}{2}, r = -\frac{1}{3}$ **20.** $a_1 = \frac{5}{7}, r = -\frac{1}{5}$

21. $a_1 = 1, r = e$ **22.** $a_1 = 2, r = e^{0.1}$

In Exercises 23–34, find the indicated term of the geometric sequence.

23. $a_1 = 16, r = \frac{1}{4}$, 5th term

24. $a_1 = 9, r = \frac{2}{3}$, 7th term

25. $a_1 = 6, r = -\frac{1}{3}$, 12th term

26. $a_1 = 1, r = -\frac{2}{3}$, 10th term

27. $a_1 = 100, r = e$, 9th term

28. $a_1 = 6, r = e^{0.1}$, 9th term

29. $a_1 = 500, r = 1.02$, 40th term

30. $a_1 = 1000, r = 1.005$, 60th term

31. $a_2 = -40, a_4 = -10$, 3rd term

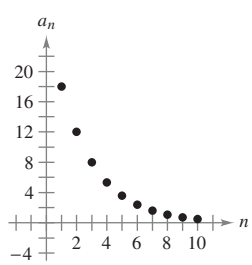
32. $a_2 = 3, a_5 = \frac{3}{64}$, 3rd term

33. $a_2 = -18, a_5 = \frac{2}{3}$, 6th term

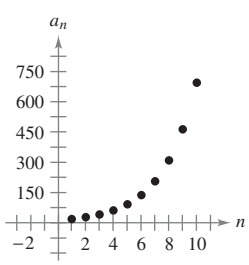
34. $a_3 = \frac{16}{3}, a_5 = \frac{64}{27}$, 7th term

In Exercises 35–38, match the geometric sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

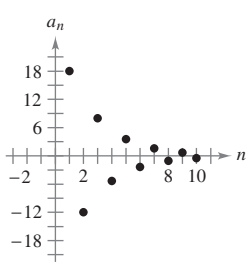
(a)



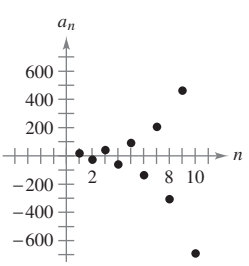
(b)



(c)



(d)



35. $a_n = 18\left(\frac{2}{3}\right)^{n-1}$ **36.** $a_n = 18\left(-\frac{2}{3}\right)^{n-1}$

37. $a_n = 18\left(\frac{3}{2}\right)^{n-1}$ **38.** $a_n = 18\left(-\frac{3}{2}\right)^{n-1}$

(T) In Exercises 39–44, use a graphing utility to graph the first 10 terms of the sequence.

39. $a_n = 12(-0.75)^{n-1}$

40. $a_n = 10(-1.5)^{n-1}$

41. $a_n = 12(-0.4)^{n-1}$

42. $a_n = 20(-1.25)^{n-1}$

43. $a_n = 2(1.3)^{n-1}$

44. $a_n = 10(1.2)^{n-1}$

In Exercises 45–58, find the sum of the finite geometric sequence.

45. $\sum_{n=1}^{10} 8(2^n)$

46. $\sum_{n=1}^{6} 3(4^n)$

47. $\sum_{n=0}^{8} 2^n$

48. $\sum_{n=0}^{8} (-2)^n$

49. $\sum_{i=1}^{10} 8\left(\frac{1}{4}\right)^{i-1}$

50. $\sum_{i=1}^{10} 5\left(\frac{1}{3}\right)^{i-1}$

51. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1}$

52. $\sum_{i=1}^{6} 30\left(\frac{1}{5}\right)^{i-1}$

53. $\sum_{n=0}^{40} 2\left(-\frac{1}{4}\right)^n$

54. $\sum_{n=0}^{50} 10\left(\frac{2}{3}\right)^{n-1}$

55. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^n$

56. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^n$

57. $\sum_{i=1}^{40} 4\left(-\frac{5}{2}\right)^{i-1}$

58. $\sum_{i=1}^{100} 8\left(\frac{7}{4}\right)^{i-1}$

In Exercises 59–64, use summation notation to write the sum.

59. $5 + 15 + 45 + \cdots + 3645$

60. $7 + 14 + 28 + \cdots + 896$

61. $2 - \frac{1}{2} + \frac{1}{8} - \cdots + \frac{1}{2048}$

62. $15 - 3 + \frac{3}{5} - \cdots - \frac{3}{625}$

63. $0.1 + 0.4 + 1.6 + \cdots + 102.4$

64. $32 + 24 + 18 + \cdots + 10.125$

In Exercises 65–72, find the sum of the infinite geometric series, if it exists.

65. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

66. $\sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^n = 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$

67. $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

68. $\sum_{n=1}^{\infty} 2\left(-\frac{2}{3}\right)^{n-1} = 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \cdots$

69. $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$

70. $\sum_{n=0}^{\infty} 4\left(\frac{1}{4}\right)^n = 4 + 1 + \frac{1}{4} + \frac{1}{16} + \cdots$

71. $\sum_{n=0}^{\infty} 3\left(\frac{1}{10}\right)^n = 3 + 0.3 + 0.03 + 0.003 + \cdots$

72. $\sum_{n=0}^{\infty} 5\left(\frac{4}{3}\right)^n = 5 + \frac{20}{3} + \frac{80}{9} + \frac{320}{27} + \cdots$

In Exercises 73–76, find the rational number representation of the repeating decimal.

73. $0.\overline{36}$

74. $0.\overline{297}$

75. $0.3\overline{18}$

76. $1.3\overline{8}$

77. Compound Interest You are saving for a motorcycle. You deposit $300 at the beginning of each month for 4 years in an account that pays 6% compounded monthly. The balance A in the account at the end of 4 years is

$$A = 300\left(1 + \frac{0.06}{12}\right)^1 + \cdots + 300\left(1 + \frac{0.06}{12}\right)^{48}.$$

(a) Is there enough money in the account after 4 years to buy a $7,000 motorcycle?

(b) Suppose the interest rate is 9%. Is there enough money in the account after 4 years to buy a $7,000 motorcycle?

78. Compound Interest You are saving to pay for an addition to your house. You deposit $200 at the beginning of each month for 3 years in an account that pays 5% compounded monthly. The balance A in the account at the end of 3 years is

$$A = 200\left(1 + \frac{0.05}{12}\right)^1 + \cdots + 200\left(1 + \frac{0.05}{12}\right)^{36}.$$

(a) Is there enough money in the account after 3 years to pay for the construction of an $8000 addition to your house?

(b) Suppose the interest rate is 7%. Is there enough money after 3 years to pay for the construction of an $8000 addition to your house?

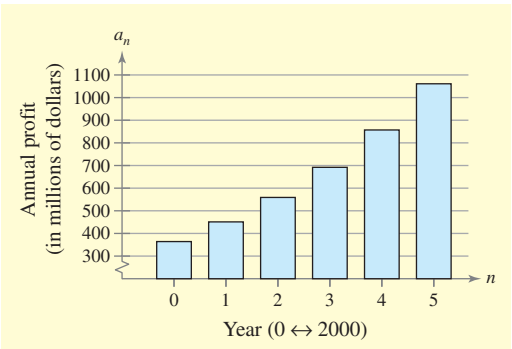
(T) **79. Compound Interest** You deposit $100 in an account at the beginning of each month for 10 years. The account pays 8% compounded monthly. Use a graphing utility to find your balance at the end of 10 years. If the interest were compounded continuously, what would be the balance?

(T) **80. Compound Interest** You deposit $50 in an account at the beginning of each month for 20 years. The account pays 6% compounded monthly. Use a graphing utility to find your balance at the end of 20 years. If the interest were compounded continuously, what would be the balance?

81. Profit The annual profits a_n (in millions of dollars) for Harley-Davidson, Inc. from 2000 to 2005 can be approximated by the model

$a_n = 364.10e^{0.214n}, \quad n = 0, 1, \ldots, 5$

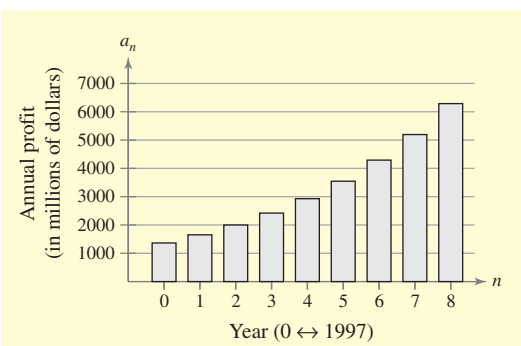
where n represents the year, with $n = 0$ corresponding to 2000 (see figure). Use the formula for the sum of a finite geometric sequence to approximate the total profit earned during this six-year period. *(Source: Harley-Davidson, Inc.)*



82. Profit The annual profits a_n (in millions of dollars) for The Home Depot, Inc. from 1997 to 2005 can be approximated by the model

$a_n = 1364.39e^{0.191n}, \quad n = 0, 1, 2, 3, \ldots, 8$

where n represents the year, with $n = 0$ corresponding to 1997 (see figure). Use the formula for the sum of a finite geometric sequence to approximate the total profit earned during this nine-year period. *(Source: The Home Depot, Inc.)*



83. Orthopedic Implants The annual sales a_n (in millions of dollars) for Stryker Corporation from 2000 to 2006 can be approximated by the model

$a_n = 2277.53e^{0.149n}, \quad n = 0, 1, 2, 3, \ldots, 6$

where n represents the year, with $n = 0$ corresponding to 2000. *(Source: Stryker Corporation)*

(a) Construct a bar graph showing the annual sales from 2000 to 2006.

(b) Use the formula for the sum of a finite geometric sequence to approximate the total sales during this seven-year period.

84. Cardiovascular Devices The annual sales a_n (in millions of dollars) for St. Jude Medical Inc. from 1999 to 2006 can be approximated by the model

$a_n = 1018.63e^{0.166n}, \quad n = 0, 1, 2, 3, \ldots, 7$

where n represents the year, with $n = 0$ corresponding to 1999. *(Source: St. Jude Medical Inc.)*

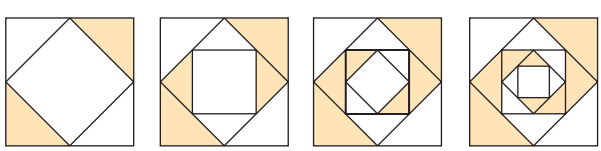
(a) Construct a bar graph showing the annual sales from 1999 to 2006.

(b) Use the formula for the sum of a finite geometric sequence to approximate the total sales during this eight-year period.

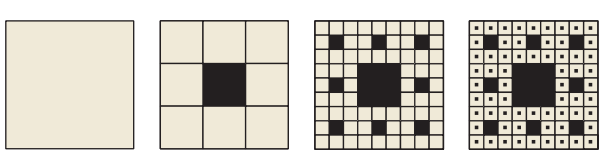
85. *MAKE A DECISION* You decide to work at a company that pays ₿0.01 for the first day, ₿0.02 for the second day, ₿0.04 for the third day, and so on. If the daily wage keeps doubling, what will be your total income for working 29 days? 30 days? 31 days? Which job pays more the first year: the job that pays ₿0.01 the first day and twice the previous day's wage for every day thereafter, or a job that pays $0,000 per year?

86. *MAKE A DECISION* You decide to work at a company that pays ₿0.01 for the first day, ₿0.03 for the second day, ₿0.09 for the third day, and so on. If the daily wage keeps tripling, what will be your total income for working 29 days? 30 days? 31 days? Which job pays more the first year: the job that pays ₿0.01 the first day and triple the previous day's wage for every day thereafter, or a job that pays ₿0,000 per year?

87. Geometry The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the resulting triangles are shaded (see figure). If this process is repeated five more times, what will be the total area of the shaded region?



88. Geometry The sides of a square are 27 inches in length. New squares are formed by dividing the original square into nine squares. The center square is then shaded (see figure). Each of the eight unshaded squares is then divided into nine smaller squares and the center square of each is shaded. If this process is repeated four more times, what will be the total area of the shaded region?



Section 7.4

The Binomial Theorem

- Find a binomial coefficient.
- Use Pascal's Triangle to determine a binomial coefficient.
- Expand a binomial using binomial coefficients and find a specific term in a binomial expansion.

Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that gives a quick method of raising a binomial to a power. To begin, look at the expansion of $(x + y)^n$ for several values of n.

$$(x + y)^0 = 1$$
$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.
2. In each expansion, x and y have symmetrical roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.
3. The degree of each term is n. For instance, in the expansion of $(x + y)^5$, the degree of each term is 5.

$$(x + y)^5 = x^5 + 5x^{\overbrace{4}^{4+1=5}}y^1 + 10x^{\overbrace{3}^{3+2=5}}y^2 + 10x^2y^3 + 5xy^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.** To find them, you can use the **Binomial Theorem.**

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$, the $(r + 1)$th term, is given by

$${}_nC_r = \frac{n!}{(n - r)!r!}.$$

The notation $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote a binomial coefficient.

Example 1 Finding Binomial Coefficients

Find each binomial coefficient.

a. ${}_8C_2$ **b.** ${}_{10}C_3$ **c.** ${}_7C_0$ **d.** ${}_8C_8$

SOLUTION

a. ${}_8C_2 = \dfrac{8!}{6! \cdot 2!} = \dfrac{(8 \cdot 7) \cdot 6!}{6! \cdot 2!} = \dfrac{8 \cdot 7}{2 \cdot 1} = 28$ Coefficient of third term in expansion of $(x + y)^8$.

b. ${}_{10}C_3 = \dfrac{10!}{7! \cdot 3!} = \dfrac{(10 \cdot 9 \cdot 8) \cdot 7!}{7! \cdot 3!} = \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$ Coefficient of fourth term in expansion of $(x + y)^{10}$.

c. ${}_7C_0 = \dfrac{7!}{7! \cdot 0!} = 1$ Coefficient of first term in expansion of $(x + y)^7$.

d. ${}_8C_8 = \dfrac{8!}{0! \cdot 8!} = 1$ Coefficient of ninth term in expansion of $(x + y)^8$.

✓CHECKPOINT 1

Find each binomial coefficient.

a. ${}_5C_2$ **b.** ${}_6C_6$ ■

DISCOVERY

Find each pair of binomial coefficients.

a. ${}_7C_0$, ${}_7C_7$

b. ${}_8C_0$, ${}_8C_8$

c. ${}_7C_1$, ${}_7C_6$

d. ${}_8C_1$, ${}_8C_7$

What do you observe about the pairs in (a) and (b)? What do you observe about the pairs in (c) and (d)? Write two conjectures from your observations. Develop a convincing argument for your two conjectures.

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) of Example 1, there is a simple pattern for evaluating binomial coefficients.

$${}_8C_2 = \underbrace{\frac{\overbrace{8 \cdot 7}^{\text{2 factors}}}{2 \cdot 1}}_{\text{2 factorial}} \quad \text{and} \quad {}_{10}C_3 = \underbrace{\frac{\overbrace{10 \cdot 9 \cdot 8}^{\text{3 factors}}}{3 \cdot 2 \cdot 1}}_{\text{3 factorial}}$$

Example 2 Finding Binomial Coefficients

Find each binomial coefficient using the pattern shown above.

a. ${}_7C_3$ **b.** $\binom{7}{4}$ **c.** ${}_{12}C_1$ **d.** $\binom{12}{11}$

SOLUTION

a. ${}_7C_3 = \dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ Coefficient of fourth term in expansion of $(x + y)^7$.

b. $\binom{7}{4} = \dfrac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35$ Coefficient of fifth term in expansion of $(x + y)^7$.

c. ${}_{12}C_1 = \dfrac{12}{1} = 12$ Coefficient of second term in expansion of $(x + y)^{12}$.

d. $\binom{12}{11} = \dfrac{12!}{1! \cdot 11!} = \dfrac{(12) \cdot 11!}{1! \cdot 11!} = \dfrac{12}{1} = 12$ Coefficient of twelfth term in expansion of $(x + y)^{12}$.

✓CHECKPOINT 2

Find each binomial coefficient using the pattern shown above Example 2.

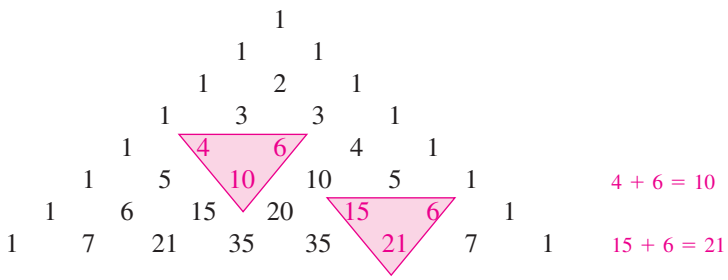
a. ${}_9C_2$ **b.** $\binom{6}{4}$ ■

It is not a coincidence that the results in parts (a) and (b) of Example 2 are the same and that the results in parts (c) and (d) are the same. In general, it is true that

$${}_nC_r = {}_nC_{n-r}.$$

Pascal's Triangle

There is a convenient way to remember a pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle.** This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).



The first and last numbers in each row of Pascal's Triangle are 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that the numbers in this triangle are precisely the same numbers that are the coefficients of binomial expansions, as follows.

$$(x + y)^0 = 1x \qquad \text{0th row}$$
$$(x + y)^1 = 1x + 1y \qquad \text{1st row}$$
$$(x + y)^2 = 1x^2 + 2xy + 1y^2 \qquad \text{2nd row}$$
$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \qquad \text{3rd row}$$
$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \qquad \vdots$$
$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$
$$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$
$$(x + y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

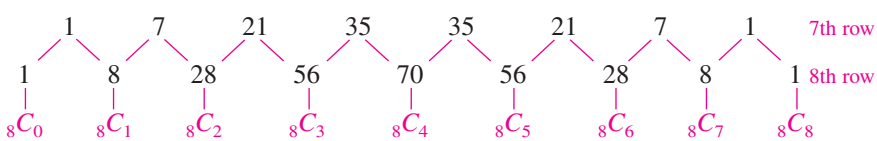
The top row in Pascal's Triangle is called the *zero row* because it corresponds to the binomial expansion $(x + y)^0 = 1$. Similarly, the next row is called the *first row* because it corresponds to the binomial expansion $(x + y)^1 = 1x + 1y$. In general, the *nth row* in Pascal's Triangle gives the coefficients of $(x + y)^n$.

Example 3 Using Pascal's Triangle

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

$${}_8C_0,\ {}_8C_1,\ {}_8C_2,\ {}_8C_3,\ {}_8C_4,\ {}_8C_5,\ {}_8C_6,\ {}_8C_7,\ {}_8C_8$$

SOLUTION



✓CHECKPOINT 3

Use the eighth row of Pascal's Triangle to find the binomial coefficients.

$${}_9C_3,\ {}_9C_5,\ {}_9C_7$$ ■

Binomial Expansions

As mentioned at the beginning of this section, when you write out the coefficients for a binomial that is raised to a power, you are **expanding a binomial.** The formulas for binomial coefficients give you an easy way to expand binomials, as demonstrated in the next three examples.

Example 4 Expanding a Binomial

Write the expansion for the expression

$(x + 1)^3$.

SOLUTION The binomial coefficients from the third row of Pascal's Triangle are

1, 3, 3, 1.

So, the expansion is as follows.

$$\begin{aligned}(x + 1)^3 &= (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)\\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

✓CHECKPOINT 4

Write the expansion for the expression

$(x + 2)^5$. ■

To expand binomials representing *differences,* rather than sums, you alternate signs. Here are two examples.

$$\begin{aligned}(x - 1)^3 &= [x + (-1)]^3\\ &= x^3 - 3x^2 + 3x - 1\\ (x - 1)^4 &= [x + (-1)]^4\\ &= x^4 - 4x^3 + 6x^2 - 4x + 1\end{aligned}$$

Example 5 Expanding a Binomial

Write the expansion for each expression.

a. $(2x - 3)^4$ **b.** $(x - 2y)^4$

SOLUTION The binomial coefficients from the fourth row of Pascal's Triangle are

1, 4, 6, 4, 1.

So, the expansions are as follows.

a. $$\begin{aligned}(2x - 3)^4 &= (1)(2x)^4 - (4)(2x)^3(3) + (6)(2x)^2(3^2) - (4)(2x)(3^3) + (1)(3^4)\\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$$

b. $$\begin{aligned}(x - 2y)^4 &= (1)x^4 - (4)x^3(2y) + (6)x^2(2y)^2 - (4)x(2y)^3 + (1)(2y)^4\\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4\end{aligned}$$

✓CHECKPOINT 5

Write the expansion for the expression

$(3x + 1)^5$. ■

Example 6 Finding a Term in a Binomial Expansion

a. Find the sixth term of $(a + 2b)^8$.

b. Find the coefficient of the term a^6b^5 in the expansion of $(3a - 2b)^{11}$.

SOLUTION

a. From the Binomial Theorem, you can see that the $(r + 1)$th term is ${}_nC_r x^{n-r}y^r$. In this case, $6 = r + 1$ and therefore $r = 5$. Because $n = 8$, $x = a$, and $y = 2b$, the sixth term in the binomial expansion is

$$\begin{aligned} {}_8C_5a^{8-5}(2b)^5 &= 56 \cdot a^3 \cdot (2b)^5 \\ &= 56(2^5)a^3b^5 \\ &= 1792a^3b^5. \end{aligned}$$

b. In this case, $n = 11$, $r = 5$, $x = 3a$, and $y = -2b$. Substitute these values to obtain

$$\begin{aligned} {}_nC_r\, x^{n-r}y^r &= {}_{11}C_5(3a)^6(-2b)^5 \\ &= (462)(729a^6)(-32b^5) \\ &= -10{,}777{,}536a^6b^5. \end{aligned}$$

So, the coefficient is $-10{,}777{,}536$.

✓CHECKPOINT 6

Find the specified nth term in the expansion of the binomial.

a. $(2a + b)^{10}$, $n = 6$

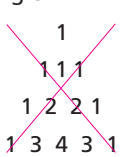
b. $(3a + 5b)^6$, $n = 4$ ■

CONCEPT CHECK

1. Can ${}_nC_r\, x^{n-3}y^4$ be a term of the binomial expansion of $(x + y)^n$? Explain.
2. The coefficient of a term in a binomial expansion is given by ${}_7C_3$. To what power is the binomial raised? Explain.
3. When finding the fourth term of a binomial expansion by evaluating

 ${}_nC_r\, x^{n-r}y^r$,

 what value should you substitute for r? Explain.
4. Describe and correct the error in setting up the first four rows of Pascal's Triangle.



Skills Review 7.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.5 and 7.1.

In Exercises 1–6, perform the indicated operations and simplify.

1. $5x^2(x^3 + 3)$ **2.** $(x + 5)(x^2 - 3)$ **3.** $(x + 4)^2$

4. $(2x - 3)^2$ **5.** $x^2y(3xy^{-2})$ **6.** $(-2z)^5$

In Exercises 7–10, evaluate the expression.

7. $5!$ **8.** $\frac{8!}{5!}$ **9.** $\frac{10!}{7!}$ **10.** $\frac{6!}{3!3!}$

Exercises 7.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, find the binomial coefficient.

1. ${}_6C_2$ **2.** ${}_9C_4$

3. ${}_{10}C_4$ **4.** ${}_{10}C_6$

5. ${}_{12}C_5$ **6.** ${}_{12}C_{12}$

7. ${}_{20}C_{15}$ **8.** ${}_{20}C_5$

9. $\binom{7}{7}$ **10.** $\binom{12}{0}$

11. $\binom{100}{98}$ **12.** $\binom{100}{2}$

In Exercises 13–18, evaluate using Pascal's Triangle.

13. ${}_4C_2$ **14.** ${}_4C_3$

15. ${}_7C_4$ **16.** ${}_6C_3$

17. $\binom{8}{5}$ **18.** $\binom{9}{8}$

In Exercises 19–38, use the Binomial Theorem to expand the expression. Simplify your answer.

19. $(x + 1)^4$ **20.** $(x + 1)^6$

21. $(x + 2)^3$ **22.** $(x + 2)^4$

23. $(x - 2)^4$ **24.** $(x - 2)^5$

25. $(x + y)^5$ **26.** $(x + y)^6$

27. $(3s + 2)^4$ **28.** $(2s + 3)^5$

29. $(3 - 2s)^6$ **30.** $(2 - 3s)^4$

31. $(x - y)^5$ **32.** $(2x - y)^5$

33. $(1 - 2x)^3$ **34.** $(1 - 3x)^4$

35. $(x^2 - y^2)^5$ **36.** $(x^2 + y^2)^6$

37. $\left(\frac{1}{x} + y\right)^5$ **38.** $\left(x - \frac{3}{y}\right)^3$

39. How are the expansions of the binomials $(x + y)^n$ and $(x - y)^n$ alike? How are they different?

40. How many terms are in the expanded form of $(x + y)^5$? $(x + y)^6$? $(x + y)^7$? $(x + y)^8$? How many terms are in $(x + y)^n$?

In Exercises 41–46, use the Binomial Theorem to expand the complex number. Simplify your answer by using the fact that $i^2 = -1$.

41. $(1 + i)^4$ **42.** $(2 - i)^5$

43. $\left(3 - \sqrt{-4}\right)^6$ **44.** $\left(5 + \sqrt{-16}\right)^3$

45. $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)^3$ **46.** $\left(\frac{3}{4} - \frac{\sqrt{2}}{4}i\right)^4$

In Exercises 47–50, expand the expression by using Pascal's Triangle to determine the coefficients.

47. $(2t - 1)^5$ **48.** $(x + 4)^5$

49. $(3 + 2z)^4$ **50.** $(3y + 2)^5$

In Exercises 51–58, find the specified nth term in the expansion of the binomial.

51. $(x + y)^{10}$, $n = 4$ **52.** $(x - y)^6$, $n = 7$

53. $(x - 6y)^5$, $n = 3$ **54.** $(x - 10z)^7$, $n = 4$

55. $(4x + 3y)^9$, $n = 8$ **56.** $(5a + 6b)^5$, $n = 5$

57. $(10x - 3y)^{12}$, $n = 9$ **58.** $(7x + 2y)^{15}$, $n = 7$

In Exercises 59–66, find the coefficient a of the term in the expansion of the binomial.

Binomial	*Term*
59. $(x + 2)^{10}$	ax^5
60. $(x + 3)^{12}$	ax^8
61. $(x - 2y)^{10}$	ax^8y^2

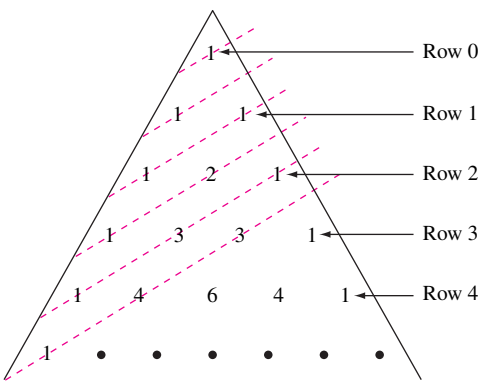
	Binomial	*Term*
62.	$(4x - y)^{10}$	ax^3y^7
63.	$(3x - 2y)^9$	ax^5y^4
64.	$(2x - 3y)^{11}$	ax^3y^8
65.	$(x^2 - 1)^8$	ax^6
66.	$(x^2 + 2)^{12}$	ax^{10}

In Exercises 67–70, use the Binomial Theorem to expand the expression.

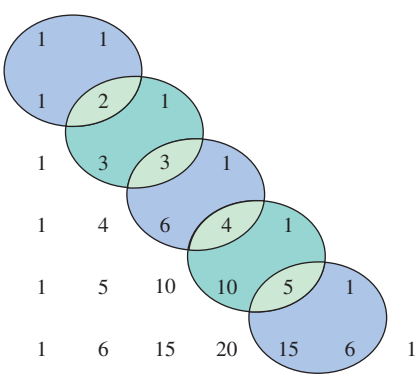
67. $\left(\frac{1}{2} + \frac{1}{2}\right)^7$ **68.** $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$

69. $(0.6 + 0.4)^5$ **70.** $(0.35 + 0.65)^6$

71. Finding a Pattern Describe the pattern formed by the sums of the numbers along the diagonal segments of Pascal's Triangle (see figure).



72. Finding a Pattern Use each of the encircled groups of numbers in the figure to form a 2×2 matrix. Find the determinant of each matrix. Describe the pattern.



73. Nonfarm Employees The total number $f(t)$ of nonfarm employees (in millions) in the United States from 1980 to 2005 can be approximated by

$$f(t) = -0.011t^2 + 2.21t + 87.0, \quad 0 \le t \le 25$$

where t represents the year, with $t = 0$ corresponding to 1980 (see figure). *(Source: U.S. Bureau of Labor Statistics)*

(a) You want to adjust this model so that $t = 0$ corresponds to 1990 rather than 1980. To do this, you shift the graph of f 10 units *to the left* and obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.

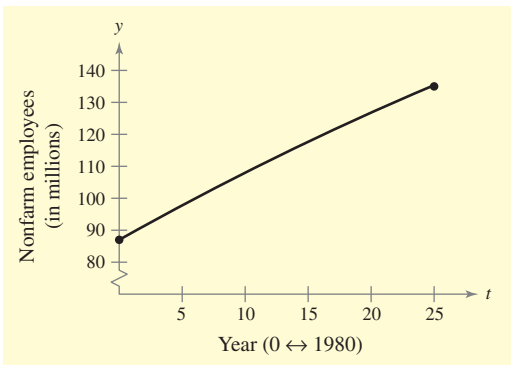


Figure for 73

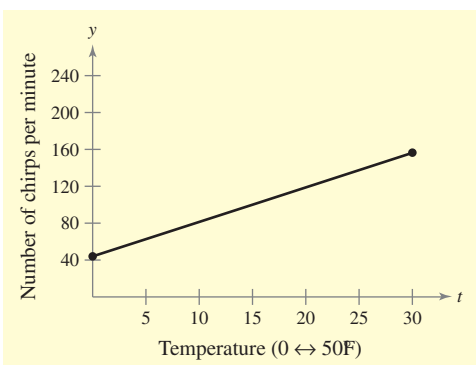
74. Crickets You conduct an experiment and determine that the numbers of chirps $f(t)$ that a cricket makes each minute at various temperatures can be approximated by

$$f(t) = 0.004t^2 + 3.73t + 44, \quad 0 \le t \le 30$$

where $t = 0$ represents 50°F (see figure).

(a) You want to adjust this model so that $t = 0$ corresponds to 60°F rather than 50°F. To do this, you shift the graph of f 10 units *to the left* and obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.



75. Extended Application To work an extended application involving fractals, visit this text's website at *college.hmco.com/info/larsonapplied.*

Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, write the first five terms of the sequence. (Assume that n begins with 1.)

1. $a_n = 2n - 3$

2. $a_n = \dfrac{n}{(n+1)^2}$

In Exercises 3 and 4, evaluate the expression.

3. $\dfrac{5!}{3!}$

4. $\dfrac{30!}{27!}$

In Exercises 5–10, write an expression for a_n. (Assume that n begins with 1.)

5. $3, \frac{3}{4}, \frac{1}{3}, \frac{3}{16}, \frac{3}{25}, \ldots$

6. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

7. Arithmetic: $a_1 = 2, d = 5$

8. Arithmetic: $a_1 = 7, d = 3$

9. Geometric: $a_1 = 10, r = 3$

10. Geometric: $a_1 = 4, r = -\frac{1}{2}$

In Exercises 11–14, determine whether the sequence is arithmetic, geometric, or neither. Explain your reasoning.

11. $a_n = 2 + 3(n - 1)$

12. $1, 1, 2, 3, 5, 8, 13, \ldots$

13. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

14. $a_n = 3(2)^n$

In Exercises 15–20, find the sum.

15. $\displaystyle\sum_{i=1}^{5} 9$

16. $\displaystyle\sum_{i=1}^{4} 2i^2$

17. $\displaystyle\sum_{n=1}^{25} (4n - 3)$

18. $\displaystyle\sum_{n=1}^{6} 2^n$

19. $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$

20. $\displaystyle\sum_{n=0}^{\infty} 5\left(\frac{1}{5}\right)^n$

21. Evaluate (a) ${}_8C_6$ and (b) ${}_{10}C_{10}$.

In Exercises 22 and 23, use the Binomial Theorem to expand the expression.

22. $(x - 5)^4$

23. $(2x + 1)^5$

24. Find the ninth term in the expansion of $(x^2 + y)^{10}$.

25. A person accepts a position with a company and will receive a salary of \$2,500 for the first year and a guaranteed raise of \$250 per year.

(a) What is the employee's salary during the sixth year of employment?

(b) What is the total compensation from the company through six full years of employment?

26. You deposit \$00 in an account at the beginning of each month for 20 years. The account pays 8% compounded monthly. What is your balance at the end of the 20 years?

Section 7.5

Counting Principles

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.

Simple Counting Problems

This section and Section 7.6 offer a brief introduction to some of the basic counting principles and their application to probability.

Example 1 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box, its number is written down, and the piece of paper is *replaced in the box*. Then, a second piece of paper is drawn from the box, and its number is written down. Finally, the two numbers are added together. In how many different ways can a sum of 12 be obtained?

SOLUTION To solve this problem, count the different ways that a sum of 12 can be obtained using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

From this list, you can see that a sum of 12 can occur in five different ways.

✓CHECKPOINT 1

In Example 1, in how many different ways can a sum of 14 be obtained? ■

Example 2 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box and, *without replacing the paper*, a second piece of paper is drawn. The numbers on the pieces of paper are written down and totaled. In how many different ways can a sum of 12 be obtained?

SOLUTION To solve this problem, you can count the different ways that a sum of 12 can be obtained using two *different* numbers from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can be obtained in four different ways.

✓CHECKPOINT 2

In Example 2, in how many different ways can a sum of 14 be obtained? ■

STUDY TIP

The counting problems in Examples 1 and 2 can be distinguished by saying that the random selection in Example 1 occurs **with replacement,** whereas the random selection in Example 2 occurs **without replacement,** which eliminates the possibility of choosing two 6's.

Counting Principles

Examples 1 and 2 describe simple counting problems in which you can *list* each possible way that an event can occur. When it is possible, this is the best way to solve a counting problem. However, some events can occur in so many different ways that it is not feasible to write out the entire list. In such cases, you must rely on formulas and counting principles. The most important of these is the **Fundamental Counting Principle.**

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1, E_2, and E_3 can occur is $m_1 \cdot m_2 \cdot m_3$.

Example 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

SOLUTION This experiment has two events. The first event is the choice of the first letter, and the second event is the choice of the second letter. Because the English alphabet contains 26 letters, it follows that the number of pairs of letters is $26 \cdot 26 = 676$.

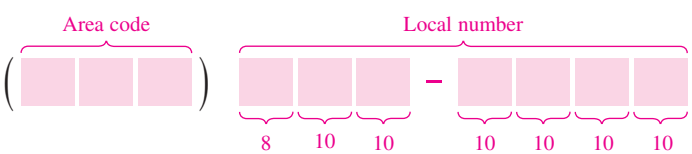
✓CHECKPOINT 3

The extension for a filename on a computer typically consists of three letters from the English alphabet. How many different extensions are possible? ■

Example 4 Using the Fundamental Counting Principle

Telephone numbers in the United States currently have 10 digits. The first three are the *area code* and the next seven are the *local telephone number.* How many different local telephone numbers are possible within each area code? (Note that at this time, a local telephone number cannot begin with 0 or 1.)

SOLUTION Because the first digit cannot be 0 or 1, there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of local telephone numbers that are possible within each area code is $8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000$.

✓CHECKPOINT 4

The catalog number for a particular product consists of one letter from the English alphabet followed by a five-digit number. How many different catalog numbers are possible? ■

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

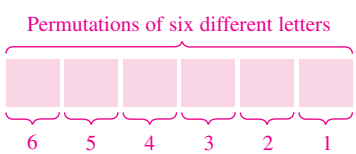
Example 5 Finding the Number of Permutations of n Elements

How many permutations of the letters A, B, C, D, E, and F are possible?

SOLUTION Consider the following reasoning.

First position:	Any of the *six* letters.
Second position:	Any of the remaining *five* letters.
Third position:	Any of the remaining *four* letters.
Fourth position:	Any of the remaining *three* letters.
Fifth position:	Any of the remaining *two* letters.
Sixth position:	The *one* remaining letter.

So, the numbers of choices for the six positions are as follows.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 720.$$

✓CHECKPOINT 5

How many permutations of the letters W, X, Y, and Z are possible? ■

Number of Permutations of n Elements

The number of permutations of n distinct elements is given by

$$n \cdot (n - 1) \cdot \cdots \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n distinct elements can be ordered.



K. Beebe/CMSP/Getty Images

Occasionally, you are interested in ordering a *subset* of a collection of elements rather than the entire collection. For example, you might want to choose (and order) r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time.**

Example 6 Counting Kidney Donors

Eight family members of a patient with end-stage kidney (renal) disease are identified as potential kidney donors. How many possible orders are there for a best match, a second-best match, and a third-best match?

SOLUTION Here are the different possibilities.

Best match:	*Eight* choices
Second-best match:	*Seven* choices
Third-best match:	*Six* choices

Using the Fundamental Counting Principle, multiply these three numbers together to obtain the following.

Different orders of donors

8 7 6

So, there are $8 \cdot 7 \cdot 6 = 336$ different orders.

✓CHECKPOINT 6

In Example 6, suppose six family members are potential donors. In how many different ways can the top three potential donors be ordered? ■

Permutations of n Elements Taken r at a Time

The number of permutations of n distinct elements taken r at a time is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$
$$ = n(n-1)(n-2) \cdot \cdots \cdot (n-r+1). $$

Using this formula, you can rework Example 6 to find the number of permutations of the eight potential donors taken three at a time.

$$ {}_8P_3 = \frac{8!}{(8-3)!} $$
$$ = \frac{8!}{5!} $$
$$ = \frac{(8 \cdot 7 \cdot 6) \cdot 5!}{5!} $$
$$ = 336 $$

The answer is 336 different orders, the same answer obtained in Example 6.

Remember that for permutations, order is important. So, if you were looking at the possible permutations of the letters A, B, C, and D taken three at a time, the permutations (A, B, D) and (B, A, D) would be different because the *order* of the elements is different.

Suppose, however, that you were asked to find the possible permutations of the letters A, A, B, and C. The total number of permutations of the four letters would be ${}_4P_4 = 4!$. However, not all of these arrangements would be *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with $n = n_1 + n_2 + n_3 + \cdots + n_k$. Then the number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!}.$$

Example 7 Distinguishable Permutations

In how many distinguishable ways can the letters in BANANA be written?

SOLUTION This word has six letters, so $n = 6$, but you have to take into account that there are three A's, two N's, and one B. So, the number of distinguishable ways the letters can be written is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3!} = \frac{6!}{3! \cdot 2! \cdot 1!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!} = 60.$$

The 60 different distinguishable permutations are as follows.

AAABNN	AAANBN	AAANNB	AABANN	AABNAN	AABNNA
AANABN	AANANB	AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN	ABNANA	ABNNAA
ANAABN	ANAANB	ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB	ANNABA	ANNBAA
BAAANN	BAANAN	BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA	NAAABN	NAAANB
NAABAN	NAABNA	NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA	NBAAAN	NBAANA
NBANAA	NBNAAA	NNAAAB	NNAABA	NNABAA	NNBAAA

✓CHECKPOINT 7

In how many distinguishable ways can the letters in MITOSIS be written? ■

Combinations

When you count the number of possible permutations of a set of elements, *order* is important. As a final topic in this section, you will look at a method of selecting subsets of a larger set in which order is *not important.* Such subsets are called **combinations of *n* elements taken *r* at a time.** For instance, the combinations

$\{A, B, C\}$ and $\{B, A, C\}$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is *not important.* So, you would count only one of the two sets. A common example of how a combination occurs is a card game in which the player is free to reorder the cards after they have been dealt.

Example 8 Combinations of *n* Elements Taken *r* at a Time

In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)

SOLUTION The following subsets represent the different combinations of three letters that can be chosen from the five letters.

$\{A, B, C\}$ $\{A, B, D\}$
$\{A, B, E\}$ $\{A, C, D\}$
$\{A, C, E\}$ $\{A, D, E\}$
$\{B, C, D\}$ $\{B, C, E\}$
$\{B, D, E\}$ $\{C, D, E\}$

From this list, you can conclude that there are 10 different ways that three letters can be chosen from the five letters.

✓CHECKPOINT 8

In how many different ways can two letters be chosen from the letters J, K, L, M, and N? (The order of the two letters is not important.) ■

Combinations of *n* Elements Taken *r* at a Time

The number of combinations of n elements taken r at a time is

$$ {}_nC_r = \frac{n!}{(n-r)!r!}. $$

Note that the formula for ${}_nC_r$ is the same one given for binomial coefficients. To see how this formula is used, solve the counting problem in Example 8. In that problem, you are asked to find the number of combinations of five elements taken three at a time. So, $n = 5$, $r = 3$, and the number of combinations is

$$ {}_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3} \cdot \cancel{2!}}{\cancel{2!} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 10 $$

which is the same answer obtained in Example 8.

TECHNOLOGY

Most graphing utilities have keys that will evaluate the formulas for the number of permutations or combinations of n elements taken r at a time. For specific instructions on how to evaluate permutations and combinations using a graphing utility, go to the text website at *college.hmco.com/info/larsonapplied.* Evaluate ${}_8P_5$ (the number of permutations of eight elements taken five at a time). The display should be 6720. Now evaluate ${}_8C_5$ (the number of combinations of eight elements taken five at a time). The display should be 56. Use your graphing utility to evaluate the following.

a. ${}_{36}P_4$ **b.** ${}_{36}C_4$
c. ${}_{25}P_3$ **d.** ${}_{25}C_3$

Example 9 Counting Card Hands

A standard poker hand consists of five cards dealt from a deck of 52. How many different poker hands are possible? (After the cards are dealt the player may reorder them, and therefore order is not important.)

SOLUTION You can find the number of different poker hands by using the formula for the number of combinations of 52 elements taken five at a time, as follows.

$$_{52}C_5 = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960 \text{ different hands}$$

✓CHECKPOINT 9

In three-card poker each player is dealt three cards from a deck of 52. How many different three-card poker hands are possible? (Order is not important.) ■

Example 10 MAKE A DECISION Choosing Between Permutations and Combinations

Decide whether each scenario should be counted using permutations or combinations.

a. Number of different arrangements of three types of flowers from an array of 20 types

b. Number of different three-digit pin numbers for a debit card

SOLUTION To determine which counting principle is necessary to solve each problem, ask yourself two questions: (1) Is the order of the elements important? If yes, then you should solve the problem using permutations. (2) Are the chosen elements a subset of a larger set in which order is not important? If yes, then you should solve the problem using combinations.

a. The order of the flowers is not important. So, the number of possible flower arrangements should be counted using combinations.

b. The order of the digits in the pin number matters. So, the number of possible pin numbers should be counted using permutations.

✓CHECKPOINT 10

Should the number of different nine-digit social security numbers be counted using permutations or combinations? ■

CONCEPT CHECK

1. You have n cards numbered from 1 to n. Describe the possible number of permutations of the numbers when taking two cards at a time.
2. Some locks are called *combination locks.* Are these locks named after mathematical combinations? Explain.
3. Is $_nC_{r+1}$ always greater than $_nC_r$? Explain.
4. From n elements, you take r at a time. For what value of r is the number of permutations equal to the number of combinations? Explain.

Skills Review 7.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3, 7.1, and 7.4.

In Exercises 1–4, evaluate the expression.

1. $13 \cdot 8^2 \cdot 2^3$

2. $10^2 \cdot 9^3 \cdot 4$

3. $\dfrac{12!}{2!(7!)(3!)}$

4. $\dfrac{25!}{22!}$

In Exercises 5 and 6, find the binomial coefficient.

5. ${}_{12}C_7$

6. ${}_{25}C_{22}$

In Exercises 7–10, simplify the expression.

7. $\dfrac{n!}{(n-4)!}$

8. $\dfrac{(2n)!}{4(2n-3)!}$

9. $\dfrac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2n)}{2^n}$

10. $\dfrac{3 \cdot 6 \cdot 9 \cdot 12 \cdot \cdots \cdot (3n)}{3^n}$

Exercises 7.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Random Selection In Exercises 1–8, determine the number of ways a computer can randomly generate the specified integer(s) from 1 through 15.

1. An odd integer

2. An even integer

3. A prime integer

4. An integer that is greater than 9

5. An integer that is divisible by 4

6. An integer that is divisible by 5

7. Two integers whose sum is 20

8. Two *distinct* integers whose sum is 20

9. Job Applicants A small college needs two additional faculty members: a chemist and a statistician. There are five applicants for the chemistry position and six applicants for the statistics position. In how many ways can these positions be filled?

10. Computer Systems A customer in a computer store can choose one of six monitors, one of three keyboards, and one of eight computers. If all the choices are compatible, how many different systems can be chosen?

11. Mixtures A chemist is making a mixture. The chemist must choose one of the solutions in five pipets and one of the solutions in three beakers. How many different mixtures can the chemist make?

12. Dissections In a physiology class, a student must dissect three different organisms. The student can select one of nine earthworms, one of four frogs, and one of seven fetal pigs. In how many ways can the student select the specimens?

13. Toboggan Ride Six people are lining up for a ride on a toboggan, but only two of the six are willing to take the first position. With that constraint, in how many ways can the six people be seated on the toboggan? Draw a diagram to illustrate the number of ways that the people can sit.

14. Course Schedule A college student needs to schedule five prescribed courses for next semester. Only three of the five courses are able to be scheduled as the first class of the day. With that constraint, in how many ways can she select her schedule? Draw a diagram to illustrate the number of ways she can select her schedule.

15. License Plate Numbers In a certain state, each automobile license plate number consists of two letters followed by a four-digit number. How many distinct license plate numbers can be formed?

16. License Plate Numbers In a certain state, each automobile license plate number consists of two letters followed by a four-digit number. To avoid confusion between "O" and "zero" and "I" and "one," the letters "O" and "I" are not used. How many distinct license plate numbers can be formed?

17. True-False Exam In how many ways can a 10-question true-false exam be answered? (Assume that no questions are omitted.)

18. True-False Exam In how many ways can a 15-question true-false exam be answered? (Assume that no questions are omitted.)

19. Three-Digit Numbers How many three-digit numbers can be formed under each condition?

(a) The leading digit cannot be 0.

(b) The leading digit cannot be 0 and no repetition of digits is allowed.

(c) The leading digit cannot be 0 and the number must be a multiple of 5.

20. Four-Digit Numbers How many four-digit numbers can be formed under each condition?

(a) The leading digit cannot be 0.

(b) The leading digit cannot be 0 and no repetition of digits is allowed.

(c) The leading digit cannot be 0 and the number must be a multiple of 5.

21. Competing Restaurants Two hamburger restaurants are competing for customers. Restaurant A offers one of six cheeses, one of three types of meat patties, one of four toppings, one of five sauces, and one of five buns. Restaurant B offers one of seven cheeses, one of three meats, one of five toppings, one of five sauces, and one of three buns. How many different burgers can be made at each restaurant? Which restaurant can truthfully advertise that it has more burger choices?

22. Competing Restaurants Two Italian restaurants are featuring a "Design Your Own Pasta Dish" sales promotion. Restaurant A offers one of six pastas, one of four sauces, one of five cheeses, and one of eight toppings. Restaurant B offers one of five pastas, one of five sauces, one of six cheeses, and one of seven toppings. How many different pasta dishes can be made at each restaurant? Which restaurant can truthfully advertise that it has more pasta dish choices?

23. Combination Lock A combination lock will open when the right choice of three numbers (from 1 to 40, inclusive) is selected. How many different lock combinations are possible?

24. Combination Lock A combination lock will open when the right choice of three numbers (from 1 to 50, inclusive) is selected. How many different lock combinations are possible?

25. Chemistry Lab In a chemistry lab, five oxidation-reduction reactions can be performed in any order. How many different orders are possible for the five reactions?

26. Biology Lab In a plant biology lab, six different experiments can be performed in any order. How many different orders are possible for the six experiments?

27. Concert Seats Four couples have reserved seats in a given row for a concert. In how many different ways can they be seated, given the following conditions?

(a) There are no restrictions.

(b) The two members of each couple wish to sit together.

28. Single File In how many orders can five girls and three boys walk through a doorway single-file, given the following conditions? (a) There are no restrictions. (b) The boys go before the girls. (c) The girls go before the boys.

In Exercises 29–40, evaluate the expression.

29. ${}_4P_3$ **30.** ${}_5P_4$

31. ${}_4C_3$ **32.** ${}_5C_4$

33. ${}_8P_3$ **34.** ${}_{10}P_5$

35. ${}_8C_3$ **36.** ${}_{10}C_5$

37. ${}_{12}P_4$ **38.** ${}_{48}P_3$

39. ${}_{12}C_4$ **40.** ${}_{48}C_3$

In Exercises 41–44, simplify the expression.

41. ${}_nP_3$ **42.** ${}_nC_3$

43. ${}_nP_{n-1}$ **44.** ${}_nC_{n-1}$

45. Choosing Officers From a pool of 15 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. In how many different ways can the offices be filled if each of the 15 candidates can hold any office?

46. Bike Race There are 10 bicyclists entered in a race. In how many different ways can the top three places be decided?

47. Forming an Experimental Group In order to conduct an experiment, five students are randomly selected from a class of 30. How many different groups of five students are possible?

48. Lab Practical A student may answer any 15 questions from a total of 20 questions on a biology lab practical. In how many ways can the student select the questions?

49. Lottery In Connecticut's Classic Lotto game, a player chooses six distinct numbers from 1 to 44. In how many ways can a player select the six numbers? (The order of selection is not important.)

50. Lottery In South Dakota's Dakota Cash game, a player chooses five distinct numbers from 1 to 35. In how many ways can a player select the five numbers? (The order of selection is not important.)

In Exercises 51–56, find the number of distinguishable permutations of the group of letters.

51. A, S, S, E, T, S **52.** A, C, C, R, U, A, L

53. S, P, E, C, I, E, S **54.** M, A, M, M, A, L

55. A, L, G, E, B, R, A

56. M, I, S, S, I, S, S, I, P, P, I

57. Number of Subsets How many subsets of four elements can be formed from a set of 100 elements?

58. Number of Subsets How many subsets of five elements can be formed from a set of 80 elements?

59. Game Show On a game show, 10 audience members are randomly chosen to be eligible contestants. Six of the 10 eligible contestants are chosen to play a game on stage. You and your friend are two of the 10 eligible contestants. In how many ways can the six players be chosen from the group of eligible contestants given that you and your friend are chosen to play a game?

60. Job Applicants An employer interviews 10 people for four openings in the research and development department of the company. Four of the 10 people are women. All 10 applicants are qualified. In how many ways can the employer fill the four positions, given the following conditions?

(a) There are no restrictions.

(b) Exactly two selections are women.

61. Poker Hand Five cards are selected from a standard deck of 52 playing cards. In how many ways can you get a full house? (A full house consists of three of one kind and two of another. For example, A-A-A-5-5 and K-K-K-10-10 are full houses.)

62. Poker Hand Five cards are selected from a standard deck of 52 playing cards. In how many ways can you get a straight flush? (A straight flush consists of five cards that are in order and of the same suit. For example, A♥, 2♥, 3♥, 4♥, 5♥ and 10♠, J♠, Q♠, K♠, A♠ are straight flushes.)

MAKE A DECISION In Exercises 63–66, decide whether the scenario should be counted using permutations or combinations. Explain your reasoning.

63. Number of ways 10 people can line up in a row for concert tickets

64. Number of ways three different roles can be filled by 10 people auditioning for a play.

65. Number of different three-topping pizzas that can be made from an assortment of 15 different toppings

66. Number of ways a jury of 12 people can be selected from a group of 50 people.

67. Forming a Committee A six-member research committee at a local college is to be formed. It will consist of one administrator, three faculty members, and two students. There are seven administrators, 12 faculty members, and 20 students in contention for the committee. How many six-member committees are possible?

68. Law Enforcement A police department uses computer imaging to create digital photographs of alleged perpetrators from eyewitness accounts. One software package contains 195 hairlines, 99 sets of eyes and eyebrows, 89 noses, 105 mouths, and 74 chins and cheek structures.

(a) Find the possible number of different faces that the software could create.

(b) An eyewitness can clearly recall the hairline and eyes and eyebrows of a suspect. How many different faces can be produced with this information?

Geometry In Exercises 69–72, find the number of diagonals of the polygon. (A line segment connecting any two nonadjacent vertices is called a *diagonal* of a polygon.)

69. Pentagon

70. Hexagon

71. Octagon

72. Decagon (10 sides)

73. ***MAKE A DECISION*** Without calculating the numbers, determine which of the following is greater. Explain.

(a) The combinations of 10 elements taken six at a time

(b) The permutations of 10 elements taken six at a time

74. ***MAKE A DECISION*** Is the value of ${}_nP_r$ always greater than the value of ${}_nC_r$? Explain.

Business Capsule



Inti St Clair/DigitalVision/Getty Images

Founded in 1999, Intelebiz specializes in software development and e-business solutions. One of the products offered by Intelebiz is an inventory management system that keeps the supply chain in balance. The system uses combinations and permutations of items and item variations to offer customers a wide variety of products. The functionality of the system supports any number of inventory items and item relationships.

75. Research Project Use your campus library, the Internet, or some other reference source to find information about a company that uses combinations or permutations to perform inventory management or some other business function. Write a brief report about the company. In the report, explain why the use of combinations or permutations is needed to perform the business function.

Section 7.6 Probability

- Determine the sample space of an experiment and find the probability of an event.
- Find the probabilities of mutually exclusive events.
- Find the probabilities of independent events.
- Find the probability of the complement of an event.

The Probability of an Event

In measuring the uncertainties of everyday life, ambiguous terminology, such as "fairly certain" and "highly unlikely," is often used. In mathematics we attempt to assign a number to the likelihood of the occurrence of an event. This measurement is called the **probability** that the event will occur. For example, if you toss a fair coin, the probability that it will land face up is $\frac{1}{2}$.

Any happening whose result is uncertain is called an **experiment.** The possible results of an experiment are **outcomes,** the set of all possible outcomes of an experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event.** For instance, when a six-sided die is tossed, the sample space can be represented by the numbers from 1 through 6, each being an *equally likely* outcome.

To describe a sample space in such a way that each outcome is equally likely, you must sometimes distinguish between various outcomes in ways that appear artificial. Example 1 illustrates such a situation.

Example 1 Finding the Sample Space

Find the sample space for each event.

a. One coin is tossed.

b. Two coins are tossed.

c. Three coins are tossed.

SOLUTION

a. Because the coin will land either heads up (denoted by H) or tails up (denoted by T), the sample space is $S = \{H, T\}$.

b. Because either coin can land heads up or tails up, the possible outcomes are

HH = heads up on both coins

HT = heads up on first coin and tails up on second coin

TH = tails up on first coin and heads up on second coin

TT = tails up on both coins.

The sample space is $S = \{HH, HT, TH, TT\}$. This list distinguishes between the two cases HT and TH, even though these two outcomes appear to be similar.

c. Following the notation of part (b), the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

✓CHECKPOINT 1

Find the sample space when two students select a drink from three possible drinks (denoted by A, B, and C) for lunch. ■

To calculate the probability of an event, count the number of outcomes in the event and in the sample space.The *number of outcomes* in event E is denoted by $n(E)$, and the number of outcomes in the sample space S is denoted by $n(S)$. The probability that event E will occur is given by $n(E)/n(S)$.

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, the **probability** of event E is

$$P(E) = \frac{n(E)}{n(S)}.$$

Because the number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, the probability of an event must be a number between 0 and 1. That is, for any event E, it must be true that $0 \le P(E) \le 1$, as shown in Figure 73.

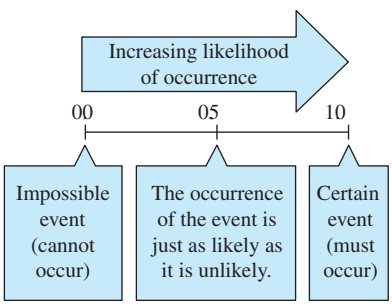


FIGURE 7.3

Properties of the Probability of an Event

Let E be an event that is a subset of a finite sample space S.

1. $0 \le P(E) \le 1$
2. If $P(E) = 0$, E *cannot occur* and is called an **impossible event.**
3. If $P(E) = 1$, E *must occur* and is called a **certain event.**

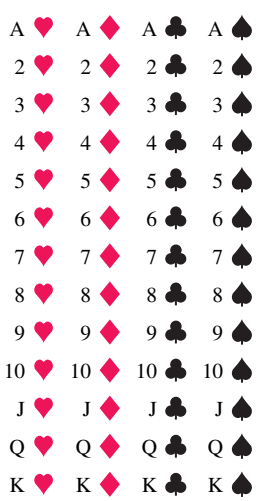


FIGURE 7.4 Standard Deck of Playing Cards

Example 2 **Finding the Probability of an Event**

a. Two coins are tossed.What is the probability that both land heads up?

b. A card is drawn from a standard deck of playing cards (see Figure 74).What is the probability that it is an ace?

SOLUTION

a. Following the procedure in Example 1(b), let $E = \{HH\}$ and $S = \{HH, HT, TH, TT\}$. The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

b. Because there are 52 cards in a standard deck of playing cards and there are four aces (one in each suit), the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

✓CHECKPOINT 2

A card is drawn from a standard deck of playing cards.What is the probability that the card is a diamond? ■

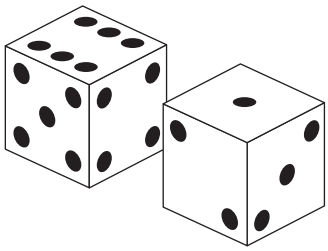


FIGURE 7.5

TECHNOLOGY

You can use the *random integer* feature of a graphing utility to simulate the tossing of a die or set of dice. For instance, the result of rolling a pair of dice five times and summing the outcomes is shown below.

```
randInt(1,6,5)+r
andInt(1,6,5)
   {11 7 4 10 5}
```

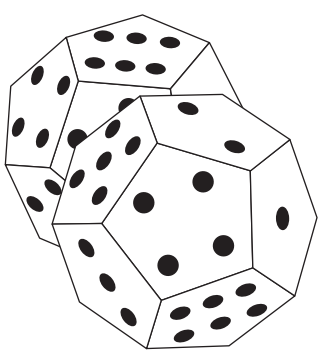


FIGURE 7.6

✓CHECKPOINT 4

Two twelve-sided dice (as described in Example 4) are tossed. What is the probability that the total of the two dice is less than 5? ■

Example 3 Finding the Probability of an Event

Two six-sided dice are tossed. What is the probability that the total of the two dice is 7? (See Figure 75.)

SOLUTION Because there are six possible outcomes on each die, you can use the Fundamental Counting Principle to conclude that there are $6 \cdot 6$ or 36 different outcomes when two dice are tossed. To find the probability of rolling a total of 7, you must first count the number of ways this can occur.

First Die	1	2	3	4	5	6
Second Die	6	5	4	3	2	1

So, a total of 7 can be rolled in six ways, which means that the probability of rolling a 7 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

✓CHECKPOINT 3

Two six-sided dice are tossed. What is the probability that the total of the two dice is 4? ■

Example 4 Finding the Probability of an Event

Twelve-sided dice can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 appears twice on each die, as shown in Figure 76. Can these dice be used in any game requiring ordinary six-sided dice without changing the probabilities of different outcomes?

SOLUTION For an ordinary six-sided die, each of the numbers 1, 2, 3, 4, 5, and 6 occurs only once, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For a twelve-sided die, each number occurs twice, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

So, the twelve-sided dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of different outcomes.

In Examples 2, 3, and 4, you simply counted the outcomes in the desired events. For larger sample spaces, however, you should use the counting principles discussed in Section 75.

Example 5 The Probability of Winning a Lottery

In Louisiana's Lotto game, a player chooses six numbers from 1 to 40. If these six numbers match the six numbers drawn (in any order) by the lottery commission, the player wins (or shares) the jackpot. What is the probability of winning the jackpot?

SOLUTION Because the order of the six numbers does not matter, use the formula for the number of combinations of 40 elements taken six at a time.

$$n(S) = {}_{40}C_6 = \frac{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 3{,}838{,}380.$$

If a person buys only one ticket, the probability of winning the jackpot is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{3{,}838{,}380}.$$

✓CHECKPOINT 5

In Pennsylvania's Cash 5 Lotto game, a player chooses five numbers from 1 to 39. The order of the numbers is not important. What is the probability of winning the jackpot? ■

STUDY TIP

You can write a probability as a fraction, a decimal, or a percent.

Example 6 Random Selection

The frequencies of respiratory diseases in 2005 among adults 18 years old and older are shown in the table below. One person is selected at random. What is the probability that the person is at most 64 years old? *(Source: National Health Interview Survey)*

Respiratory Diseases: 2005	
18–44 years	110,431
45–64 years	72,296
65–74 years	18,446
75 years and over	16,600

SOLUTION From the table, the total number of adults with a respiratory disease is 217,773. Because there are $110{,}431 + 72{,}296 = 182{,}727$ adults with a respiratory disease that are at most 64 years old, the probability that the person is at most 64 years old is

$$P(E) = \frac{n(E)}{n(S)} = \frac{182{,}727}{217{,}773} \approx 0839.$$

✓CHECKPOINT 6

In Example 6, what is the probability that the randomly selected adult with a respiratory disease is at least 45 years old? ■

Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** if A and B have no outcomes in common.In the terminology of sets, the **intersection of A and B** is the empty set and

$$P(A \cap B) = 0.$$

For instance, if two dice are tossed, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive.To find the probability that one or the other of two mutually exclusive events will occur, you can *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, the probability of A or B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Example 7 The Probability of a Union

One card is selected from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

SOLUTION Because the deck has 13 hearts, the probability of selecting a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, because the deck has 12 face cards, the probability of selecting a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Because three of the cards are hearts and face cards (see Figure 77), it follows that the events A and B are *not* mutually exclusive, and therefore

$$P(A \cap B) = \frac{3}{52}.$$

Finally, applying the formula for the probability of the union of two events, you can conclude that the probability of selecting a heart or a face card is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &\approx 0423. \end{aligned}$$

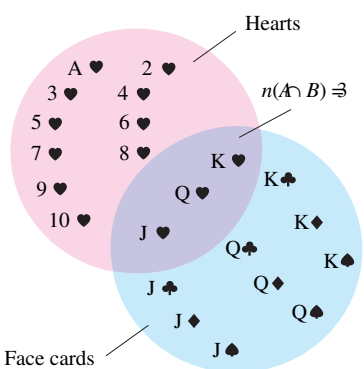


FIGURE 7.7

✓ **CHECKPOINT 7**

One card is selected from a standard deck of 52 playing cards. What is the probability that the card is either a spade or an eight? ■

Example 8 Probability of Mutually Exclusive Events

The personnel department of a company has compiled data showing the number of years of service for each employee. The results are shown in the table.

Years of service	Number of employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	37
35–39	21
40–44	8

If an employee is chosen at random, what is the probability that the employee has had 9 or fewer years of service?

SOLUTION To begin, add the number of employees and find that the total is 529. Next, let event A represent choosing an employee with 0 to 4 years of service and let event B represent choosing an employee with 5 to 9 years of service. Then

$$P(A) = \frac{157}{529} \quad \text{and} \quad P(B) = \frac{89}{529}.$$

Because A and B have no outcomes in common, you can conclude that these two events are mutually exclusive and that

$$P(A \cup B) = P(A) + P(B) = \frac{157}{529} + \frac{89}{529}$$

$$= \frac{246}{529}$$

$$\approx 0.465.$$

So, the probability of choosing an employee who has 9 or fewer years of service is about 0.465.

✓CHECKPOINT 8

In Example 8, what is the probability that the employee has 30 or more years of service? ■

Independent Events

Two events are **independent** if the occurrence of one has no effect on the occurrence of the other. For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. To find the probability that two independent events will occur, *multiply* the probabilities of each.

Probability of Independent Events

If A and B are independent events, the probability that both A and B will occur is

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

Example 9 Probability of Independent Events

A random number generator on a computer selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?

SOLUTION The probability of selecting an integer from 1 to 5 is

$$P(A) = \frac{5}{20} = \frac{1}{4}.$$

So, the probability that all three integers are less than or equal to 5 is

$$P(A) \cdot P(A) \cdot P(A) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$$

$$= \frac{1}{64}.$$

✓CHECKPOINT 9

A random number generator on a computer selects two integers from 1 to 20. What is the probability that one number is less than or equal to 10 and one number is greater than 10? ■

Example 10 Probability of Independent Events

In 2005, approximately 752%f the population of the United States was 18 years old or older.In a survey, 10 people were chosen at random from the population. What is the probability that all 10 were 18 years old or older? *(Source: U.S. Census Bureau)*

SOLUTION Let A represent choosing a person who was 18 years old or older. The probability of choosing a person who was 18 years old or older is 0752, the probability of choosing a second person who was 18 years old or older is 0752, and so on. Because these events are independent, you can conclude that the probability that all 10 people were 18 years old or older is

$$\underbrace{P(A) \cdot P(A) \cdot \ldots \cdot P(A)}_{\text{10 occurrences}} = [P(A)]^{10}$$

$$= (0752)^{10}$$

$$\approx 00578$$

✓CHECKPOINT 10

In 2005, approximately 124%f the population of the United States was 65 years old or older.In a survey, 5 people were chosen at random from the population.What is the probability that all 5 were 65 years old or older? *(Source: U.S. Census Bureau)* ■

DISCOVERY

You are in a class with 22 other people. What is the probability that at least two out of the 23 people will have a birthday on the same day of the year?

The complement of the probability that at least two people have the same birthday is the probability that all 23 birthdays are different.So, first find the probability that all 23 people have different birthdays and then find the probability of the complement.

Now, determine the probability that in a room with 50 people at least two people have the same birthday.

The Complement of an Event

The **complement of an event A** is the collection of all outcomes in the sample space that are *not* in A. The complement of event A is denoted by A'. Because $P(A \text{ or } A') = 1$ and because A and A' are mutually exclusive, it follows that $P(A) + P(A') = 1$. So, the probability of A' is given by

$$P(A') = 1 - P(A).$$

For instance, if the probability of *winning* a certain game is $P(A) = \frac{1}{4}$, then the probability of *losing* the game is $P(A') = 1 - \frac{1}{4} = \frac{3}{4}$.

Probability of a Complement

Let A be an event and let A' be its complement.If the probability of Ais $P(A)$, then the probability of the complement is

$$P(A') = 1 - P(A).$$

Example 11 **Finding the Probability of a Complement**

A digital piano manufacturer has determined that a certain machine averages one faulty piano for every 1000 it produces.What is the probability that an order of 200 pianos will have one or more faulty pianos?

SOLUTION To solve this problem as stated, you would need to find the probabilities of having exactly one faulty piano, exactly two faulty pianos, exactly three faulty pianos, and so on.However, using complements, you can simply find the probability that all pianos are perfect and then subtract this value from 1. Because the probability that any given piano is perfect is 999/1000, the probability that all 200 pianos are perfect is

$$P(A) = \left(\frac{999}{1000}\right)^{200} \approx 08186.$$

So, the probability that at least one piano is faulty is

$$P(A') = 1 - P(A) \approx 01814.$$

✓CHECKPOINT 11

A microscope manufacturer has determined that a certain model averages one faulty microscope for every 2000 it produces.What is the probability that an order of 300 microscopes will have one or more faulty microscopes? ■

CONCEPT CHECK

1. Event E is neither impossible nor certain. What is true about $P(E)$?
2. Explain how you can be sure that event A and event B are not mutually exclusive given that $P(A) = 0.4$ and $P(B) = 0.8$.
3. Events A and B are independent events. Explain why the probability of both A and B occurring cannot be greater than the probability of event A occurring.
4. Events A, B, and C are three events such that $P(A) + P(B) = 1 - P(C)$ and $A \cup B$ is the complement of C. What is the relationship of events A and B?

Skills Review 7.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.3, 7.1, and 7.4.

In Exercises 1–8, evaluate the expression.

1. $\frac{1}{4} + \frac{5}{8} - \frac{5}{16}$

2. $\frac{4}{15} + \frac{3}{5} - \frac{1}{3}$

3. $\frac{5 \cdot 4}{5!}$

4. $\frac{5!2!}{27!}$

5. $\frac{4!}{8!2!}$

6. $\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{9!}$

7. $\frac{{}_5C_3}{{}_{10}C_3}$

8. $\frac{{}_{10}C_2 \cdot {}_{10}C_2}{{}_{20}C_4}$

In Exercises 9 and 10, evaluate the expression. (Round your answer to three decimal places.)

9. $\left(\frac{99}{100}\right)^{100}$

10. $1 - \left(\frac{89}{100}\right)^{50}$

Exercises 7.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, determine the sample space for the experiment.

1. A coin and a six-sided die are tossed.

2. A six-sided die is tossed twice and the sum of the points is recorded.

3. A taste tester has to rank three varieties of yogurt (A, B, and C) according to preference.

4. Two marbles are selected from a sack containing two red marbles, two blue marbles, and one black marble. The color of each marble is recorded.

5. Two county supervisors are selected from five supervisors (A, B, C, D and E) to study a recycling plan.

6. A sales representative makes presentations of a product in three homes.In each home, there may be a sale (denote by S) or there may be no sale (denote by F).

Choosing an Integer In Exercises 7–10, a computer randomly generates an integer from 1 through 50. Find the probability of the event.

7. An odd number is generated.

8. A number less than 25 is generated.

9. A prime number is generated.

10. A multiple of 5 is generated.

Heads or Tails? In Exercises 11–14, a coin is tossed three times. Find the probability of the event.

11. Getting exactly one head

12. Getting a tail on the second toss

13. Getting at least one tail

14. Getting at least two heads

Tossing a Die In Exercises 15–20, two six-sided dice are tossed. Find the probability of the event.

15. The sum is 5.

16. The sum is less than 10.

17. The sum is at least 7.

18. The sum is 2, 3, or 8.

19. The sum is odd and no more than 7.

20. The sum is odd or a prime number.

Drawing a Card In Exercises 21–26, a card is selected from a standard deck of 52 cards. Find the probability of the event.

21. Getting a red card

22. Getting a face card

23. Not getting a face card

24. Getting a red card that is not a face card

25. Getting a card that is a 6 or less (aces are low)

26. Getting a card that is a 9 or higher (aces are low)

Drawing Marbles In Exercises 27–32, two marbles are drawn (without replacement) from a bag containing two green, three yellow, and four red marbles. Find the probability of the event.

27. Drawing exactly one red marble

28. Drawing exactly one yellow marble

29. Drawing two green marbles

30. Drawing two red marbles

31. Drawing none of the yellow marbles

32. Drawing marbles of different colors

In Exercises 33 and 34, you are given the probability that an event *will* happen. Find the probability that the event *will not* happen.

33. $p = 03$ **34.** $p = 072$

In Exercises 35 and 36, you are given the probability that an event *will not* happen. Find the probability that the event *will* happen.

35. $p = 01$ **36.** $p = 068$

37. Winning an Election Three people have been nominated for president of a college class. From a small poll, it is estimated that the probability of Jane winning the election is 046, and the probability of Larry winning the election is 032. What is the probability of the third candidate winning the election?

38. Winning an Election Taylor, Moore, and Jenkins are candidates for a public office. It is estimated that Moore and Jenkins have about the same probability of winning, and Taylor is believed to be twice as likely to win as either of the others. Find each candidate's probability of winning the election.

39. College Bound In a high school graduating class of 198 students, 43 are on the honor roll. Of these, 37 are going on to college, and of the other 155 students, 102 are going on to college. A student is selected at random from the class. What is the probability that the person chosen is (a) going to college, (b) not going to college, and (c) on the honor roll, but not going to college?

40. Alumni Association The alumni office of a college sends a survey to selected members of the class of 2005. The numbers of graduates, and the numbers of graduates who attended graduate school, are shown in the table. An alumnus is selected at random. What is the probability that the person is (a) female, (b) male, and (c) female and did not attend graduate school?

	Number of graduates	Number of graduates that attended graduate school
Women	672	124
Men	582	198

41. Random Number Generator Two integers (from 1 to 30, inclusive) are chosen by a random number generator on a computer. What is the probability that (a) both numbers are even, (b) one number is even and one is odd, (c) both numbers are less than 10, and (d) the same number is chosen twice?

42. Random Number Generator Two integers (from 1 to 40, inclusive) are chosen by a random number generator on a computer. What is the probability that (a) both numbers are even, (b) one number is even and one is odd, (c) both numbers are less than 30, and (d) the same number is chosen twice?

43. Preparing for a Test A biology instructor gives her class a list of eight study problems, from which she will select five to be answered on an exam. A student knows how to solve six of the problems. Find the probability that the student will be able to answer all five questions on the exam.

44. Preparing for a Test A chemistry instructor gives his class a list of 20 study problems, from which he will select 10 to be answered on an exam. A student knows how to solve 15 of the problems. Find the probability that the student will be able to answer all 10 questions on the exam.

45. Drawing Cards from a Deck One card is selected at random from a standard deck of 52 playing cards. Find (a) the probability that the card is an even-numbered card, (b) the probability that the card is a heart or a diamond, and (c) the probability that the card is a nine or a face card.

46. Drawing Cards from a Deck Two cards are selected at random from a standard deck of 52 playing cards. Find the probability that two hearts are selected under each condition.

(a) The cards are drawn in sequence, with the first card being replaced and the deck reshuffled prior to the second drawing.

(b) The two cards are drawn consecutively, without replacement.

47. Game Show On a game show, you are given five different digits to arrange in the proper order to represent the price of a car. If you are correct, you win the car. Find the probability of winning under each condition.

(a) You must guess the position of each digit.

(b) You know the first digit but must guess the remaining four.

(c) You know the first and last digits but must guess the remaining three.

48. Game Show On a game show, you are given six different digits to arrange in the proper order to represent the price of a house. If you are correct, you win the house. Find the probability of winning under each condition.

(a) You must guess the position of each digit.

(b) You know the first digit but must guess the remaining five.

(c) You know the first and last digits but must guess the remaining four.

49. Letter Mix-Up Five letters and envelopes are addressed to five different people. The letters are randomly inserted into the envelopes. What is the probability that (a) exactly one will be inserted in the correct envelope and (b) at least one will be inserted in the correct envelope?

50. Payroll Mix-Up Three paychecks and envelopes are addressed to three different people. The paychecks get mixed up and are randomly inserted into the envelopes. What is the probability that (a) exactly one will be inserted in the correct envelope and (b) at least one will be inserted in the correct envelope?

51. Poker Hand Five cards are drawn from a standard deck of 52 cards. What is the probability of getting a full house? (A full house consists of three of one kind and two of another. For example, A-A-A-5-5 and K-K-K-10-10 are full houses.)

52. Poker Hand Five cards are drawn from a standard deck of 52 cards. What is the probability of getting a straight flush? (A straight flush consists of five cards that are in order and of the same suit. For example, A♥, 2♥, 3♥, 4♥, 5♥ and 10♠, J♠, Q♠, K♠, A♠ are straight flushes.)

53. Defective Units A shipment of 1000 compact disc players contains four defective units. A retail outlet has ordered 20 units.

(a) What is the probability that all 20 units are good?

(b) What is the probability that at least one unit is defective?

54. Defective Units A shipment of 12 stereos contains three defective units. Four of the units are shipped to a retail store. What is the probability that (a) all four units are good, (b) exactly two units are good, and (c) at least two units are good?

55. Cash Scholarship A high school is composed of 31% freshmen, 26% sophomores, 25% juniors, and 18% seniors. A single student is picked randomly by lottery for a cash scholarship. What is the probability that the scholarship winner is either a freshman, a sophomore, or a junior?

56. Houses for Sale by Region In 2006, about 1006% of houses for sale in the United States were in the northeast, about 1806% were in the midwest, about 4972% were in the south, and about 2216% were in the west. A house is selected at random from the houses for sale in the United States. What is the probability that the house is in the midwest or the south? *(Source: U.S. Census Bureau)*

57. Making a Sale A sales representative makes a sale at approximately one-third of the offices she calls on. On a given day, she goes to four offices. What is the probability that she will make a sale at (a) all four offices, (b) none of the offices, and (c) at least one office?

58. Making a Sale A sales representative makes a sale at approximately one-fourth of the businesses he calls on. On a given day, he goes to five businesses. What is the probability that he will make a sale at (a) all five businesses, (b) none of the businesses, and (c) at least one of the businesses?

59. A Boy or a Girl? Assume that the probability of the birth of a child of a particular gender is 50%. In a family with six children, what is the probability that (a) all six children are girls, (b) all six children are of the same gender, and (c) there is at least one girl?

60. A Boy or a Girl? Assume that the probability of the birth of a child of a particular gender is 50%. In a family with four children, what is the probability that (a) all four children are boys, (b) all four children are of the same gender, and (c) there is at least one boy?

Probability In Exercises 61–64, consider n independent trials of an experiment in which each trial has two possible outcomes, called success and failure. The probability of a success on each trial is p, and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k\,p^k\,q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of k successes in the n trials of the experiment.

61. A fair coin is tossed eight times. To find the probability of obtaining five heads, evaluate the term

$${}_8C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

62. The probability of a baseball player getting a hit during any given time at bat is $\frac{1}{5}$. To find the probability that the player gets four hits during the next 10 times at bat, evaluate the term

$${}_{10}C_4\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^6$$

in the expansion of $\left(\frac{1}{5} + \frac{4}{5}\right)^{10}$.

63. The probability of a sales representative making a sale with any one customer is $\frac{1}{4}$. The sales representative makes 10 contacts a day.To find the probability of making four sales, evaluate the term

$${}_{10}C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^6$$

in the expansion of $\left(\frac{1}{4}+\frac{3}{4}\right)^{10}$.

64. To find the probability that the baseball player in Exercise 62 makes five hits during the next 10 times at bat if the probability of a hit is $\frac{1}{3}$, evaluate the term

$${}_{10}C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5$$

in the expansion of $\left(\frac{1}{3}+\frac{2}{3}\right)^{10}$.

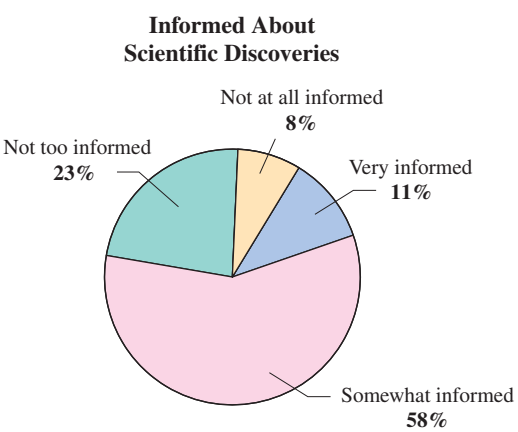
65. Use information from news sources, the Internet, or an original experiment to write an example of two mutually exclusive events.

66. Use information from news sources, the Internet, or an original experiment to write an example of two independent events.

Scientific Discoveries In Exercises 67 and 68, use the following information. In a survey, Americans were asked how well informed they were about new scientific discoveries. The results are shown in the pie graph. Two people from the survey were chosen at random. *(Source: Pew Internet & American Life Project)*

67. What is the probability that both people considered themselves *very informed* about scientific discoveries?

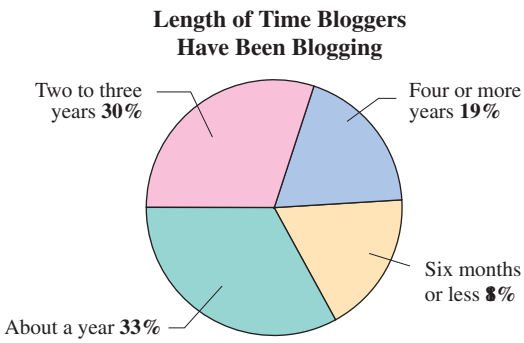
68. What is the probability that neither person felt *very informed* about scientific discoveries?



Blogging In Exercises 69 and 70, use the following information. In a survey, bloggers were asked how long they had been blogging. The results are shown in the pie graph. Two bloggers from the survey were chosen at random. *(Source: Pew Internet & American Life Project)*

69. What is the probability that neither blogger had been blogging for more than three years?

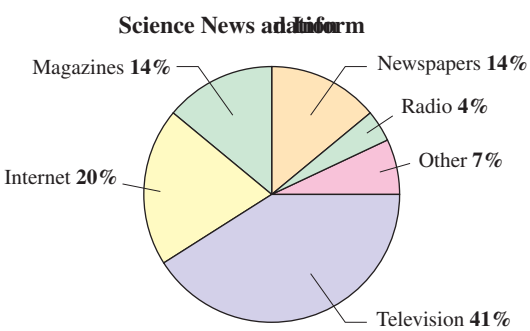
70. What is the probability that both bloggers had been blogging for more than one year?



Science News In Exercises 71 and 72, use the following information. A survey asked Americans what their main source of science news and information was. The results are shown in the pie graph. Three people from the survey were chosen at random. *(Source: Pew Internet & American Life Project)*

71. What is the probability that the main source of science news and information for all three people was either television or the Internet?

72. What is the probability that none of the three people considered newspapers or the Internet to be their main source of science news and information?



73. ***MAKE A DECISION*** For a given n and r, is it possible to order the following from least to greatest? Explain.

$$n!,\ {}_nP_r,\ \frac{n!}{r!},\ {}_nC_r$$

Section 7.7

Mathematical Induction

- Use mathematical induction to prove a formula.
- Find the sums of powers of integers.
- Find a formula for a finite sum.
- Use finite differences to find a linear or quadratic model.

Introduction

In this section, you will study a form of mathematical proof called **mathematical induction.** It is important that you clearly see the logical need for it, so take a closer look at the following problem.

$$S_1 = 1 = 1^2$$
$$S_2 = 1 + 3 = 2^2$$
$$S_3 = 1 + 3 + 5 = 3^2$$
$$S_4 = 1 + 3 + 5 + 7 = 4^2$$
$$S_5 = 1 + 3 + 5 + 7 + 9 = 5^2$$

Judging from the pattern formed by these first five sums, it appears that the sum of the first n odd integers is

$$S_n = 1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2.$$

Although this particular formula *is* valid, it is important for you to see that recognizing a pattern and then simply *jumping to the conclusion* that the pattern must be true for all values of n is *not* a logically valid method of proof.There are many examples in which a pattern appears to be developing for small values of n and then at some point the pattern fails.One of the most famous cases of this was the conjecture by the French mathematician Pierre de Fermat (1601–1665), who speculated that all numbers of the form

$$F_n = 2^{2^n} + 1, \quad n = 0, 1, 2, \ldots$$

are prime.For $n = 0, 1, 2, 3,$and 4, the conjecture is true.

$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65{,}537$$

The size of the next Fermat number $(F_5 = 4{,}294{,}967{,}297)$ is so great that it was difficult for Fermat to determine whether it was prime or not. However, another well-known mathematician, Leonhard Euler (1707–1783), later found the factorization

$$F_5 = 4{,}294{,}967{,}297$$
$$= 641(6{,}700{,}417)$$

which proved that F_5 is not prime and therefore Fermat's conjecture was false.

Just because a rule, pattern, or formula seems to work for several values of n, you cannot simply decide that it is valid for all values of n without going through a *legitimate proof.* Mathematical induction is one method of proof.

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If

1. P_1 is true, and
2. the truth of P_k implies the truth of P_{k+1} for every positive integer k, then P_n must be true for all positive integers n.

It is important to recognize that both parts of the Principle of Mathematical Induction are necessary. To apply the Principle of Mathematical Induction, you need to be able to determine the statement P_{k+1} for a given statement P_k.

Example 1 A Preliminary Example

Find P_{k+1} for the following.

a. $P_k{:}S_k = \dfrac{k^2(k+1)^2}{4}$

b. $P_k{:}S_k = 1 + 5 + 9 + \cdots + [4(k-1) - 3] + (4k - 3)$

c. $P_k{:}3^k \geq 2k + 1$

SOLUTION

a. $P_{k+1}{:}S_{k+1} = \dfrac{(k+1)^2(k+1+1)^2}{4}$ Replace k by $k + 1$.

$= \dfrac{(k+1)^2(k+2)^2}{4}$ Simplify.

b. $P_{k+1}{:}S_{k+1} = 1 + 5 + 9 + \cdots + \{4[(k+1) - 1] - 3\} + [4(k+1) - 3]$

$= 1 + 5 + 9 + \cdots + (4k - 3) + (4k + 1)$

c. $P_{k+1}{:}3^{k+1} \geq 2(k+1) + 1$

$3^{k+1} \geq 2k + 3$

✓CHECKPOINT 1

Find P_{k+1} for $P_k{:}S_k = k(2k + 1)$. ■

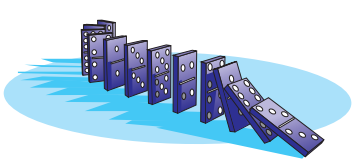


FIGURE 7.8

A well-known illustration used to explain why the Principle of Mathematical Induction works is the unending line of dominoes shown in Figure 78. If the line actually contains infinitely many dominoes, it is clear that you could not knock the entire line down by knocking down only *one domino* at a time. However, suppose it were true that each domino would knock down the next one as it fell. Then you could knock them all down simply by pushing the first one and starting a chain reaction. Mathematical induction works in the same way. If the truth of P_k implies the truth of P_{k+1} and if P_1 is true, the chain reaction proceeds as follows: P_1 implies P_2, P_2 implies P_3, P_3 implies P_4, and so on.

Using Mathematical Induction

Example 2 Using Mathematical Induction

Use mathematical induction to prove the formula

$$S_n = 1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$$

for all integers $n \geq 1$.

SOLUTION Mathematical induction consists of two distinct parts. First, you must show that the formula is true when $n = 1$.

1. When $n = 1$, the formula is valid, because

$$S_1 = 1 = 1^2.$$

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for *some* integer k. The second step is to use this assumption to prove that the formula is valid for the next integer, $k + 1$.

2. Assuming that the formula

$$S_k = 1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \cdots + (2k - 1) + [2(k + 1) - 1] \\ &= [1 + 3 + 5 + 7 + \cdots + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + (2k + 1) && \text{Group terms to form } S_k. \\ &= k^2 + 2k + 1 && \text{Replace } S_k \text{ by } k^2. \\ &= (k + 1)^2 && \text{Factor.} \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* integers $n \geq 1$.

STUDY TIP

When using mathematical induction to prove a *summation* formula (such as the one in Example 2), it is helpful to think of S_{k+1} as $S_{k+1} = S_k + a_{k+1}$, where a_{k+1} is the $(k + 1)$th term of the original sum.

✓CHECKPOINT 2

Use mathematical induction to prove the formula

$$S_n = 4 + 5 + 6 + 7 + \cdots + (n + 3) = \frac{n(n + 7)}{2}$$

for all integers $n \geq 1$. ■

It occasionally happens that a statement involving natural numbers is not true for the first $k - 1$ positive integers but is true for all values of $n \geq k$. In these instances, you use a slight variation of the Principle of Mathematical Induction in which you verify P_k rather than P_1. This variation is called the **Extended Principle of Mathematical Induction.** To see the validity of this variation, note from Figure 78 that all but the first $k - 1$ dominoes can be knocked down by knocking over the kth domino. This suggests that you can prove a statement P_n to be true for $n \geq k$ by showing that P_k is true and that P_k implies P_{k+1}. In Exercises 41-44 of this section, you are asked to apply this extension of mathematical induction.

Example 3 Using Mathematical Induction

Use mathematical induction to prove the formula

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 1$.

SOLUTION

1. When $n = 1$, the formula is valid, because

$$S_1 = 1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1(2)(3)}{6}.$$

2. Assuming that

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

you must show that

$$S_{k+1} = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

To do this, write the following.

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} \\ &= (1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{By assumption} \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{Add expressions.} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{Factor out } (k+1). \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} && \text{Simplify.} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} && \text{Completely factor.} \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* integers $n \geq 1$.

✓CHECKPOINT 3

Use mathematical induction to prove the formula

$$S_n = 3 + 6 + 9 + 12 + \cdots + 3n = \frac{3n(n+1)}{2}$$

for all integers $n \geq 1$. ■

When proving a formula by mathematical induction, the only statement that you *need* to verify is P_1. For the inequality in Example 4, you need to show that the inequality holds for $n = 1$ and $n = 2$ because in Step 2 of the mathematical induction, $k > 1$ (as opposed to $k \geq 1$) is the restriction on k.

Example 4 Proving an Inequality by Mathematical Induction

Prove that $n < 2^n$ for all positive integers n.

SOLUTION

1. For $n = 1$ and $n = 2$, the formula is true, because

$$1 < 2^1 \text{ and } 2 < 2^2.$$

2. Assuming that

$$k < 2^k$$

you need to show that $k + 1 < 2^{k+1}$. For $n = k$, you have

$$2^{k+1} = 2(2^k) > 2(k) = 2k. \qquad \text{By assumption}$$

Because $2k = k + k > k + 1$ for all $k > 1$, it follows that

$$2^{k+1} > 2k > k + 1$$

or

$$k + 1 < 2^{k+1}.$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that $n < 2^n$ for all positive integers n.

✓CHECKPOINT 4

Prove that $n! > 2n$ for all integers $n > 3$. ■

Sums of Powers of Integers

The formula in Example 3 is one of a collection of useful summation formulas. This and other formulas dealing with the sums of various powers of the first n positive integers are as follows.

Sums of Powers of Integers

1. $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n + 1)}{2}$

2. $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

3. $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$

4. $1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4 = \dfrac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}$

5. $1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5 = \dfrac{n^2(n + 1)^2(2n^2 + 2n - 1)}{12}$

Each of these formulas for sums can be proven by mathematical induction.

Example 5 **Finding a Sum of Powers of Integers**

Find $\sum_{n=1}^{7} n^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$.

SOLUTION Using the formula for the sum of the cubes of the first n positive integers, you obtain the following.

$$\begin{aligned}\sum_{n=1}^{7} n^3 &= 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 \\ &= \frac{7^2(7+1)^2}{4} \\ &= \frac{49(64)}{4} \\ &= 784\end{aligned}$$

Check this sum by adding the numbers, as shown below.

$$1 + 8 + 27 + 64 + 125 + 216 + 343 = 784$$

✓CHECKPOINT 5

Find $\sum_{n=1}^{10} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$. ■

Pattern Recognition

Although choosing a formula on the basis of a few observations does *not* guarantee the validity of the formula, pattern recognition *is* important. Once you have a pattern that you think works, you can try using mathematical induction to prove your formula.

Finding a Formula for the *n*th Term of a Sequence

To find a formula for the nth term of a sequence, consider the following guidelines.

1. Calculate the first several terms of the sequence. It is often a good idea to write the terms in both simplified and factored forms.
2. Try to find a recognizable pattern for the terms and write a formula for the nth term of the sequence. This is your *hypothesis* or *conjecture*. You might try computing one or two more terms in the sequence to test your hypothesis.
3. Use mathematical induction to prove your hypothesis.

Example 6 **Finding a Formula for a Finite Sum**

Find a formula for the following finite sum.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{n(n+1)}$$

SOLUTION Begin by writing out the first few sums.

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}$$

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3} = \frac{2}{2+1}$$

$$S_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{9}{12} = \frac{3}{4} = \frac{3}{3+1}$$

$$S_4 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{48}{60} = \frac{4}{5} = \frac{4}{4+1}$$

From this sequence, it appears that the formula for the kth sum is

$$S_k = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{k(k+1)}$$

$$= \frac{k}{k+1}.$$

To prove the validity of this hypothesis, use mathematical induction, as follows. Note that you have already verified the formula for $n = 1$, so you can begin by assuming that the formula is valid for $n = k$ and trying to show that it is valid for $n = k + 1$.

$$S_{k+1} = \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{By assumption}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} \qquad \text{Add fractions.}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} \qquad \text{Distributive Property}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} \qquad \text{Factor numerator.}$$

$$= \frac{k+1}{k+2} \qquad \text{Simplify.}$$

So, the hypothesis is valid.

✓CHECKPOINT 6

Find a formula for the following finite sum.

$$3 + 5 + 7 + 9 + \cdots + (2n+1)$$ ■

Finite Differences

The **first differences** of a sequence are found by subtracting consecutive terms. The **second differences** are found by subtracting consecutive first differences.The first and second differences of the sequence 3, 5, 8, 12, 17, 23, . . . are as follows.

n:	1	2	3	4	5	6
a_n:	3	5	8	12	17	23
First differences:		2	3	4	5	6
Second differences:			1	1	1	1

For this sequence, the second differences are all the same nonzero number.When this happens, the sequence has a perfect *quadratic* model.If the first differences are all the same nonzero number, the sequence has a *linear* model. That is, it is arithmetic.

Example 7 Finding an Appropriate Model

Decide whether the sequence 6, 12, 26, 48, 78, 116, . . . can be represented perfectly by a linear or a quadratic model.If so, find the model.

SOLUTION The first and second differences are as follows.

n:	1	2	3	4	5	6
a_n:	6	12	26	48	78	116
First differences:		6	14	22	30	38
Second differences:			8	8	8	8

For this sequence, the second differences are all the same. So, the sequence follows a quadratic model perfectly and you know that the model has the form

$$a_n = an^2 + bn + c.$$

By substituting 1, 2, and 3 for n, you can obtain a system of three linear equations in three variables.

$a_1 = a(1)^2 + b(1) + c = 6$ Substitute 1 for n.

$a_2 = a(2)^2 + b(2) + c = 12$ Substitute 2 for n.

$a_3 = a(3)^2 + b(3) + c = 26$ Substitute 3 for n.

You now have a system of three equations in a, b, and c.

$$\begin{cases} a + b + c = 6 & \text{Equation 1} \\ 4a + 2b + c = 12 & \text{Equation 2} \\ 9a + 3b + c = 26 & \text{Equation 3} \end{cases}$$

Using the techniques discussed in Chapter 5, you can find that the solution of this system is

$$a = 4,\ b = -6, \text{ and } c = 8.$$

So, the quadratic model is

$$a_n = 4n^2 - 6n + 8.$$

✓CHECKPOINT 7

Decide whether the sequence

3, 8, 17, 30, 47, 68, . . .

can be represented perfectly by a linear or a quadratic model.If so, find the model. ■

Example 8 Finding an Appropriate Model

Decide whether the sequence 2, 9, 16, 23, 30, 37, . . . can be represented perfectly by a linear or a quadratic model. If so, find the model.

SOLUTION The first differences are as follows.

n:	1	2	3	4	5	6
a_n:	2	9	16	23	30	37
First differences:		7	7	7	7	7

For this sequence, the first differences are all the same. So, the sequence follows a linear model perfectly and you know that the model has the form

$$a_n = an + b.$$

By substituting 1 and 2 for n, you can obtain a system of two linear equations in two variables.

$$a_1 = a(1) + b = 2 \qquad \text{Substitute 1 for } n.$$

$$a_2 = a(2) + b = 9 \qquad \text{Substitute 2 for } n.$$

You now have a system of two equations in a and b.

$$\begin{cases} a + b = 2 & \text{Equation 1} \\ 2a + b = 9 & \text{Equation 2} \end{cases}$$

Using the techniques discussed in Chapter 5, you can find that the solution of this system is $a = 7$ and $b = -5$. So, the linear model is

$$a_n = 7n - 5.$$

✓CHECKPOINT 8

Decide whether the sequence 6, 10, 14, 18, 22, 26, . . . can be represented perfectly by a linear or a quadratic model. If so, find the model. ■

CONCEPT CHECK

1. Determine whether the following statement is true or false. Explain your reasoning.

 The statement $n^n > n$ is true for all positive integers.

2. Let P_n be a statement involving the positive integer n. You know P_5 is true and the truth of P_k implies the truth of P_{k+1} for every positive integer k. For which integers n does this prove that P_n is true?

3. A sequence has a linear model. What is true about the second differences of this sequence? Explain.

4. Explain how you could use a sum of powers of integers formula to find the sum

 $$6^3 + 7^3 + 8^3 + \cdots + n^3$$

 where n is an integer greater than 8. (*Hint:* Think of the sum as a difference of two sums.)

Skills Review 7.7

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3, 0.4, 0.6, 0.7, and 7.1.

In Exercises 1–4, find the sum.

1. $\sum_{k=3}^{6} (2k - 3)$ **2.** $\sum_{j=1}^{5} (j^2 - j)$ **3.** $\sum_{k=2}^{5} \frac{1}{k}$ **4.** $\sum_{i=1}^{2} \left(1 + \frac{1}{i}\right)$

In Exercises 5–10, simplify the expression.

5. $\frac{2(k + 1) + 3}{5}$ **6.** $\frac{3(k + 1) - 2}{6}$ **7.** $2 \cdot 2^{2(k+1)}$

8. $\frac{3^{2k}}{3^{2(k+1)}}$ **9.** $\frac{k + 1}{k^2 + k}$ **10.** $\frac{\sqrt{32}}{\sqrt{50}}$

Exercises 7.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find P_{k+1} for the given P_k.

1. $P_k = \frac{5}{k(k + 1)}$ **2.** $P_k = \frac{3}{(k + 2)(k + 3)}$

3. $P_k = \frac{k^2(k - 1)^2}{4}$ **4.** $P_k = \frac{k^2}{2(k + 1)^2}$

In Exercises 5–18, use mathematical induction to prove the formula for every positive integer n.

5. $2 + 4 + 6 + 8 + \cdots + 2n = n(n + 1)$

6. $3 + 7 + 11 + 15 + \cdots + (4n - 1) = n(2n + 1)$

7. $2 + 7 + 12 + 17 + \cdots + (5n - 3) = \frac{n}{2}(5n - 1)$

8. $1 + 4 + 7 + 10 + \cdots + (3n - 2) = \frac{n}{2}(3n - 1)$

9. $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$

10. $2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1}) = 3^n - 1$

11. $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n + 1)}{2}$

12. $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}$

13. $\frac{1}{1(1 + 1)} + \frac{1}{2(2 + 1)} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$

14. $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{n}\right) = n + 1$

15. $\sum_{i=1}^{n} i^5 = \frac{n^2(n + 1)^2(2n^2 + 2n - 1)}{12}$

16. $\sum_{i=1}^{n} i^4 = \frac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}$

17. $\sum_{i=1}^{n} i(i + 1) = \frac{n(n + 1)(n + 2)}{3}$

18. $\sum_{i=1}^{n} \frac{1}{(2i - 1)(2i + 1)} = \frac{n}{2n + 1}$

In Exercises 19–28, find the sum using the formulas for the sums of powers of integers.

19. $\sum_{n=1}^{20} n$ **20.** $\sum_{n=1}^{50} n$

21. $\sum_{n=1}^{6} n^2$ **22.** $\sum_{n=1}^{10} n^3$

23. $\sum_{n=1}^{5} n^4$ **24.** $\sum_{n=1}^{8} n^5$

25. $\sum_{n=1}^{6} (n^2 - n)$ **26.** $\sum_{n=1}^{10} (n^3 - n^2)$

27. $\sum_{i=1}^{6} (6i - 8i^3)$ **28.** $\sum_{j=1}^{5} (2j^3 - j^2)$

In Exercises 29–34, find a formula for the sum of the first n terms of the sequence.

29. $1, 5, 9, 13, \ldots$ **30.** $25, 22, 19, 16, \ldots$

31. $1, \frac{9}{10}, \frac{81}{100}, \frac{729}{1000}, \ldots$ **32.** $3, -\frac{9}{2}, \frac{27}{4}, -\frac{81}{8}, \ldots$

33. $\frac{1}{4}, \frac{1}{12}, \frac{1}{24}, \frac{1}{40}, \ldots, \frac{1}{2n(n + 1)}, \ldots$

34. $\frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \frac{1}{5 \cdot 6}, \ldots, \frac{1}{(n + 1)(n + 2)}, \ldots$

In Exercises 35–40, determine whether the inequality is true for $n = 1$, 2, 3, and 4.

35. $n^2 < n^3$ **36.** $4n^4 \leq n^6$

37. $(n + 1)^2 \leq n^3$

38. $(n - 2)^3 < n^2$

39. $n - 1 \geq \dfrac{n + 1}{n^2}$

40. $\dfrac{(n - 1)^2}{n} < n! - n$

In Exercises 41–44, prove the inequality for the indicated integer values of n.

41. $n! > 2^n, \quad n \geq 4$

42. $\left(\frac{4}{3}\right)^n > n, \quad n \geq 7$

43. $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}, \quad n \geq 2$

44. $\left(\dfrac{x}{y}\right)^{n+1} < \left(\dfrac{x}{y}\right)^n, n \geq 1$ and $0 < x < y$

In Exercises 45–52, use mathematical induction to prove the given property for all positive integers n.

45. $(ab)^n = a^n b^n$

46. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

47. If $x_1 \neq 0, x_2 \neq 0, \ldots, x_n \neq 0$, then
$(x_1 x_2 x_3 \cdots x_n)^{-1} = x_1^{-1} x_2^{-1} x_3^{-1} \cdots x_n^{-1}$.

48. If $x_1 > 0, x_2 > 0, \ldots, x_n > 0$, then
$\ln(x_1 x_2 x_3 \cdots x_n) =$
$\ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_n$.

49. Generalized Distributive Law:
$x(y_1 + y_2 + \cdots + y_n) = xy_1 + xy_2 + \cdots + xy_n$

50. $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for all $n \geq 1$.

51. A factor of $(n^3 + 3n^2 + 2n)$ is 3.

52. A factor of $(2^{2n-1} + 3^{2n-1})$ is 5.

53. Writing In your own words, explain what is meant by a proof by mathematical induction.

54. Think About It What conclusion can be drawn from the given information about the sequence of statements P_n?

(a) P_3 is true and P_k implies P_{k+1}.

(b) $P_1, P_2, P_3, \ldots, P_{50}$ are all true.

(c) P_1, P_2, and P_3 are all true, but the truth of P_k does not imply the truth of P_{k+1}.

(d) P_2 is true and P_{2k} implies P_{2k+2}.

In Exercises 55–60, decide whether the sequence can be represented perfectly by a linear or a quadratic model. If so, find the model.

55. 5, 13, 21, 29, 37, 45, . . .

56. 2, 9, 16, 23, 30, 37, . . .

57. 6, 15, 30, 51, 78, 111, . . .

58. 0, 6, 16, 30, 48, 70, . . .

59. -2, 1, 6, 13, 22, 33, . . .

60. -1, 8, 23, 44, 71, 104, . . .

In Exercises 61–70, write the first five terms of the sequence, where $a_1 = f(1)$. Then calculate the first and second differences of the sequence. Does the sequence have either a linear model or a quadratic model? If so, find the model.

61. $f(1) = 0$
$a_n = a_{n-1} + 3$

62. $f(1) = 2$
$a_n = n - a_{n-1}$

63. $f(1) = 3$
$a_n = a_{n-1} - n$

64. $f(1) = -3$
$a_n = -2a_{n-1}$

65. $a_0 = 0$
$a_n = a_{n-1} + n$

66. $a_0 = 2$
$a_n = (a_{n-1})^2$

67. $f(1) = 2$
$a_n = a_{n-1} + 2$

68. $f(1) = 0$
$a_n = a_{n-1} + 2n$

69. $a_0 = 1$
$a_n = a_{n-1} + n^2$

70. $a_0 = 0$
$a_n = a_{n-1} - 1$

In Exercises 71–74, find a quadratic model for the sequence with the indicated terms.

71. $a_0 = 3, a_1 = 3, a_4 = 15$

72. $a_0 = 7, a_1 = 6, a_3 = 10$

73. $a_1 = 0, a_2 = 8, a_4 = 30$

74. $a_0 = -3, a_2 = -5, a_6 = -57$

75. *MAKE A DECISION: BRAKING DISTANCE* The table shows the distance d_n (in feet) required to stop a car from the moment its brakes are applied at a speed of n miles per hour on dry, level pavement. *(Source: The University of Oklahoma Police Department)*

Speed, n	20	30	40	50	60
Braking distance, d_n	25	57	101	158	227

(a) Let d_n be the nth term of a sequence. Find the first and second differences of the given values of d_n.

(d) Will a linear or quadratic model fit the data best? Algebraically find a model of this type using $n = 20$, 30, and 40.

(c) Use the *regression* feature of a graphing utility to find a model of the same type for all the data.

(d) Graph the models from parts (b) and (c) with a scatter plot of the data. How closely does each model fit the data points?

(e) When you drive a car, do you assume an equal risk for each 10 mile-per-hour increase in speed? Explain your reasoning.

Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 606.
Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 7.1

	Review Exercises
■ Use sequence notation to find the terms of a sequence.	*1–12*
■ Evaluate a factorial expression.	*13–16*

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$$

As a special case, zero factorial is defined as $0! = 1$.

■ Find the sum of a finite sequence and use summation notation to write the sum of a sequence.	*17–24*

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

■ Use a sequence to solve an application problem.	*25, 26*

Section 7.2

■ Recognize and find the nth term of an arithmetic sequence.	*27–36*
■ Find a formula for an arithmetic sequence.	*37–40*

$a_n = dn + c$, where $c = a_1 - d$

$a_{n+1} = a_n + d$

$a_n = a_1 + (n-1)d$

■ Find the nth partial sum of an arithmetic sequence.	*41–44*

$$S_n = \frac{n}{2}(a_1 + a_n)$$

■ Use an arithmetic sequence to solve an application problem.	*45, 46*

Section 7.3

■ Recognize and find the nth term of a geometric sequence.	*47–58*

$$a_n = a_1 r^{n-1}$$

■ Find the sum of a finite geometric sequence.	*59–62*

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right)$$

■ Find the sum of an infinite geometric series.	*63–66*

If $|r| < 1$, then $S = \sum_{i=0}^{\infty} a_1 r^i = \frac{a_1}{1 - r}$.

■ Use a geometric sequence to solve an application problem.	*67–70*

*Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

Section 7.4

	Review Exercises
■ Find a binomial coefficient.	*71–74*
Binomial Theorem: In the expansion of $(x + y)^n$, the coefficient of $x^{n-r}y^r$, the $(r + 1)$th term, is given by $${}_nC_r = \frac{n!}{(n - r)!r!}.$$	
■ Use Pascal's Triangle to determine a binomial coefficient.	*75, 76*
■ Expand a binomial using binomial coefficients and find a specific term in a binomial expansion.	*77–86*

Section 7.5

■ Use the Fundamental Counting Principle to solve counting problems.	*87–92*
■ Use permutations to solve counting problems.	*93–98*
$${}_nP_r = \frac{n!}{(n - r)!}$$	
■ Use combinations to solve counting problems.	*99, 100*
$${}_nC_r = \frac{n!}{(n - r)!r!}$$	

Section 7.6

■ Determine the sample space of an experiment and find the probability of an event.	*101–109, 112*
$$P(E) = \frac{n(E)}{n(S)}$$	
■ Find the probabilities of mutually exclusive events.	*103, 105*
A and B mutually exclusive: $P(A \cup B) = P(A) + P(B)$	
■ Find the probabilities of independent events.	*110, 111*
A and B not mutually exclusive: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ A and B independent: $P(A \text{ and } B) = P(A) \cdot P(B)$	
■ Find the probability of the complement of an event.	*105, 110–114*
$P(A') = 1 - P(A)$	

Section 7.7

■ Use mathematical induction to prove a formula.	*115–118*
■ Find the sums of powers of integers.	*119–122*
■ Find a formula for a finite sum.	*123, 124*
■ Use finite differences to find a linear or quadratic model.	*125–128*

Study Strategies

- **Studying Relationships Between Concepts** To reinforce your understanding of the concepts presented in this chapter, review how they are related.For instance, look back at the similarities and distinctions between concepts such as sequences and series, geometric and arithmetic sequences, combinations and binomial coefficients, combinations and permutations, counting methods and probability, and so on.
- **Using Mathematical Induction** Proofs can be complicated.When using mathematical induction, write out a plan to identify how you will meet the two conditions necessary for the proof.If you get lost in the proof, review the plan to get yourself back on track.Revise the plan if you need to.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, write the first five terms of the sequence. (Assume that n begins with 1.)

1. $a_n = 4n + 1$

2. $a_n = 3n - 2$

3. $a_n = \dfrac{n}{n+1}$

4. $a_n = \dfrac{3n-1}{n}$

5. $a_n = (-1)^{n+1}\dfrac{2^n}{n+1}$

6. $a_n = (-1)^n\dfrac{3^n-1}{3^n+1}$

In Exercises 7–12, write an expression for the *most apparent* nth term of the sequence. (Assume that n begins with 1.)

7. $1, 2, 4, 8, \ldots$

8. $4, -4, 4, -4, \ldots$

9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

10. $-\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots$

11. $3, -9, 27, -81, 243, \ldots$

12. $-2, 4, -8, 16, -32, \ldots$

In Exercises 13–16, evaluate the expression.

13. $\dfrac{16!}{12!}$

14. $\dfrac{24!}{22!}$

15. $\dfrac{4! \cdot 5!}{8!}$

16. $\dfrac{9! \cdot 6!}{6! \cdot 7!}$

In Exercises 17–20, find the sum.

17. $\sum_{i=1}^{6}(3i - 2)$

18. $\sum_{i=1}^{6}(4i + 5)$

19. $\sum_{k=1}^{6}(k - 1)(k + 2)$

20. $\sum_{k=1}^{8}k^2 + 1$

In Exercises 21–24, use summation notation to write the sum.

21. $\dfrac{3}{1+1} + \dfrac{3}{1+2} + \dfrac{3}{1+3} + \dfrac{3}{1+4} + \cdots + \dfrac{3}{1+12}$

22. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{60}$

23. $2 - 4 + 8 - 16 + 32 - 64 + \cdots + 8192$

24. $2 + 8 + 18 + 32 + \cdots + 98$

25. Compound Interest On your next birthday, you plan to deposit \$4000 in an account that earns 6% compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 4000\left(1 + \frac{006}{4}\right)^n, \quad n = 1, 2, 3, \ldots.$$

(a) Compute the first eight terms of the sequence.

(b) Find the balance in the account after 10 years by computing the 40th term of the sequence.

(c) Find the balance in the account after 30 years by computing the 120th term of the sequence.

26. Compound Interest A deposit of \$0,000 is made in an account that earns 8% compounded monthly. The balance in the account after n months is given by

$$A_n = 10{,}000\left(1 + \frac{008}{12}\right)^n, \quad n = 1, 2, 3, \ldots.$$

(a) Compute the first 10 terms of the sequence.

(b) Find the balance in the account after 10 years by computing the 120th term of the sequence.

In Exercises 27–30, determine whether the sequence is arithmetic. If it is, find the common difference.

27. $3, 8, 13, 18, \ldots$

28. $1, 3, 9, 27, \ldots$

29. $4, \frac{7}{2}, 3, \frac{5}{2}, 2, \ldots$

30. $64, 56, 48, 40, 32, \ldots$

In Exercises 31–36, write the first five terms of the arithmetic sequence.

31. $a_1 = 9, d = 5$

32. $a_1 = 8, d = -4$

33. $a_5 = 14, a_8 = 29$

34. $a_9 = 0, a_{15} = -9$

35. $a_1 = 24, a_{n+1} = a_n - 7$

36. $a_1 = 16, a_{n+1} = a_n + 4$

In Exercises 37–40, find a formula for a_n for the arithmetic sequence.

37. $a_1 = 2, d = 3$

38. $a_1 = 15, d = -3$

39. $2, 7, 12, 17, 22, \ldots$

40. $39, 33, 27, 21, 15, \ldots$

In Exercises 41–44, find the indicated nth partial sum of the arithmetic sequence.

41. $3, 9, 15, 21, 27, \ldots, \quad n = 8$

42. $-8, -4, 0, 4, 8, \ldots, n \quad = 12$

43. $a_1 = 9, a_{15} = 93, n = 15$

44. $a_1 = 27, a_{40} = 300, n = 20$

45. Job Offers You have job offers from two companies. The position with Company A has a salary of $2,500 for the first year with a guaranteed annual raise of $600 per year. The position with Company B has a salary of $28,000 for the first year with a guaranteed annual raise of $500 per year.

(a) What will your salary be during the sixth year of employment at Company A? at Company B?

(b) How much will Company A have paid you at the end of 6 years?

(c) How much will Company B have paid you at the end of 6 years?

46. Seating You are setting up 20 rows of chairs for a concert. There will be 20 chairs in the first row, 23 chairs in the second row, 26 chairs in the third row, and so on. How many chairs do you need?

In Exercises 47–50, determine whether the sequence is geometric. If it is, find its common ratio and write a formula for a_n.

47. $1, -3, 9, -27, \ldots$

48. $3, 9, 15, 21, \ldots$

49. $16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

50. $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}, \ldots$

In Exercises 51–54, write the first five terms of the geometric sequence.

51. $a_1 = 3, r = 2$

52. $a_1 = 2, r = e$

53. $a_1 = 10, r = -\frac{1}{5}$

54. $a_1 = 1, r = \frac{3}{4}$

In Exercises 55–58, find the indicated term of the geometric sequence.

55. $a_1 = 8, r = \frac{1}{2}$, 40th term

56. $a_1 = 100, r = -\frac{1}{10}$, 5th term

57. $a_2 = -100, a_5 = 125,$ 10th term

58. $a_1 = 10, a_3 = 11025,$ 19th term

In Exercises 59–62, find the sum of the finite geometric sequence.

59. $\sum_{n=1}^{10} 4(2^n)$

60. $\sum_{n=0}^{20} 2\left(\frac{2}{3}\right)^n$

61. $\sum_{i=1}^{15} 20(01\)^{i-1}$

62. $\sum_{i=1}^{30} 100(106\)^{i-1}$

In Exercises 63–66, find the sum of the infinite geometric series, if it exists.

63. $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

64. $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

65. $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$

66. $\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$

67. *MAKE A DECISION: COMPOUND INTEREST* You deposit $100 at the beginning of each month for 10 years in an account that pays 6% compounded monthly. The balance A at the end of 10 years is as follows.

$$A = 100\left(1 + \frac{006}{12}\right)^1 + \cdots + 100\left(1 + \frac{006}{12}\right)^{120}$$

(a) Find the balance after 10 years.

(b) Is the balance after 20 years twice what it is after 10 years? Explain your reasoning.

68. Compound Interest You deposit $300 in an account at the beginning of each month for 10 years. The account pays 6% compounded monthly. What will your balance be at the end of 10 years? Suppose instead that the interest were compounded continuously. What is your balance at the end of the 10 years?

69. Profit The annual profits a_n (in millions of dollars) for a pharmaceuticals company from 1990 to 2008 can be approximated by the model

$$a_n = 1653e^{\,015n}, \quad n = 0, 1, 2, 3, \ldots, 18$$

where n represents the year, with $n = 0$ corresponding to 1990.

(a) Construct a bar graph that represents the company's profits during the 19-year period.

(b) Use the formula for the sum of a finite geometric sequence to approximate the total profit earned during the 19-year period.

70. Sales The annual sales a_n (in millions of dollars) for a furniture company from 1990 to 2008 can be approximated by the model

$$a_n = 225e^{0135n}, \quad n = 0, 1, 2, 3, \ldots, 18$$

where n represents the year, with $n = 0$ corresponding to 1990. Use the formula for the sum of a finite geometric sequence to approximate the total sales during the 19-year period.

In Exercises 71–74, find the binomial coefficient.

71. ${}_7C_4$

72. ${}_{12}C_8$

73. $\binom{20}{20}$

74. $\binom{20}{10}$

In Exercises 75 and 76, evaluate using Pascal's Triangle.

75. ${}_{12}C_5$

76. $\binom{7}{5}$

In Exercises 77–82, use the Binomial Theorem to expand the expression. Simplify your answer.

77. $(x + 1)^6$

78. $(x - 1)^7$

79. $(3x + y)^5$

80. $(2x - y)^6$

81. $(x - 2y)^8$

82. $(x^2 + 4)^5$

In Exercises 83 and 84, find the specified nth term in the expansion of the binomial.

83. $(a - 3b)^5,\ n = 3$

84. $(2x + y)^{10},\ n = 8$

In Exercises 85 and 86, find the coefficient a of the term in the expansion of the binomial.

	Binomial	*Term*
85.	$(x + 2)^{12}$	ax^7
86.	$(2x - 5y)^9$	ax^4y^5

87. Computer Systems A customer in a computer store can choose one of six monitors, one of five keyboards, and one of seven computers. All of the choices are compatible. How many different systems can be chosen?

88. Chemistry Experiment To set up an experiment, a chemist can use one of two beakers, one of two graduated cylinders, one of six hot plates, and one of three stirring rods. How many different equipment setups are possible for the experiment?

89. Genetics A geneticist is using gel electrophoresis to analyze DNA. The DNA is divided into five samples. Each sample is treated with a different restriction enzyme and then injected into one of five wells that have been formed in a bed of gel. In how many orders can the five samples be injected into the wells?

90. Aircraft Boarding Ten people are boarding an aircraft. Two have tickets for first class and board before the other eight people. In how many orders can the ten people board the aircraft?

91. Four-Digit Numbers How many four-digit numbers can be formed under each condition?

(a) The leading digit cannot be 0, 1, or 2.

(b) The leading digit cannot be 0, 1, or 2, and no repetition of digits is allowed.

(c) The leading digit cannot be 0 and the number must be divisible by 2.

92. Five-Digit Numbers How many five-digit numbers can be formed under each condition?

(a) The leading digit cannot be 0.

(b) The leading digit cannot be 0 and no repetition of digits is allowed.

(c) The leading digit cannot be 0 and the number must be odd.

In Exercises 93–96, evaluate the expression.

93. ${}_9P_3$

94. ${}_{30}P_1$

95. ${}_8P_8$

96. ${}_{15}P_2$

In Exercises 97 and 98, find the number of distinguishable permutations of the group of letters.

97. E, N, T, R, E, P, R, E, N, E, U, R

98. P, H, O, T, O, S, Y, N, T, H, E, S, I, S

99. Zebrafish A researcher has a sample of 24 zebrafish embryos and must choose a sample of them for study. In how many ways can the researcher choose a sample of each size?

(a) 5 embryos

(b) 10 embryos

(c) 18 embryos

100. Job Applicants An employer interviews 10 people for five openings in the marketing department of a company. Seven of the 10 people are women. All 10 applicants are qualified and seven of them are women. In how many ways can the employer fill the five openings, given the following conditions?

(a) The people are chosen at random.

(b) Exactly three of those hired are women.

(c) All five of those hired are women.

In Exercises 101 and 102, determine the sample space for the experiment.

101. Guessing the last digit in a telephone number

102. Tossing four coins

Drawing a Card In Exercises 103 and 104, a card is selected from a standard deck of 52 cards. Find the probability of the event.

103. Getting a 10, jack, or queen

104. Getting a black card that is not a face card

105. Education In a high school graduating class of 72 students, 28 are on the honor roll. Of these, 18 are going on to college, and of the other 44 students, 12 are going on to college. A student is selected at random from the class. What is the probability that the person chosen is (a) going to college, (b) not going to college, and (c) on the honor roll, but not going to college?

106. Preparing for a Test An instructor gives a class a list of 12 study problems, from which eight will be selected to construct an exam. A student knows how to solve 10 of the problems. Find the probability that the student will be able to answer (a) all eight questions on the exam, (b) exactly seven questions on the exam, and (c) at least seven questions on the exam.

107. Game Show On a game show, you are given five different digits to arrange in the proper order to represent the price of a car. If you are correct, you win the car. Find the probability of winning under each condition.

(a) You must guess the position of each digit.

(b) You know the first two digits, but must guess the remaining three.

108. Payroll Mix-Up Four paychecks and envelopes are addressed to four different people. The paychecks get mixed up and are randomly inserted into the envelopes. What is the probability that (a) exactly one will be inserted in the correct envelope and (b) at least one will be inserted in the correct envelope?

109. Poker Hand Five cards are drawn from a standard deck of 52 playing cards. What is the probability of getting two pair? (Two pair consists of two of one kind, two of another kind, and one of a third kind. For example, K-K-9-9-5 is two pair.)

110. A Boy or a Girl? Assume that the probability of the birth of a child of a particular gender is 50%. In a family with five children, what is the probability that (a) all the children are boys, (b) all the children are of the same gender, and (c) there is at least one boy?

111. A Boy or a Girl? Assume that the probability of the birth of a child of a particular gender is 50%. In a family with seven children, what is the probability that (a) all the children are girls, (b) all the children are of the same gender, and (c) there is at least one girl?

112. Defective Units A shipment of 1000 radar detectors contains five defective units. A retail outlet has ordered 50 units.

(a) What is the probability that all 50 units are good?

(b) What is the probability that at least one unit is defective?

In Exercises 113 and 114, you are given the probability that an event will happen. Find the probability that the event will not happen.

113. $p = 057$ **114.** $p = 023$

In Exercises 115–118, use mathematical induction to prove the formula for every positive integer n.

115. $1 + 4 + \cdots + (3n - 2) = \frac{n}{2}(3n - 1)$

116. $1 + \frac{3}{2} + 2 + \frac{5}{2} + \cdots + \frac{1}{2}(n + 1) = \frac{n}{4}(n + 3)$

117. $\sum_{i=0}^{n-1} ar^i = \frac{a(1 - r^n)}{1 - r}$

118. $\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2}[2a + (n - 1)d]$

In Exercises 119–122, find the sum using the formulas for the sums of powers of integers.

119. $\sum_{n=1}^{30} n$ **120.** $\sum_{n=1}^{5} n^2$

121. $\sum_{n=1}^{8} n^3$ **122.** $\sum_{n=1}^{6} (n^2 + n)$

In Exercises 123 and 124, find a formula for the sum of the first n terms of the sequence.

123. $1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}, \ldots$

124. $5, -\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}, \ldots$

In Exercises 125 and 126, decide whether the sequence can be represented perfectly by a linear or a quadratic model. If so, find the model.

125. 5, 10, 15, 25, 30, 35, . . .

126. 1, 6, 13, 22, 33, 46, . . .

MAKE A DECISION In Exercises 127 and 128, write the first five terms of the sequence, where $a_1 = f(1)$. Then calculate the first and second differences of the sequence. Does the sequence have either a linear model or a quadratic model? If so, find the model.

127. $a_0 = 2;\ a_n = a_{n-1} + n$

128. $f(1) = 1;\ a_n = a_{n-1} + 2n$

Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, write the first five terms of the sequence. (Assume that n begins with 1.)

1. $a_n = 2n + 1$ **2.** $a_n = (-1)^n n^2$ **3.** $a_n = (n + 1)!$ **4.** $a_n = \left(\frac{1}{2}\right)^n$

In Exercises 5–8, determine whether the sequence is arithmetic, geometric, or neither. If possible, find its common difference or common ratio.

5. $a_n = 5n - 2$ **6.** $a_n = 2^n + 2$ **7.** $a_n = 5(2^n)$ **8.** $a_n = 3n^2 - 2$

In Exercises 9–11, find the sum.

9. $\sum_{n=1}^{50} (2n + 1)$ **10.** $\sum_{n=1}^{20} 3\left(\frac{3}{2}\right)^n$ **11.** $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

12. A deposit of $8000 is made in an account that pays 5% compounded monthly. The balance in the account after n months is given by

$$A = 8000\left(1 + \frac{005}{12}\right)^n, \quad n = 1, 2, 3, \ldots.$$

Find the balance in the account after 10 years.

13. Use the Binomial Theorem to expand $(2x - 3)^5$.

14. Determine the coefficient of the x^7 term in the expansion of $(x - 3)^{12}$.

In Exercises 15 and 16, evaluate each expression.

15. (a) ${}_9P_2$ (b) ${}_{70}P_3$ **16.** (a) ${}_{11}C_4$ (b) ${}_{66}C_5$

17. A customer in an electronics store can choose one of five CD players, one of six speaker systems, and one of four satellite radios to design a sound system. How many different systems can be designed?

18. An ecology quiz consists of 12 multiple choice questions with four possible answers for each question. In how many ways can the quiz be completed? (Assume that no question is omitted.)

19. How many different five-digit numbers can be formed if the leading digit cannot be 0 and the number must be odd?

20. In how many ways can eight people sit in an eight-passenger van?

21. In how many ways can three toppings for a pizza be selected from a list of 12 toppings?

22. You roll three six-sided dice. What is the probability that you roll a five on each die?

23. The weather report calls for 55% chance of rain. According to this report, what is the probability that it will not rain?

24. Use mathematical induction to prove the following formula.

$$5 + 10 + 15 + \cdots + 5n = \frac{5n(n + 1)}{2}$$

Cumulative Test: Chapters 5–7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, solve the system of equations using the indicated method.

1. *Elimination*

$$\begin{cases} 2x - 3y + z = 18 \\ 3x - 2z = -4 \\ x - y + 3z = 20 \end{cases}$$

2. *Matrices*

$$\begin{cases} 3x - 4y + 2z = -32 \\ 2x + 3y = 8 \\ y - 3z = 19 \end{cases}$$

In Exercises 3 and 4, sketch the graph of the solution set of the system of inequalities.

3. $\begin{cases} 2x + y \geq -3 \\ x - 3y \leq 2 \end{cases}$

4. $\begin{cases} 3x - 2y \geq 6 \\ 4x + 7y < 42 \end{cases}$

In Exercises 5–8, use the matrices at the left to perform the indicated matrix operation.

5. $2A - C$ **6.** AB **7.** BD **8.** A^{-1}

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 0 & -4 \\ 3 & 0 & 1 \\ 2 & -1 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$$

Matrices for 5–9

9. Find the determinants (if possible) of the matrices A, B, C, and D at the left.

10. Find the point of equilibrium for a system with demand equation

$p = 80 - 0004x$

and supply equation

$p = 35 + 0011x$.

$$A = \begin{bmatrix} 1000 & 3000 & 2000 \\ 2000 & 4000 & 5000 \\ 3000 & 1000 & 1000 \end{bmatrix}$$

$B = [\$5 \quad \$2 \quad \$7 \quad]$

Matrices for 11

11. A factory produces three different models of a product, which are shipped to three different warehouses. The number of units of model i that are shipped to warehouse j is represented by a_{ij} in matrix A at the left. The price per unit for each model is represented by matrix B. Find the product BA and state what each entry of the product represents.

12. Write the first five terms of the sequence given by $a_n = 7n - 3$.

13. Write the first five terms of the geometric sequence for which $a_1 = 2$ and $r = \frac{1}{3}$.

14. Find the sum $\sum_{i=0}^{\infty} 13 \left(\frac{1}{10}\right)^{i-1}$

15. Use the Binomial Theorem to expand $(2x + 3y)^6$.

16. In how many ways can the letters in MATRIX be arranged?

17. A shipment of 2500 microwave ovens contains 20 defective units. You order three units from the shipment. What is the probability that all three are good?

18. On a game show, the digits 7, 8, and 9 must be arranged in the proper order to form the price of an appliance. If the digits are arranged correctly, the contestant wins the appliance. What is the probability of winning if the contestant knows that the price is at least \$800?

19. Write the first five terms of the sequence for which $a_1 = 2$ and $a_n = a_{n-1} + n$. Then calculate the first and second differences of the sequence. Does the sequence have a linear model, a quadratic model, or neither?

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A An Introduction to Graphing Utilities

Graphing utilities such as graphing calculators and computers with graphing software are very valuable tools for visualizing mathematical principles, verifying solutions to equations, exploring mathematical ideas, and developing mathematical models. Although graphing utilities are extremely helpful in learning mathematics, their use does not mean that learning algebra is any less important. In fact, the combination of knowledge of mathematics and the use of graphing utilities enables you to explore mathematics more easily and to a greater depth. If you are using a graphing utility in this course, it is up to you to learn its capabilities and to practice using this tool to enhance your mathematical learning.

In this text, there are many opportunities to use a graphing utility, some of which are described below.

Uses of a Graphing Utility

A graphing utility can be used to

- check or validate answers to problems obtained using algebraic methods.
- discover and explore algebraic properties, rules, and concepts.
- graph functions and approximate solutions of equations involving functions.
- efficiently perform complicated mathematical procedures such as those found in many real-life applications.
- find mathematical models for sets of data.

In this appendix, the features of graphing utilities are discussed from a generic perspective. To learn how to use the features of a specific graphing utility, consult your user's manual or the website for this text, found at *college.hmco.com/info/larsonapplied*, where you will find specific keystrokes for most graphing utilities. Your college library may also have a video on how to use your graphing utility.

The Equation Editor

Many graphing utilities are designed to act as "function graphers." In this course, functions and their graphs are studied in detail. A function can be thought of as a rule that describes the relationship between two variables. These rules are frequently written in terms of x and y. For example, the equation $y = 3x + 5$ represents y as a function of x.

Many graphing utilities have an *equation editor* that requires that an equation be written in "$y =$" form in order to be entered, as shown in Figure A.1. (You should note that your *equation editor* screen may not look like the screen shown in Figure A.1.)

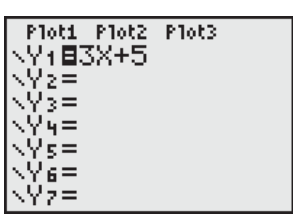


FIGURE A.1

Table Feature

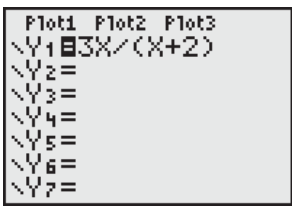


FIGURE A.2

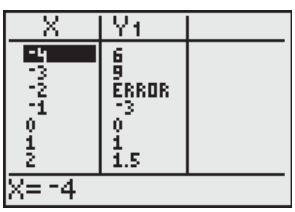


FIGURE A.3

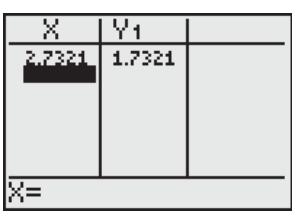


FIGURE A.4

Most graphing utilities are capable of displaying a table of values with x-values and one or more corresponding y-values. These tables can be used to check solutions of an equation and to generate ordered pairs to assist in graphing an equation by hand.

To use the *table* feature, enter an equation in the *equation editor* in "$y =$" form. The table may have a setup screen, which allows you to select the starting x-value and the table step or x-increment. You may then have the option of automatically generating values for x and y or building your own table using *ask* mode. In *ask* mode, you enter a value for x and the graphing utility displays the y-value.

For example, enter the equation

$$y = \frac{3x}{x + 2}$$

in the *equation editor*, as shown in Figure A2. In the table setup screen, set the table to start at $x = -4$ and set the table step to 1. When you view the table, notice that the first x-value is -4 and each value after it increases by 1. Also notice that the Y_1 column gives the resulting y-value for each x-value, as shown in Figure A3. The table shows that the y-value when $x = -2$ is ERROR. This means that the equation is undefined when $x = -2$.

With the same equation in the *equation editor,* set the independent variable in the table to *ask* mode. In this mode, you do not need to set the starting x-value or the table step, because you are entering any value you choose for x. You may enter any real value for x—integers, fractions, decimals, irrational numbers, and so forth. If you enter $x = 1 + \sqrt{3}$, the graphing utility may rewrite the number as a decimal approximation, as shown in Figure A4. You can continue to build your own table by entering additional x-values in order to generate y-values.

If you have several equations in the *equation editor*, the table may generate y-values for each equation.

Creating a Viewing Window

A **viewing window** for a graph is a rectangular portion of the coordinate plane. A viewing window is determined by the following six values.

Xmin = the smallest value of x
Xmax = the largest value of x
Xscl = the number of units per tick mark on the x-axis
Ymin = the smallest value of y
Ymax = the largest value of y
Yscl = the number of units per tick mark on the y-axis

When you enter these six values in a graphing utility, you are setting the viewing window. On some graphing utilities, there is a seventh value on the viewing window labeled Xres. This sets the pixel resolution. For instance, when Xres = 1, functions are evaluated and graphed at each pixel on the x-axis. Some graphing utilities have a standard viewing window, as shown in Figure A5.

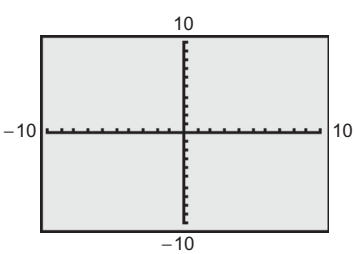
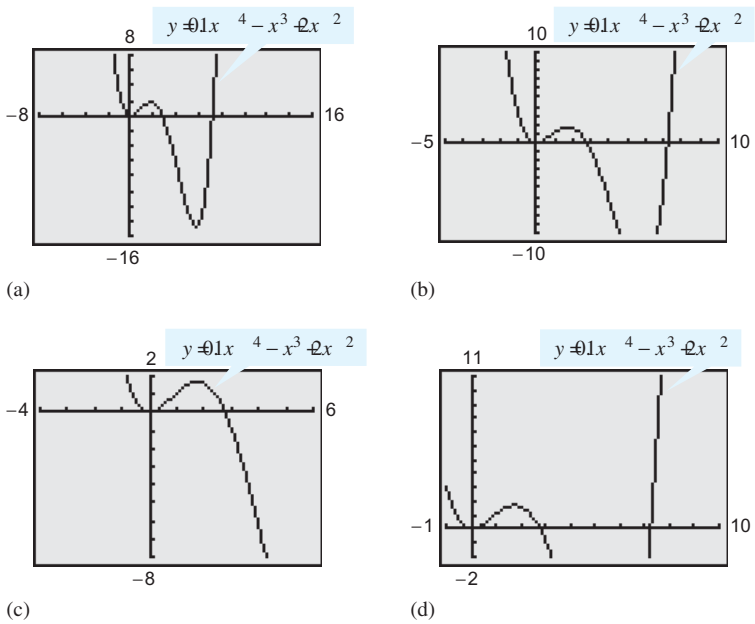


FIGURE A.5

By choosing different viewing windows for a graph, it is possible to obtain very different impressions of the graph's shape. For instance, Figure A6 shows four different viewing windows for the graph of

$$y = 0.1x^4 - x^3 + 2x^2.$$

Of these, the view shown in part (a) is the most complete.



(a) (b) (c) (d)

FIGURE A.6

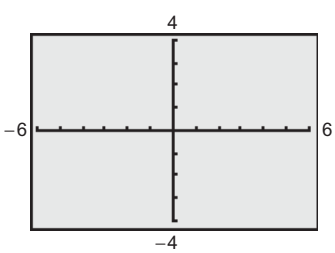


FIGURE A.7

On most graphing utilities, the display screen is two-thirds as high as it is wide. On such screens, you can obtain a graph with a true geometric perspective by using a **square setting** in which

$$\frac{\text{Ymax} - \text{Ymin}}{\text{Xmax} - \text{Xmin}} = \frac{2}{3}.$$

One such setting is shown in Figure A7. Notice that the x and y tick marks are equally spaced on a square setting, but not on a standard setting.

To see how the viewing window affects the geometric perspective, graph the semicircles $y_1 = \sqrt{9 - x^2}$ and $y_2 = -\sqrt{9 - x^2}$ using a standard viewing window. Then graph y_1 and y_2 using a square viewing window. Note the difference in the shapes of the circles.

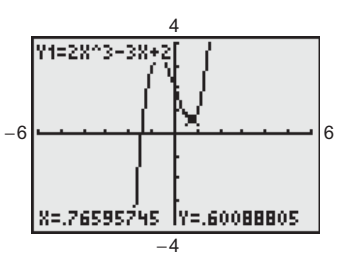


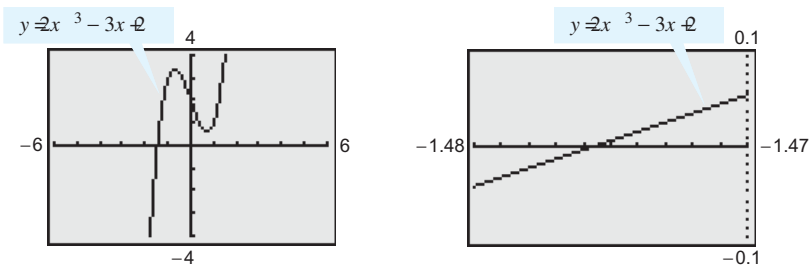
FIGURE A.8

Zoom and Trace Features

When you graph an equation, you can move from point to point along its graph using the *trace* feature. As you trace the graph, the coordinates of each point are displayed, as shown in Figure A8. The *trace* feature combined with the *zoom* feature enables you to obtain better and better approximations of desired points on a graph.

For instance, you can use the *zoom* feature of a graphing utility to approximate the x-intercept(s) of a graph. Suppose you want to approximate the

x-intercept(s) of the graph of $y = 2x^3 - 3x + 2$. Begin by graphing the equation, as shown in Figure A9(a).From the viewing window shown, the graph appears to have only one x-intercept.This intercept lies between -2and -1.By zooming in on the intercept, you can improve the approximation, as shown in Figure A9(b).To three decimal places, the solution is $x \approx -1476$.



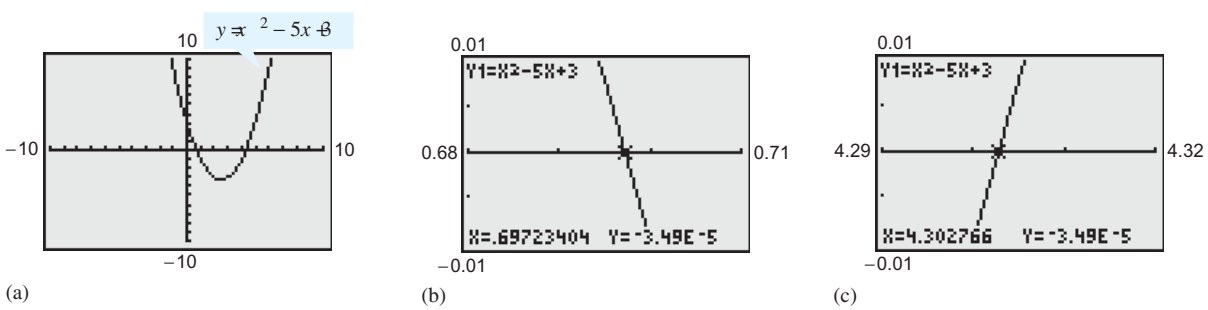
(a) (b)

FIGURE A.9

Here are some suggestions for using the *zoom* feature.

1. With each successive zoom-in, adjust the x-scale so that the viewing window shows at least one tick mark on each side of the x-intercept.
2. The error in your approximation will be less than the distance between two tick marks.
3. The *trace* feature can usually be used to add one more decimal place of accuracy without changing the viewing window.

Figure A10(a) shows the graph of $y = x^2 - 5x + 3$. Figures A10(b) and A10(c) show "zoom-in views" of the two x-intercepts.Using these views and the *trace* feature, you can approximate the x-intercepts to be $x \approx 0697$ and $x \approx 4303$.



(a) (b) (c)

FIGURE A.10

Zero or Root Feature

Using the *zero* or *root* feature, you can find the real zeros of functions of the various types studied in this text—polynomial, exponential, and logarithmic. To find the zeros of a function such as $f(x) = \frac{3}{4}x - 2$, first enter the function as $y_1 = \frac{3}{4}x - 2$ in the *equation editor*. Then use the *zero* or *root* feature, which may require entering lower and upper bound estimates of the zero, as shown in Figures A11(a) and A11(b).

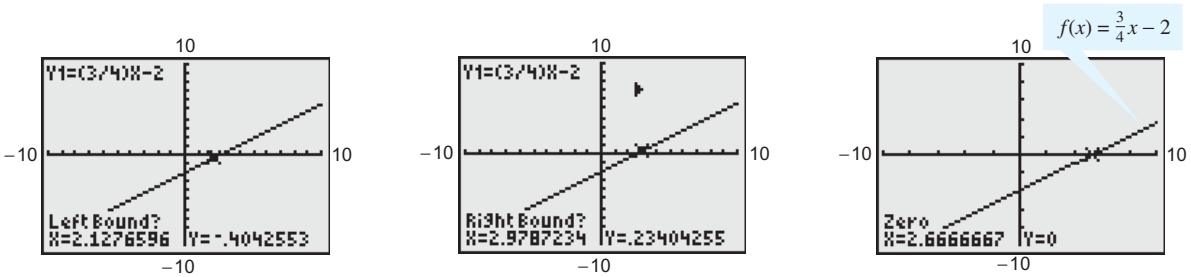


FIGURE A.11

In Figure A11(c), you can see that the zero is $x = 26666667 \approx 2\frac{2}{3}$.

Intersect Feature

To find the point(s) of intersection of two graphs, you can use the *intersect* feature.For instance, to find the point(s) of intersection of the graphs of

$$y_1 = -x + 2 \quad \text{and} \quad y_2 = x + 4$$

enter these two functions in the *equation editor* and use the *intersect* feature, as shown in Figure A12.

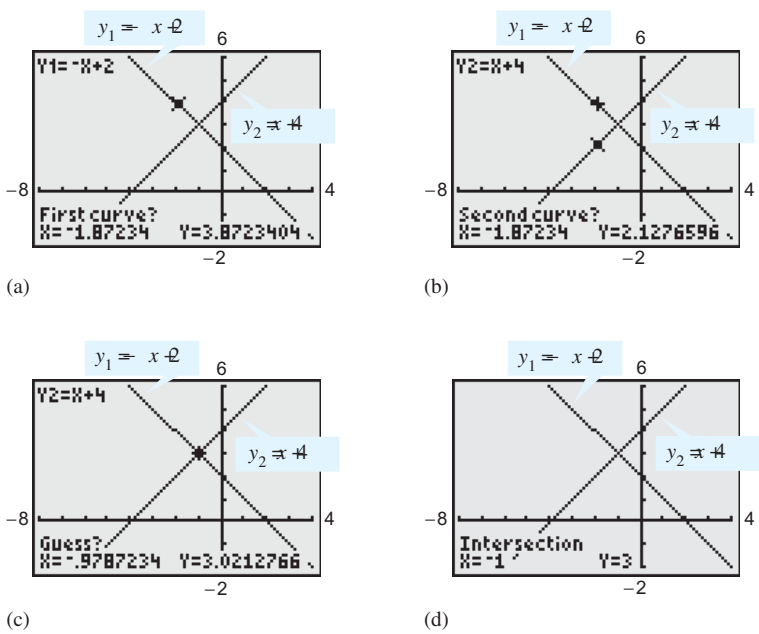


FIGURE A.12

From Figure A12(d), you can see that the point of intersection is $(-1, 3)$.

Regression Feature

Throughout the text, you are asked to use the *regression* feature of a graphing utility to find models for sets of data. Most graphing utilities have built-in regression programs for the following.

Regression	*Form of Model*
Linear	$y = ax + b$ or $y = a + bx$
Quadratic	$y = ax^2 + bx + c$
Cubic	$y = ax^3 + bx^2 + cx + d$
Quartic	$y = ax^4 + bx^3 + cx^2 + dx + e$
Logarithmic	$y = a + b\ln(x)$
Exponential	$y = ab^x$
Power	$y = ax^b$
Logistic	$y = \dfrac{c}{1 + ae^{-bx}}$
Sine	$y = a\sin(bx + c) + d$

For example, you can find a linear model for the retail sales y (in billions of dollars) of health and personal care stores in the United States from 1999 to 2005 using the data shown in the table. *(Source: U.S. Census Bureau)*

Year	1999	2000	2001	2002	2003	2004	2005
Sales, y	1428	1554	1667	1801	1922	1986	2084

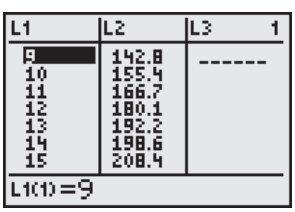


FIGURE A.13

First let x represent the year, with $x = 9$ corresponding to 1999, and enter the data in the *list editor,* as shown in Figure A13.Note that the list in the first column contains the years and the list in the second column contains the retail sales of health and personal care stores that correspond to the years. Now use your graphing utility's *linear regression* feature to obtain the coefficients a and b for the model $y = ax + b$, as shown in Figure A14.So, a linear model for the data is given by

$$y = 1103x + 454.$$

STUDY TIP

In this text, when regression models are found, the number of decimal places in the constant term of the model is the same as the number of decimal places in the data, and the number of decimal places increases by one for terms of increasing powers of the independent variable.

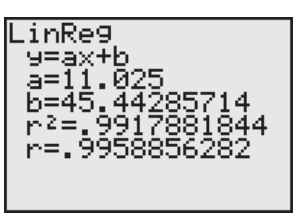


FIGURE A.14

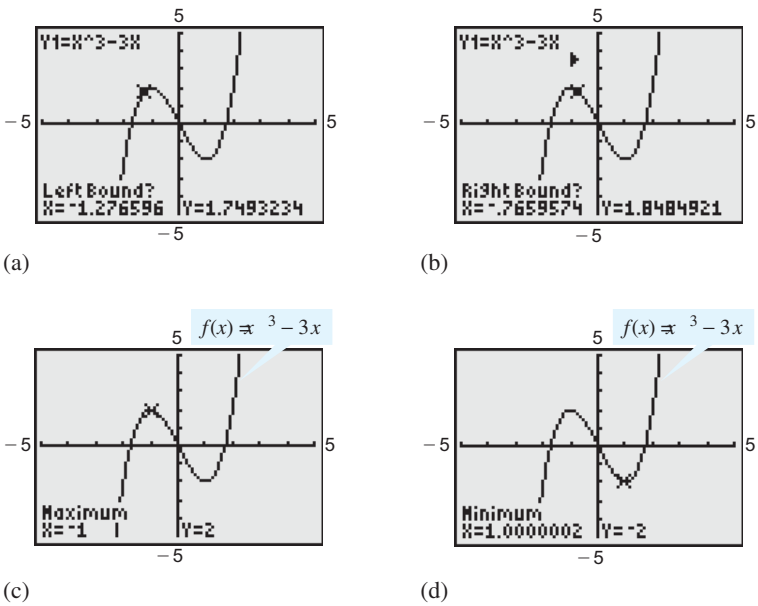
When you run some regression programs, you may obtain an "r-value," which gives a measure of how well the model fits the data.The closer the value of $|r|$ is to 1, the better the fit.For the data in the table above, you can see from Figure A14 that $r \approx 0996$, which implies that the model is a very good fit for the data.

Maximum and Minimum Features

The *maximum* and *minimum* features of a graphing utility find the relative extrema of a function.To find the relative maximum of a function such as

$$f(x) = x^3 - 3x$$

first enter the function as $y_1 = x^3 - 3x$ in the *equation editor*. Then use the *maximum* feature, which may require entering lower and upper bound estimates for the maximum, as shown in Figures A15(a) and A15(b).



(a) (b) (c) (d)

FIGURE A.15

In Figure A15(c), you can see that the relative maximum occurs at $(-1, 2)$.The relative minimum of the function can be found in a similar manner.From Figure A15(d), you can see that the relative minimum occurs at $(1, -2)$.

B Conic Sections

B.1 Conic Sections

- Recognize the four basic conics: circles, parabolas, ellipses, and hyperbolas.
- Recognize, graph, and write equations of parabolas (vertex at origin).
- Recognize, graph, and write equations of ellipses (center at origin).
- Recognize, graph, and write equations of hyperbolas (center at origin).

Introduction to Conic Sections

Conic sections were discovered during the classical Greek period, which lasted from 600 to 300 B.C. By the beginning of the Alexandrian period, enough was known of conics for Apollonius (262–190 B.C.) to produce an eight-volume work on the subject.

This early Greek study was largely concerned with the geometric properties of conics. It was not until the early seventeenth century that the broad applicability of conics became apparent.

A **conic section** (or simply **conic**) can be described as the intersection of a plane and a double-napped cone. Notice in Figure B.1 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure B.2.

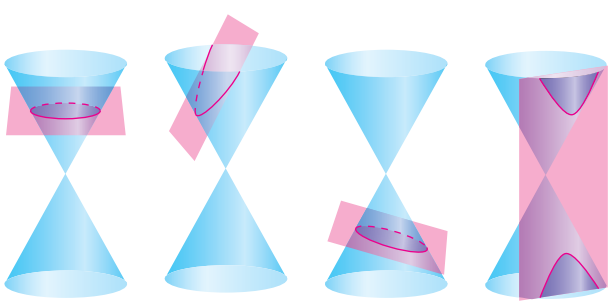


FIGURE B.1 Conic Sections

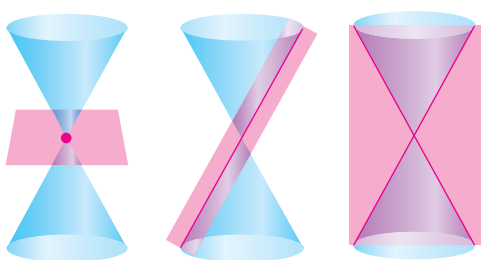


FIGURE B.2 Degenerate Conics

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach in which each of the conics is defined as a **locus,** or collection, of points satisfying a certain geometric property. For example, in Section 2.1 you saw how the definition of a circle as *the collection of all points* (x, y) *that are equidistant from a fixed point* (h, k) led easily to the standard equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$.

You will restrict your study of conics in this section to parabolas with vertices at the origin and ellipses and hyperbolas with centers at the origin. In the following section, you will look at the more general cases.

Parabolas

In Section 31 you determined that the graph of the quadratic function given by

$$f(x) = ax^2 + bx + c$$

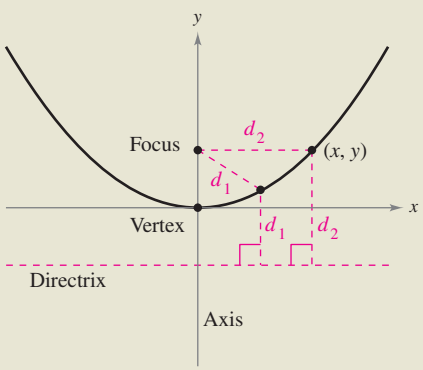
is a parabola that opens upward or downward. The following definition of a parabola is more general in the sense that it is independent of the orientation of the parabola.

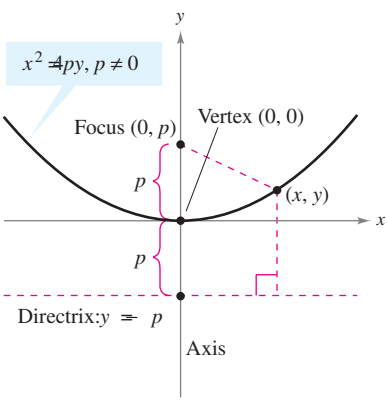
STUDY TIP

Note that the term *parabola* is a technical term used in mathematics and does not simply refer to *any* U-shaped curve.

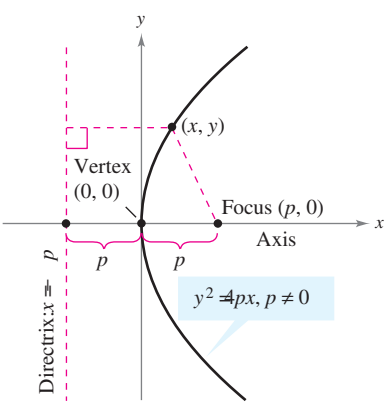
Definition of a Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line called the **directrix** and a fixed point called the **focus** (not on the line). The midpoint between the focus and the directrix is called the **vertex,** and the line passing through the focus and the vertex is called the **axis** of the parabola.





Parabola with vertical axis



Parabola with horizontal axis

FIGURE B.3

Using this definition, you can derive the following standard form of the equation of a parabola.

Standard Equation of a Parabola (Vertex at Origin)

The **standard form of the equation of a parabola** with vertex at $(0, 0)$ and directrix $y = -p$ is given by

$$x^2 = 4py, \quad p \neq 0.$$ Vertical axis

For directrix $x = -p$, the equation is given by

$$y^2 = 4px, \quad p \neq 0.$$ Horizontal axis

The focus is on the axis p units (directed distance) from the vertex. See Figure B3.

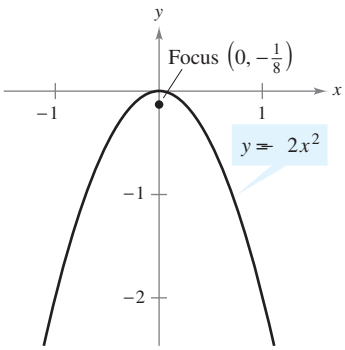


FIGURE B.4

✓CHECKPOINT 1

Find the focus of the parabola whose equation is $y^2 = 2x$. ■

Example 1 Finding the Focus of a Parabola

Find the focus of the parabola whose equation is $y = -2x^2$.

SOLUTION Because the squared term in the equation involves x, you know that the axis is vertical, and the equation is of the form

$x^2 = 4py.$ Standard form, vertical axis

You can write the original equation in this form as follows.

$-2x^2 = y$ Write original equation.

$x^2 = -\frac{1}{2}y$ Divide each side by -2.

$x^2 = 4\left(-\frac{1}{8}\right)y$ Write in standard form.

So, $p = -\frac{1}{8}$. Because p is negative, the parabola opens downward and the focus of the parabola is $(0, p) = \left(0, -\frac{1}{8}\right)$, as shown in Figure B4.

Example 2 Finding the Standard Equation of a Parabola

Write the standard form of the equation of the parabola with vertex at the origin and focus at $(2, 0)$.

SOLUTION The axis of the parabola is horizontal, passing through $(0, 0)$ and $(2, 0)$, as shown in Figure B5.So, the standard form is

$y^2 = 4px.$ Standard form, horizontal axis

Because the focus is $p = 2$ units from the vertex, the equation is

$y^2 = 4(2)x$ Standard form

$y^2 = 8x.$

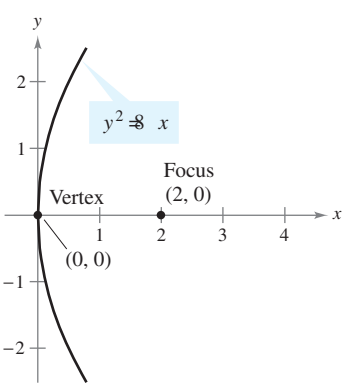


FIGURE B.5

✓CHECKPOINT 2

Write the standard form of the equation of the parabola with vertex at the origin and focus at $(0, 2)$. ■

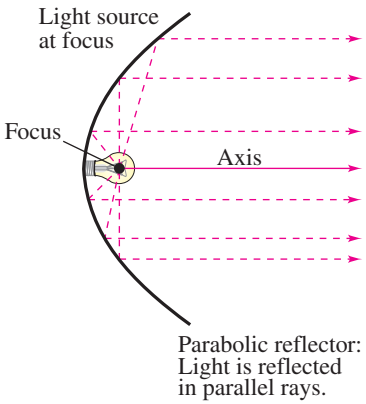


FIGURE B.6

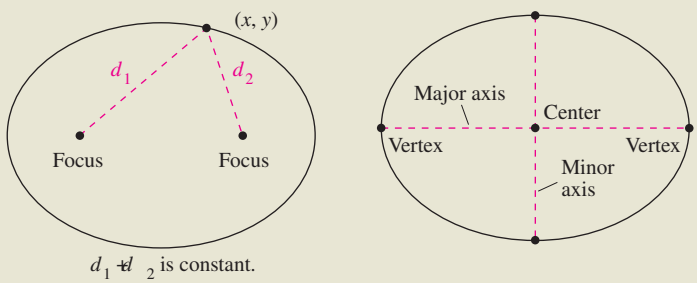
Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola about its axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light rays emanating from the focus of the parabolic reflector used in a flashlight are all reflected parallel to one another, as shown in Figure B6.

Ellipses

The second basic type of conic is called an **ellipse,** and it is defined as follows.

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) in a plane the sum of whose distances from two distinct fixed points, called **foci,** is constant.



The line through the foci intersects the ellipse at two points, called the **vertices.** The chord joining the vertices is called the **major axis,** and its midpoint is called the **center** of the ellipse. The chord perpendicular to the major axis at the center is called the **minor axis** of the ellipse.

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure B7. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

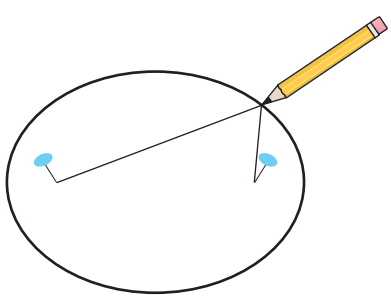


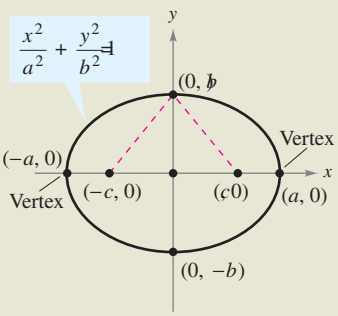
FIGURE B.7

The standard form of the equation of an ellipse takes one of two forms, depending on whether the major axis is horizontal or vertical.

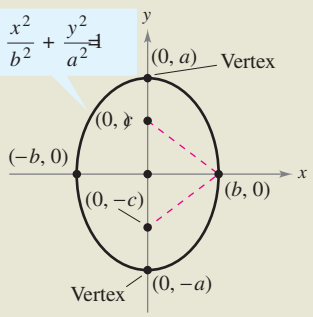
Standard Equation of an Ellipse (Center at Origin)

The **standard form of the equation of an ellipse** with the center at the origin and major and minor axes of lengths $2a$ and $2b$ (where $0 < b < a$) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$



Major axis is horizontal.
Minor axis is vertical.



Major axis is vertical.
Minor axis is horizontal.

The vertices and foci lie on the major axis, a and c units, respectively, from the center.Moreover, a, b,and care related by the equation

$$c^2 = a^2 - b^2.$$

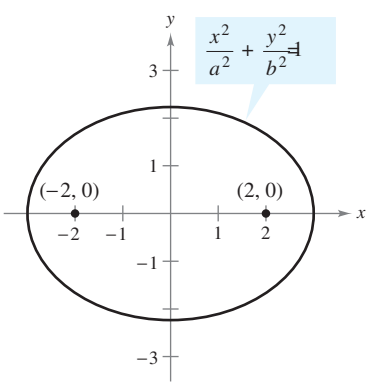


FIGURE B.8

✓CHECKPOINT 3

Find the standard form of the equation of the ellipse that has a major axis of length 8 and foci at $(0, -3)$ and $(0, 3)$. ■

Example 3 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse that has a major axis of length 6 and foci at $(-2, 0)$ and $(2, 0)$, as shown in Figure B8.

SOLUTION Because the foci occur at $(-2, 0)$ and $(2, 0)$, the center of the ellipse is $(0, 0)$, and the major axis is horizontal. So, the ellipse has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ Standard form, horizontal major axis

Because the length of the major axis is 6, you have $2a = 6$, which implies that $a = 3$. Moreover, the distance from the center to either focus is $c = 2$. Finally, you have

$$b^2 = a^2 - c^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Substituting $a^2 = 3^2$ and $b^2 = \left(\sqrt{5}\right)^2$ yields the following equation in standard form.

$$\frac{x^2}{3^2} + \frac{y^2}{\left(\sqrt{5}\right)^2} = 1$$ Standard form

This equation simplifies to $\frac{x^2}{9} + \frac{y^2}{5} = 1.$

Example 4 Sketching an Ellipse

Sketch the ellipse given by $4x^2 + y^2 = 36$, and identify the vertices.

SOLUTION Begin by writing the equation in standard form.

$$4x^2 + y^2 = 36 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36} \qquad \text{Divide each side by 36.}$$

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1 \qquad \text{Write in standard form.}$$

$$\frac{x^2}{9} + \frac{y^2}{36} = 1 \qquad \text{Simplify.}$$

Because the denominator of the y^2-term is larger than the denominator of the x^2-term, you can conclude that the major axis is vertical. Moreover, because $a = 6$, the vertices are $(0, -6)$ and $(0, 6)$. Finally, because $b = 3$, the endpoints of the minor axis are $(-3, 0)$ and $(3, 0)$, as shown in Figure B9.

STUDY TIP

The endpoints of the minor axis of an ellipse are commonly referred to as the *co-vertices*. In Figure B9, the co-vertices are $(-3, 0)$ and $(3, 0)$.

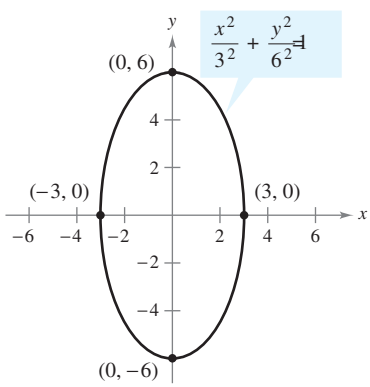


FIGURE B.9

Note that from the standard form of the equation, you can sketch the ellipse by locating the endpoints of the two axes. Because 3^2 is the denominator of the x^2-term, move three units to the *right and left* of the center to locate the endpoints of the horizontal axis. Similarly, because 6^2 is the denominator of the y^2-term, move six units *upward and downward* from the center to locate the endpoints of the vertical axis.

✓CHECKPOINT 4

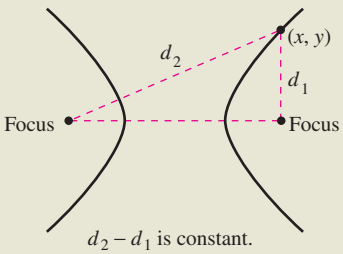
Sketch the ellipse given by $x^2 + 4y^2 = 64$, and identify the vertices. ■

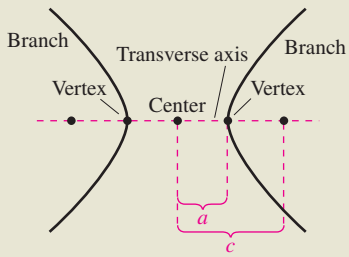
Hyperbolas

The definition of a **hyperbola** is similar to that of an ellipse. The distinction is that, for an ellipse, the *sum* of the distances between the foci and a point on the ellipse is constant, whereas for a hyperbola, the *difference* of the distances between the foci and a point on the hyperbola is constant.

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane the difference of whose distances from two distinct fixed points, called **foci,** is constant.

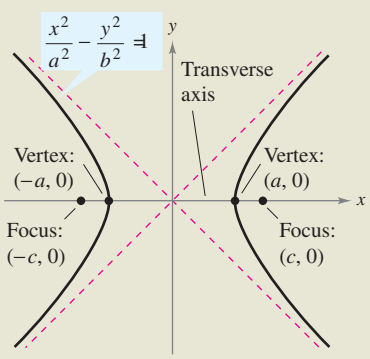


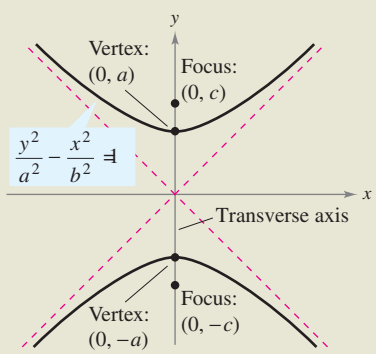


The graph of a hyperbola has two disconnected parts, called **branches.** The line through the two foci intersects the hyperbola at two points, called **vertices.** The line segment connecting the vertices is called the **transverse axis,** and the midpoint of the transverse axis is called the **center** of the hyperbola.

Standard Equation of a Hyperbola (Center at Origin)

The **standard form of the equation of a hyperbola** with the center at the origin (where $a \neq 0$ and $b \neq 0$) is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.





The vertices and foci are, respectively, a and c units from the center. Moreover, a, b, and c are related by the equation $b^2 = c^2 - a^2$.

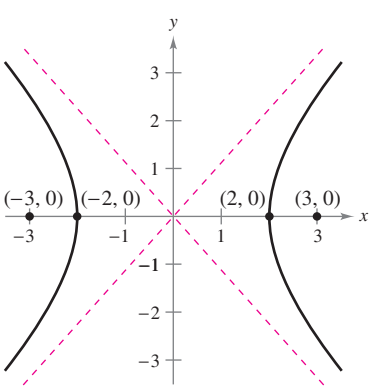


FIGURE B.10

Example 5 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci at $(-3, 0)$ and $(3, 0)$ and vertices at $(-2, 0)$ and $(2, 0)$, as shown in Figure B10.

SOLUTION From the graph, $c = 3$ because the foci are three units from the center.Moreover, $a = 2$because the vertices are two units from the center.So, it follows that

$$b^2 = c^2 - a^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Because the transverse axis is horizontal, the standard form of the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$ Standard form, horizontal transverse axis

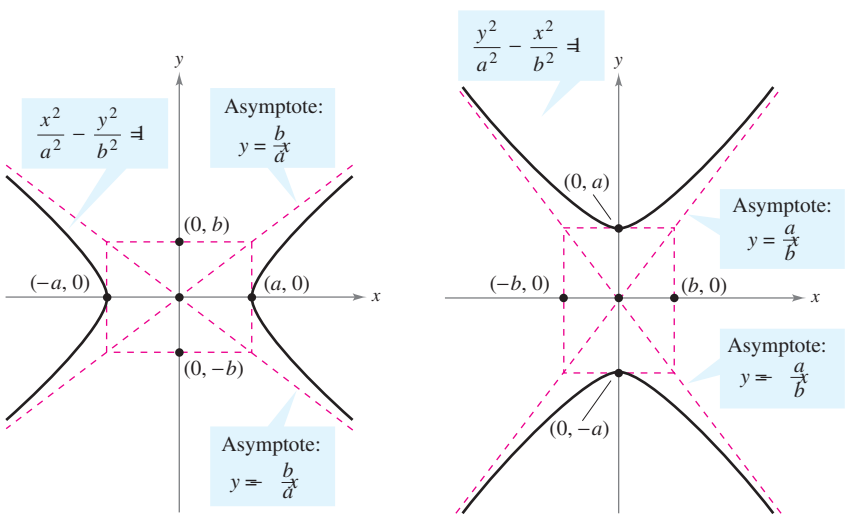
Finally, substituting $a^2 = 2^2$ and $b^2 = \left(\sqrt{5}\right)^2$ you have

$$\frac{x^2}{2^2} - \frac{y^2}{\left(\sqrt{5}\right)^2} = 1.$$ Write in standard form.

✓CHECKPOINT 5

Find the standard form of the equation of the hyperbola with foci at $(0, -4)$ and $(0, 4)$ and vertices at $(0, -3)$ and $(0, 3)$. ■

An important aid in sketching the graph of a hyperbola is the determination of its *asymptotes,* as shown in Figure B11.Each hyperbola has two asymptotes that intersect at the center of the hyperbola.Furthermore, the asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$.The line segment of length $2b$, joining $(0, b)$ and $(0, -b)$ [or $(-b, 0)$ and $(b, 0)$], is referred to as the **conjugate axis** of the hyperbola.



Transverse axis is horizontal. Transverse axis is vertical.

FIGURE B.11

Asymptotes of a Hyperbola (Center at Origin)

The **asymptotes of a hyperbola** with center at $(0, 0)$ are

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$ Transverse axis is horizontal.

or

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x.$$ Transverse axis is vertical.

Example 6 Sketching a Hyperbola

Sketch the hyperbola whose equation is $4x^2 - y^2 = 16$.

SOLUTION

$$4x^2 - y^2 = 16$$ Write original equation.

$$\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16}$$ Divide each side by 16.

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$$ Write in standard form.

Because the x^2-term is positive, you can conclude that the transverse axis is horizontal and the vertices occur at $(-2, 0)$ and $(2, 0)$. Moreover, the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$, and you can sketch the rectangle shown in Figure B12.Finally, by drawing the asymptotes through the corners of this rectangle, you can complete the sketch shown in Figure B13.

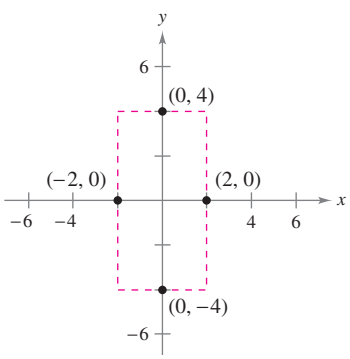


FIGURE B.12

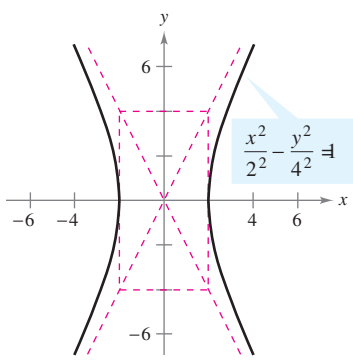


FIGURE B.13

✓CHECKPOINT 6

Sketch the hyperbola whose equation is $9y^2 - x^2 = 9$. ■

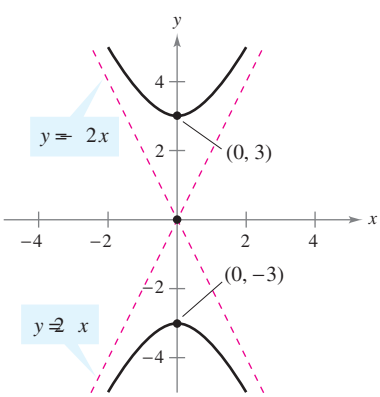


FIGURE B.14

Example 7 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola that has vertices at $(0, -3)$ and $(0, 3)$ and asymptotes $y = -2x$ and $y = 2x$, as shown in Figure B14.

SOLUTION Because the transverse axis is vertical, the asymptotes are of the form

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x.$$ Transverse axis is vertical.

Using the fact that $y = 2x$ and $y = -2x$, you can determine that $a/b = 2$. Because $a = 3$, you can determine that $b = \frac{3}{2}$. Finally, you can conclude that the hyperbola has the following equation.

$$\frac{y^2}{3^2} - \frac{x^2}{(3/2)^2} = 1$$ Write in standard form.

✓CHECKPOINT 7

Find the standard form of the equation of the hyperbola that has vertices at $(-5, 0)$ and $(5, 0)$ and asymptotes $y = -x$ and $y = x$. ■

Skills Review B.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.7, 1.3, and 2.1.

In Exercises 1–4, rewrite the equation so that it has no fractions.

1. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

2. $\frac{x^2}{32} + \frac{4y^2}{32} = \frac{32}{32}$

3. $\frac{x^2}{1/4} - \frac{y^2}{4} = 1$

4. $\frac{3x^2}{1/9} + \frac{4y^2}{9} = 1$

In Exercises 5–8, solve for c. (Assume $c > 0$.)

5. $c^2 = 3^2 - 1^2$

6. $c^2 = 2^2 + 3^2$

7. $c^2 + 2^2 = 4^2$

8. $c^2 - 1^2 = 2^2$

In Exercises 9 and 10, find the distance between the point and the origin.

9. $(0, -4)$

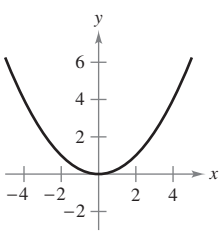
10. $(-2, 0)$

Exercises B.1

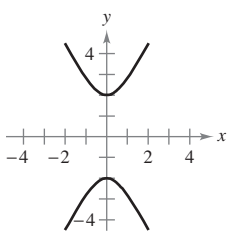
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

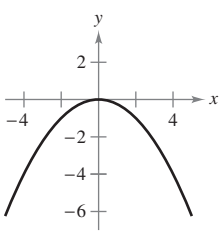
(a)



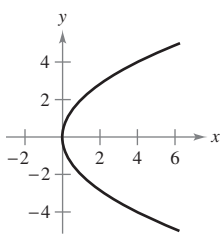
(b)



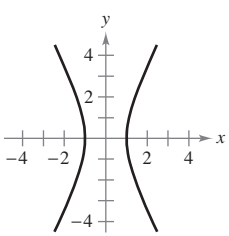
(c)



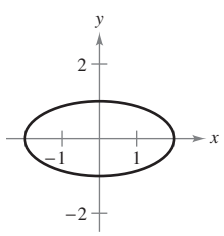
(d)



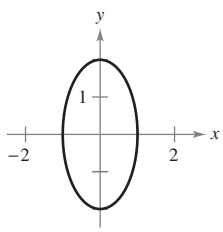
(e)



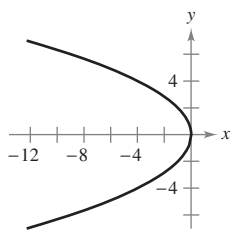
(f)



(g)



(h)



1. $x^2 = 4y$

2. $x^2 = -4y$

3. $y^2 = 4x$

4. $y^2 = -4x$

5. $\frac{x^2}{1} + \frac{y^2}{4} = 1$

6. $\frac{x^2}{4} + \frac{y^2}{1} = 1$

7. $\frac{x^2}{1} - \frac{y^2}{4} = 1$

8. $\frac{y^2}{4} - \frac{x^2}{1} = 1$

In Exercises 9–16, find the vertex and focus of the parabola and sketch its graph.

9. $y = 4x^2$

10. $y = \frac{1}{2}x^2$

11. $y^2 = -6x$

12. $y^2 = 3x$

13. $x^2 + 8y = 0$

14. $x^2 + 12y = 0$

15. $x + y^2 = 0$

16. $y^2 - 8x = 0$

In Exercises 17–26, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin.

17. Focus: $\left(0, -\frac{3}{2}\right)$

18. Focus: $(0, -2)$

19. Focus: $(-2, 0)$

20. Focus: $\left(\frac{5}{2}, 0\right)$

21. Directrix: $y = -1$

22. Directrix: $y = 2$

23. Directrix: $x = 3$

24. Directrix: $x = -2$

25. Passes through the point $(4, 6)$; horizontal axis

26. Passes through the point $(-2, -2)$; vertical axis

In Exercises 27–34, find the center and vertices of the ellipse and sketch its graph.

27. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

28. $\frac{x^2}{144} + \frac{y^2}{169} = 1$

29. $\frac{x^2}{\frac{25}{9}} + \frac{y^2}{\frac{16}{9}} = 1$

30. $\frac{x^2}{4} + \frac{y^2}{\frac{1}{4}} = 1$

31. $\frac{x^2}{9} + \frac{y^2}{5} = 1$

32. $\frac{x^2}{28} + \frac{y^2}{64} = 1$

33. $5x^2 + 3y^2 = 15$

34. $x^2 + 4y^2 = 4$

In Exercises 35–42, find the standard form of the equation of the ellipse with the given characteristics and center at the origin.

35. Vertices: $(0, \pm 2)$; minor axis of length 2

36. Vertices: $(\pm 2, 0)$; minor axis of length 3

37. Vertices: $(\pm 5, 0)$; foci: $(\pm 2, 0)$

38. Vertices: $(0, \pm 10)$; foci: $(0, \pm 4)$

39. Foci: $(\pm 5, 0)$; major axis of length 12

40. Foci: $(\pm 2, 0)$; major axis of length 8

41. Vertices: $(0, \pm 5)$; passes through the point $(4, 2)$

42. Major axis vertical; passes through the points $(0, 4)$ and $(2, 0)$

In Exercises 43–50, find the center and vertices of the hyperbola and sketch its graph.

43. $x^2 - y^2 = 1$

44. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

45. $\frac{y^2}{1} - \frac{x^2}{4} = 1$

46. $\frac{y^2}{9} - \frac{x^2}{1} = 1$

47. $\frac{y^2}{25} - \frac{x^2}{144} = 1$

48. $\frac{x^2}{36} - \frac{y^2}{4} = 1$

49. $2x^2 - 3y^2 = 6$

50. $3y^2 - 5x^2 = 15$

In Exercises 51–58, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin.

51. Vertices: $(0, \pm 2)$; foci: $(0, \pm 4)$

52. Vertices: $(\pm 3, 0)$; foci: $(\pm 5, 0)$

53. Vertices: $(\pm 1, 0)$; asymptotes: $y = \pm 3x$

54. Vertices: $(0, \pm 3)$; asymptotes: $y = \pm 3x$

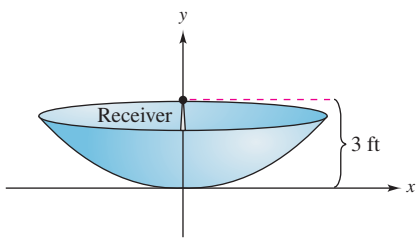
55. Foci: $(0, \pm 4)$; asymptotes: $y = \pm \frac{1}{2}x$

56. Foci: $(\pm 10, 0)$; asymptotes: $y = \pm \frac{3}{4}x$

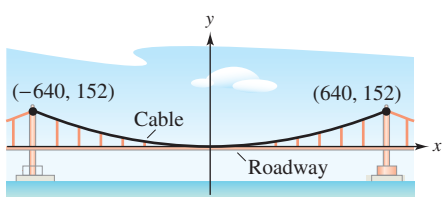
57. Vertices: $(0, \pm 3)$; passes through the point $(-2, 5)$

58. Vertices: $(\pm 2, 0)$; passes through the point $\left(3, \sqrt{3}\right)$

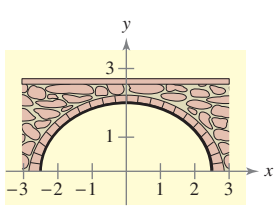
59. Satellite Antenna The receiver in a parabolic television dish antenna is 3 feet from the vertex and is located at the focus (see figure). Write an equation for a cross section of the reflector. (Assume that the dish is directed upward and the vertex is at the origin)



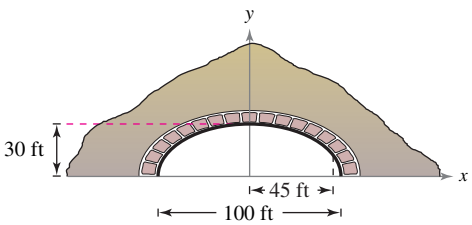
60. Suspension Bridge Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway (see figure). The cables touch the roadway at the midpoint between the towers. Write an equation for the parabolic shape of each cable.



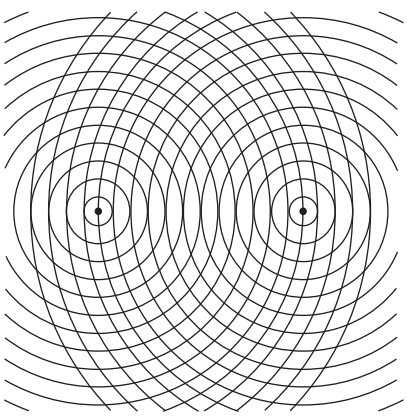
61. Architecture A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 5 feet along the base (see figure). The contractor draws the outline of the ellipse by the method shown in Figure B7. Where should the tacks be placed, and what should be the length of the piece of string?



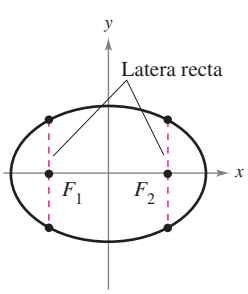
62. Mountain Tunnel A semielliptical arch over a tunnel for a road through a mountain has a major axis of 100 feet, and its height at the center is 30 feet (see figure).Determine the height of the arch 5 feet from the edge of the tunnel.



63. Sketch a graph of the ellipse that consists of all points (x, y) such that the sum of the distances between (x, y) and two fixed points is 15 units and the foci are located at the centers of the two sets of concentric circles, as shown in the figure.



64. Think About It A line segment through a focus of an ellipse with endpoints on the ellipse and perpendicular to its major axis is called a **latus rectum** of the ellipse. Therefore, an ellipse has two latera recta. Knowing the length of the latera recta is helpful in sketching an ellipse because this information yields other points on the curve (see figure). Show that the length of each latus rectum is $2b^2/a$.



In Exercises 65–68, sketch the ellipse using the latera recta (see Exercise 64).

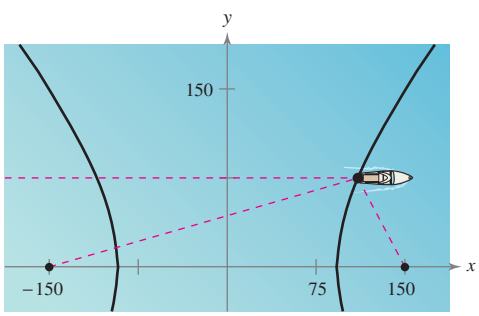
65. $\frac{x^2}{4} + \frac{y^2}{1} = 1$

66. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

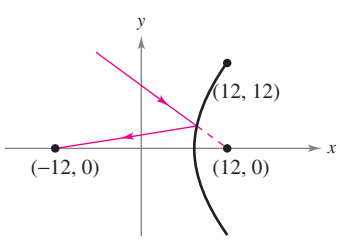
67. $9x^2 + 4y^2 = 36$

68. $5x^2 + 3y^2 = 15$

69. Navigation Long-range navigation for aircraft and ships is accomplished by synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second).The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci.Assume that two stations 300 miles apart are positioned on a rectangular coordinate system at points with coordinates $(-150, 0)$ and $(150, 0)$ and that a ship is traveling on a path with coordinates $(x, 75)$ (see figure).Find the x-coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0001 second).



70. Hyperbolic Mirror A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at one focus will be reflected to the other focus (see figure).The focus of the hyperbolic mirror has coordinates (12, 0). Find the vertex of the mirror if its mount at the top edge of the mirror has coordinates (12, 12).



B.2 Conic Sections and Translations

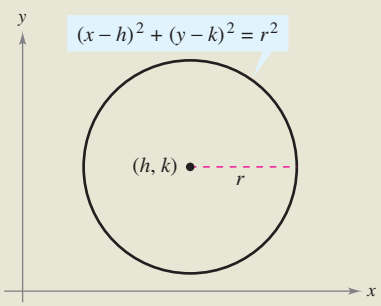
- Recognize equations of conics that have been shifted vertically or horizontally in the plane.
- Write and graph equations of conics that have been shifted vertically or horizontally in the plane.

Vertical and Horizontal Shifts of Conics

In Section B1, you studied conic sections whose graphs were in **standard position.** In this section, you will study the equations of conic sections that have been shifted vertically or horizontally in the plane.The following summary lists the standard forms of the equations of the four basic conics.

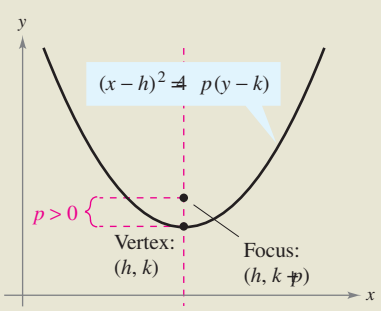
Standard Forms of Equations of Conics

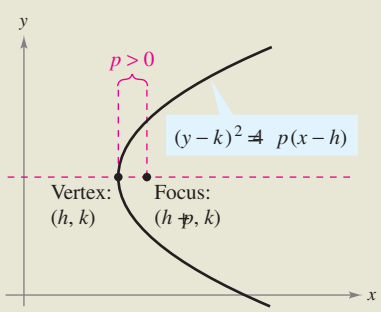
Circle: Center $= (h, k)$; Radius $= r$



Parabola: Vertex $= (h, k)$

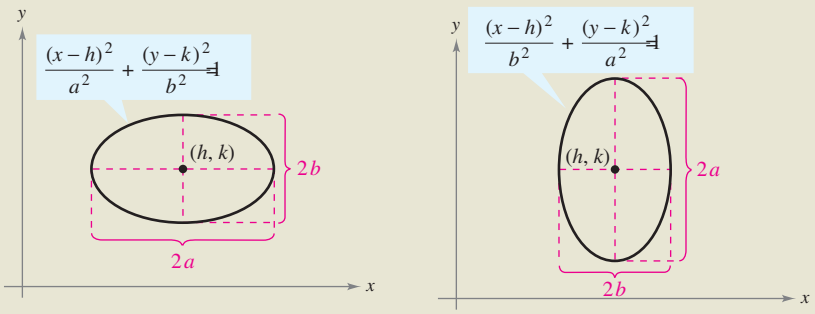
Directed distance from vertex to focus $= p$



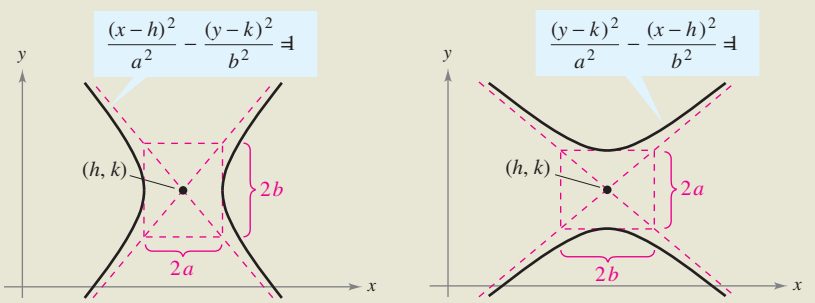


Standard Forms of Equations of Conics *(continued)*

Ellipse: Center $= (h, k)$
Major axis length $= 2a$; Minor axis length $= 2b$



Hyperbola: Center $= (h, k)$
Transverse axis length $= 2a$; Conjugate axis length $= 2b$



Example 1 Equations of Conic Sections

Describe the translation of the graph of each conic.

a. $(x - 1)^2 + (y + 2)^2 = 3^2$ **b.** $(x - 2)^2 = 4(-1)(y - 3)$

c. $\dfrac{(x - 3)^2}{1^2} - \dfrac{(y - 2)^2}{3^2} = 1$ **d.** $\dfrac{(x - 2)^2}{3^2} + \dfrac{(y - 1)^2}{2^2} = 1$

SOLUTION

a. The graph of $(x - 1)^2 + (y + 2)^2 = 3^2$ is a circle whose center is the point $(1, -2)$ and whose radius is 3, as shown in Figure B15(a) on the following page.Note that the graph of the circle has been shifted one unit to the right and two units downward from standard position.

b. The graph of $(x - 2)^2 = 4(-1)(y - 3)$ is a parabola whose vertex is the point $(2, 3)$. The axis of the parabola is vertical.The focus is one unit above or below the vertex.Moreover, because $p = -1$,it follows that the focus lies *below* the vertex, as shown in Figure B15(b).Note that the graph of the parabola has been shifted two units to the right and three units upward from standard position.

c. The graph of

$$\frac{(x-3)^2}{1^2} - \frac{(y-2)^2}{3^2} = 1$$

is a hyperbola whose center is the point $(3, 2)$. The transverse axis is horizontal with a length of $2(1) = 2$. The conjugate axis is vertical with a length of $2(3) = 6$, as shown in Figure B.15(c). Note that the graph of the hyperbola has been shifted three units to the right and two units upward from standard position.

d. The graph of

$$\frac{(x-2)^2}{3^2} + \frac{(y-1)^2}{2^2} = 1$$

is an ellipse whose center is the point $(2, 1)$. The major axis of the ellipse is horizontal with a length of $2(3) = 6$. The minor axis of the ellipse is vertical with a length of $2(2) = 4$, as shown in Figure B.15(d). Note that the graph of the ellipse has been shifted two units to the right and one unit upward from standard position.

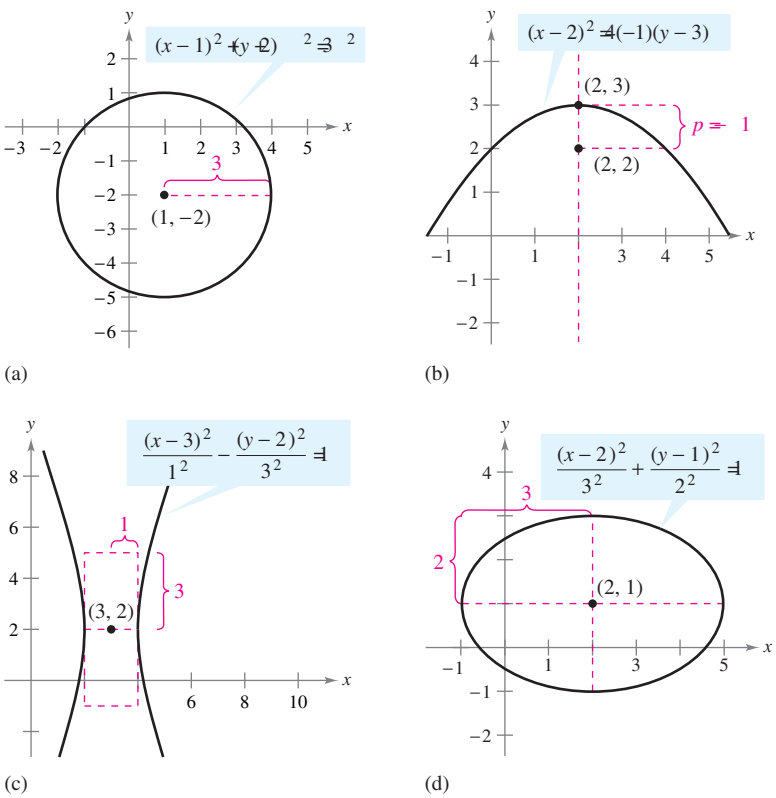


FIGURE B.15

✓CHECKPOINT 1

Describe the translation of the graph of

$$(x + 4)^2 + (y - 3)^2 = 5^2. \quad \blacksquare$$

Writing Equations of Conics in Standard Form

To write the equation of a conic in standard form, complete the square, as demonstrated in Examples 2, 3, and 4.

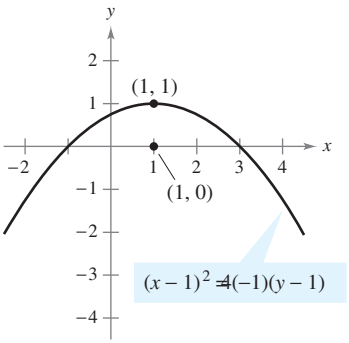


FIGURE B.16

✓CHECKPOINT 2

Find the vertex and focus of the parabola given by

$$x^2 + 2x - 4y + 5 = 0.$$ ■

Example 2 Finding the Standard Form of a Parabola

Find the vertex and focus of the parabola given by $x^2 - 2x + 4y - 3 = 0$.

SOLUTION

$x^2 - 2x + 4y - 3 = 0$	Write original equation.
$x^2 - 2x = -4y + 3$	Group terms.
$x^2 - 2x + 1 = -4y + 3 + 1$	Add 1 to each side.
$(x - 1)^2 = -4y + 4$	Write in completed square form.
$(x - 1)^2 = 4(-1)(y - 1)$	Write in standard form.

From this standard form, it follows that $h = 1$, $k = 1$, and $p = -1$. Because the axis is vertical and p is negative, the parabola opens downward. The vertex and focus are $(h, k) = (1, 1)$ and $(h, k + p) = (1, 0)$. The graph of this parabola is shown in Figure B16.

In Examples 1(b) and 2, p is the *directed distance* from the vertex to the focus. Because the axis of the parabola is vertical and $p = -1$, the focus is one unit *below* the vertex, and the parabola opens downward.

Example 3 Sketching an Ellipse

Sketch the ellipse given by the equation $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

SOLUTION

$x^2 + 4y^2 + 6x - 8y + 9 = 0$	Write original equation.
$(x^2 + 6x + \quad) + (4y^2 - 8y + \quad) = -9$	Group terms.
$(x^2 + 6x + \quad) + 4(y^2 - 2y + \quad) = -9$	Factor 4 out of y-terms.
$(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -9 + 9 + 4(1)$	Add 9 and $4(1) = 4$ to each side.
$(x + 3)^2 + 4(y - 1)^2 = 4$	Write in completed square form.
$\frac{(x + 3)^2}{4} + \frac{(y - 1)^2}{1} = 1$	Divide each side by 4.
$\frac{(x + 3)^2}{2^2} + \frac{(y - 1)^2}{1^2} = 1$	Write in standard form.

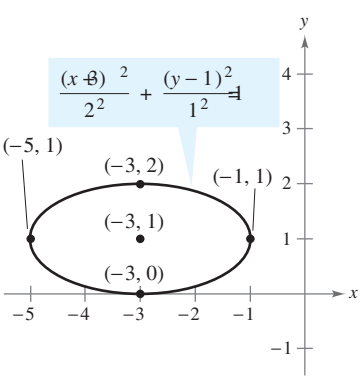


FIGURE B.17

From this standard form, it follows that the center is $(h, k) = (-3, 1)$. Because the denominator of the x-term is $a^2 = 2^2$, the endpoints of the major axis lie two units to the right and left of the center. Similarly, because the denominator of the y-term is $b^2 = 1^2$, the endpoints of the minor axis lie one unit up and down from the center. The ellipse is shown in Figure B17.

✓CHECKPOINT 3

Sketch the ellipse given by the equation

$$x^2 + 9y^2 - 4x + 18y + 4 = 0.$$ ■

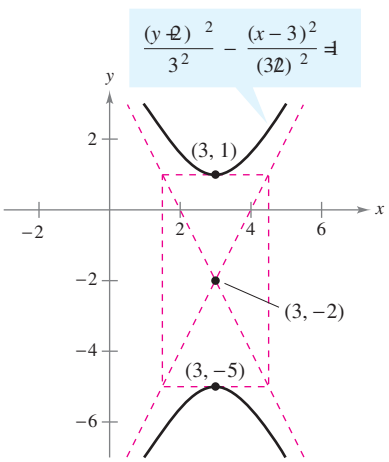


FIGURE B.18

✓CHECKPOINT 4

Sketch the hyperbola given by the equation

$4x^2 - y^2 + 16x - 2y - 1 = 0.$ ■

Example 4 Sketching a Hyperbola

Sketch the hyperbola given by the equation

$$y^2 - 4x^2 + 4y + 24x - 41 = 0.$$

SOLUTION

$y^2 - 4x^2 + 4y + 24x - 41 = 0$	Write original equation.
$(y^2 + 4y + \quad) - (4x^2 - 24x + \quad) = 41$	Group terms.
$(y^2 + 4y + \quad) - 4(x^2 - 6x + \quad) = 41$	Factor 4 out of x-terms.
$(y^2 + 4y + 4) - 4(x^2 - 6x + 9) = 41 + 4 - 4(9)$	Add 4 and subtract $4(9) = 36$ from each side.
$(y + 2)^2 - 4(x - 3)^2 = 9$	Write in completed square form.
$\frac{(y + 2)^2}{9} - \frac{4(x - 3)^2}{9} = 1$	Divide each side by 9.
$\frac{(y + 2)^2}{9} - \frac{(x - 3)^2}{9/4} = 1$	Change 4 to $1/(1/4)$.
$\frac{(y + 2)^2}{3^2} - \frac{(x - 3)^2}{(3/2)^2} = 1$	Write in standard form.

From the standard form, it follows that the transverse axis is vertical and the center lies at $(h, k) = (3, -2)$. Because the denominator of the y-term is $a^2 = 3^2$, you know that the vertices lie three units above and below the center.

Vertices: $(3, -5)$ and $(3, 1)$

To sketch the hyperbola, draw a rectangle whose top and bottom pass through the vertices. Because the denominator of the x-term is $b^2 = \left(\frac{3}{2}\right)^2$, locate the sides of the rectangle $\frac{3}{2}$ units to the right and left of the center, as shown in Figure B.18. Finally, sketch the asymptotes by drawing lines through the opposite corners of the rectangle. Using these asymptotes, you can complete the graph of the hyperbola, as shown in Figure B.18.

To find the foci in Example 4, first find c.

$$c^2 = a^2 + b^2$$

$$c^2 = 9 + \frac{9}{4}$$

$$c^2 = \frac{45}{4}$$

$$c = \frac{3\sqrt{5}}{2}$$

Because the transverse axis is vertical, the foci lie c units above and below the center.

Foci: $\left(3, -2 + \frac{3\sqrt{5}}{2}\right)$ and $\left(3, -2 - \frac{3\sqrt{5}}{2}\right)$

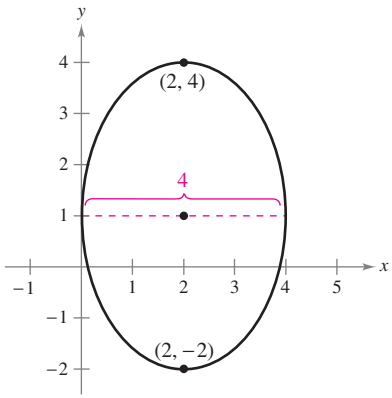


FIGURE B.19

Example 5 **Writing the Equation of an Ellipse**

Write the standard form of the equation of the ellipse whose vertices are $(2, -2)$ and $(2, 4)$. The length of the minor axis of the ellipse is 4, as shown in Figure B19.

SOLUTION The center of the ellipse lies at the midpoint of its vertices. So, the center is

$$(h, k) = \left(\frac{2 + 2}{2}, \frac{4 + (-2)}{2}\right) = (2, 1).$$ Center

Because the vertices lie on a vertical line and are six units apart, it follows that the major axis is vertical and has a length of $2a = 6$. So, $a = 3$. Moreover, because the minor axis has a length of 4, it follows that $2b = 4$, which implies that $b = 2$. Therefore, you can conclude that the standard form of the equation of the ellipse is as follows.

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$ Major axis is vertical.

$$\frac{(x - 2)^2}{2^2} + \frac{(y - 1)^2}{3^2} = 1$$ Write in standard form.

✓CHECKPOINT 5

Write the standard form of the equation of the ellipse whose vertices are $(-1, 4)$ and $(4, 4)$. The length of the minor axis of the ellipse is 3. ■

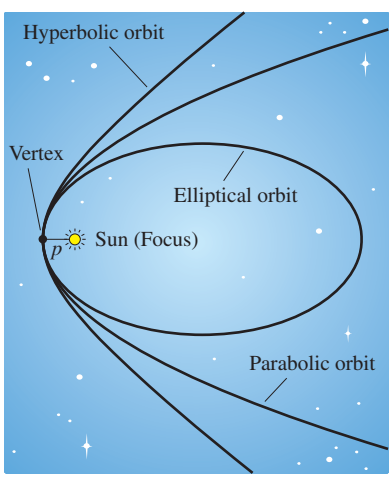


FIGURE B.20

An interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. For example, Halley's comet has an elliptical orbit, and reappearance of this comet can be predicted every 76 years. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure B20.

If p is the distance between the vertex and the focus (in meters), and v is the speed of the comet at the vertex (in meters per second), then the type of orbit is determined as follows.

1. Ellipse: $v < \sqrt{\frac{2GM}{p}}$
2. Parabola: $v = \sqrt{\frac{2GM}{p}}$
3. Hyperbola: $v > \sqrt{\frac{2GM}{p}}$

In each of these equations, $M = 1989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 667 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

Skills Review B.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section B.1.

In Exercises 1–12, identify the conic represented by the equation.

1. $\frac{x^2}{4} - \frac{y^2}{4} = 1$

2. $\frac{x^2}{9} + \frac{y^2}{1} = 1$

3. $2x + y^2 = 0$

4. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

5. $\frac{x^2}{4} + \frac{y^2}{16} = 1$

6. $4x^2 + 4y^2 = 25$

7. $\frac{y^2}{4} - \frac{x^2}{2} = 1$

8. $x^2 - 6y = 0$

9. $3x - y^2 = 0$

10. $\frac{9x^2}{16} + \frac{y^2}{4} = 1$

11. $3y^2 - 3x^2 = 48$

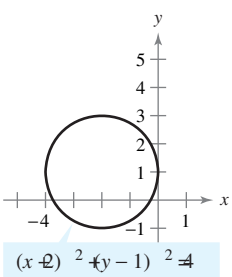
12. $\frac{x^2}{9/4} + \frac{y^2}{4} = 1$

Exercises B.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

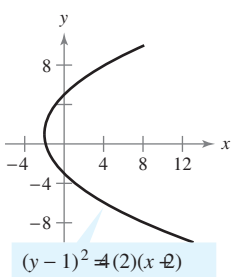
In Exercises 1–6, describe the translation of the graph of the conic from the standard position.

1.



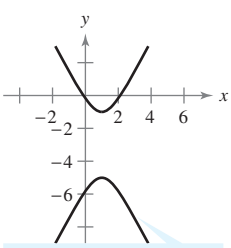
$(x-2)^2 + (y-1)^2 = 4$

2.



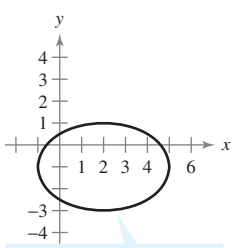
$(y-1)^2 = 4(2)(x-2)$

3.



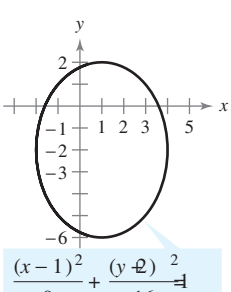
$\frac{(y-3)^2}{4} - (x-1)^2 = 1$

4.



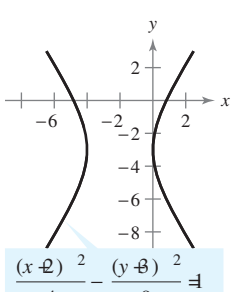
$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1$

5.



$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{16} = 1$

6.



$\frac{(x-2)^2}{4} - \frac{(y-3)^2}{9} = 1$

In Exercises 7–16, find the vertex, focus, and directrix of the parabola. Then sketch its graph.

7. $(x-1)^2 + 8(y+2) = 0$

8. $(x+3) + (y-2)^2 = 0$

9. $\left(y + \frac{1}{2}\right)^2 = 2(x-5)$

10. $\left(x + \frac{1}{2}\right)^2 = 4(y-3)$

11. $y = \frac{1}{4}(x^2 - 2x + 5)$

12. $y = -\frac{1}{6}(x^2 + 4x - 2)$

13. $4x - y^2 - 2y - 33 = 0$

14. $y^2 + x + y = 0$

15. $y^2 + 6y + 8x + 25 = 0$

16. $x^2 - 2x + 8y + 9 = 0$

In Exercises 17–24, find the standard form of the equation of the parabola with the given characteristics.

17. Vertex: $(3, 2)$;focus: $(1, 2)$

18. Vertex: $(-1, 2)$;focus: $(-1, 0)$

19. Vertex: $(0, 4)$;directrix: $y = 2$

20. Vertex: $(-2, 1)$;directrix: $x = 1$

21. Focus: $(2, 2)$;directrix: $x = -2$

22. Focus: $(0, 0)$;directrix: $y = 4$

23. Vertex: $(0, 4)$;passes through $(-2, 0)$and $(2, 0)$

24. Vertex: $(2, 4)$;passes through $(0, 0)$and $(4, 0)$

In Exercises 25–32, find the center, foci, and vertices of the ellipse. Then sketch its graph.

25. $\frac{(x-1)^2}{9} + \frac{(y-5)^2}{25} = 1$

26. $(x + 2)^2 + \dfrac{(y + 4)^2}{1/4} = 1$

27. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

28. $9x^2 + 4y^2 - 36x + 8y + 31 = 0$

29. $16x^2 + 25y^2 - 32x + 50y + 16 = 0$

30. $9x^2 + 25y^2 - 36x - 50y + 61 = 0$

31. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

32. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

In Exercises 33–42, find the standard form of the equation of the ellipse with the given characteristics.

33. Vertices: $(0, 2), (4, 2)$; minor axis of length 2

34. Vertices: $(3, 1), (3, 9)$; minor axis of length 6

35. Foci: $(0, 0), (4, 0)$; major axis of length 8

36. Foci: $(0, 0), (0, 8)$; major axis of length 16

37. Center: $(2, -1)$;vertex: $\left(2, \frac{1}{2}\right)$; minor axis of length 2

38. Center: $(4, 0)$;vertex: $(4, -4)$; minor axis of length 6

39. Center: $(3, 2)$; $a = 3c$; foci: $(1, 2), (5, 2)$

40. Center: $(0, 4)$; $a = 2c$; vertices: $(-4, 4), (4, 4)$

41. Vertices: $(5, 0), (5, 12)$; endpoints of minor axis: $(0, 6), (10, 6)$

42. Vertices: $(-12, 0), (0, 0)$; endpoints of minor axis: $(-6, 3), (-6, -3)$

In Exercises 43–52, find the center, vertices, and foci of the hyperbola. Then sketch its graph, using asymptotes as a sketching aid.

43. $\dfrac{(x - 1)^2}{4} - \dfrac{(y + 2)^2}{1} = 1$

44. $\dfrac{(x + 1)^2}{144} - \dfrac{(y - 4)^2}{25} = 1$

45. $(y + 6)^2 - (x - 2)^2 = 1$

46. $\dfrac{(y - 1)^2}{1/4} - \dfrac{(x + 3)^2}{1/9} = 1$

47. $9x^2 - y^2 - 36x - 6y + 18 = 0$

48. $x^2 - 9y^2 + 36y - 72 = 0$

49. $9y^2 - x^2 + 2x + 54y + 62 = 0$

50. $16y^2 - x^2 + 2x + 64y + 63 = 0$

51. $x^2 - 9y^2 + 2x - 54y - 107 = 0$

52. $9x^2 - y^2 + 54x + 10y + 55 = 0$

In Exercises 53–60, find the standard form of the equation of the hyperbola with the given characteristics.

53. Vertices: $(2, 0), (6, 0)$; foci: $(0, 0), (8, 0)$

54. Vertices: $(2, 3), (2, -3)$; foci: $(2, 5), (2, -5)$

55. Vertices: $(4, 1), (4, 9)$; foci: $(4, 0), (4, 10)$

56. Vertices: $(-2, 1), (2, 1)$; foci: $(-3, 1), (3, 1)$

57. Vertices: $(2, 3), (2, -3)$; passes through $(0, 5)$

58. Vertices: $(-2, 1), (2, 1)$; passes through $(4, 3)$

59. Vertices: $(0, 2), (6, 2)$; asymptotes: $y = \frac{2}{3}x$, $y = 4 - \frac{2}{3}x$

60. Vertices: $(3, 0), (3, 4)$; asymptotes: $y = \frac{2}{3}x$, $y = 4 - \frac{2}{3}x$

In Exercises 61–68, identify the conic by writing the equation in standard form. Then sketch its graph.

61. $x^2 + y^2 - 6x + 4y + 9 = 0$

62. $x^2 + 4y^2 - 6x + 16y + 21 = 0$

63. $4x^2 - y^2 - 4x - 3 = 0$

64. $y^2 - 4y - 4x = 0$

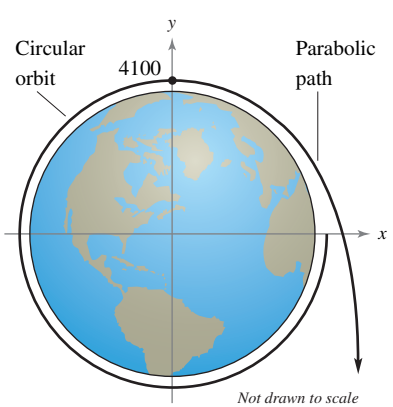
65. $4x^2 + 3y^2 + 8x - 24y + 51 = 0$

66. $4y^2 - 2x^2 - 4y + 8x - 15 = 0$

67. $25x^2 - 10x - 200y - 119 = 0$

68. $4x^2 + 4y^2 - 16y + 15 = 0$

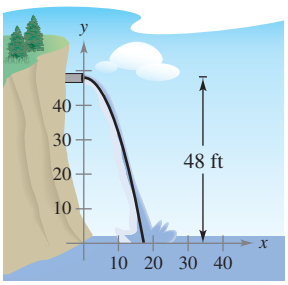
69. Satellite Orbit A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour.If this velocity is multiplied by $\sqrt{2}$,then the satellite will have the minimum velocity necessary to escape Earth's gravity, and it will follow a parabolic path with the center of Earth as the focus (see figure).



(a) Find the escape velocity of the satellite.

(b) Find an equation of its path (assume the radius of Earth is 4000 miles).

70. Fluid Flow Water is flowing from a horizontal pipe 48 feet above the ground. The falling stream of water has the shape of a parabola whose vertex $(0, 48)$ is at the end of the pipe (see figure). The stream of water strikes the ground at the point $\left(10\sqrt{3}, 0\right)$. Find the equation of the path taken by the water.



In Exercises 71–78, e is called the **eccentricity** of an ellipse and is defined by $e = c/a$. It measures the flatness of the ellipse.

71. Find an equation of the ellipse with vertices $(\pm 5, 0)$ and eccentricity $e = \frac{3}{5}$.

72. Find an equation of the ellipse with vertices $(0, \pm 8)$ and eccentricity $e = \frac{1}{2}$.

73. Planetary Motion Earth moves in an elliptical orbit with the sun at one of the foci (see figure). The length of half of the major axis is $a = 92956 \times 10^7$ miles and the eccentricity is 0017. Find the shortest distance (perihelion) and the greatest distance (aphelion) between Earth and the sun.

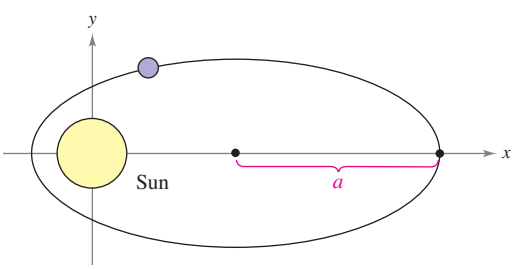
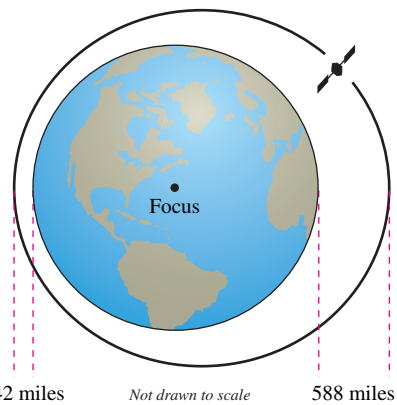


Figure for 73–75

74. Planetary Motion The dwarf planet Pluto moves in an elliptical orbit with the sun at one of the foci (see figure). The length of half of the major axis is 3670×10^9 miles and the eccentricity is 0249. Find the shortest distance and the greatest distance between Pluto and the sun.

75. Planetary Motion Saturn moves in an elliptical orbit with the sun at one of the foci (see figure). The shortest distance and the greatest distance between Saturn and the sun are 13495×10^9 kilometers and 15040×10^9 kilometers, respectively. Find the eccentricity of the orbit.

76. Satellite Orbit The first artificial satellite to orbit Earth was Sputnik I (launched by the former Soviet Union in 1957). Its highest point above Earth's surface was 588 miles, and its lowest point was 142 miles (see figure). Assume that the center of Earth is one of the foci of the elliptical orbit and that the radius of Earth is 4000 miles. Find the eccentricity of the orbit.



77. Show that the equation of an ellipse can be written as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

Note that as e approaches zero, the ellipse approaches a circle of radius a.

78. Comet Orbit Halley's comet has an elliptical orbit with the sun at one focus. The eccentricity of the orbit is approximately 097. The length of the major axis of the orbit is approximately 3588 astronomical units. (An astronomical unit is about 93 million miles) Find the standard form of the equation of the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

79. Comet Orbit The comet Encke has an elliptical orbit with the sun at one focus. Encke's path ranges from 034 to 408 astronomical units from the sun. Find the standard form of the equation of the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

80. Australian Football In Australia, football by *Australian Rules* (or rugby) is played on elliptical fields. The fields can be a maximum of 170 yards wide and a maximum of 200 yards long. Let the center of a field of maximum size be represented by the point $(0, 85)$. Find the standard form of the equation of the ellipse that represents this field. *(Source: Australian Football League)*

Answers to Odd–Numbered Exercises and Tests

CHAPTER 0

SECTION 0.1 *(page 8)*

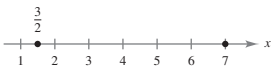
1. (a) Natural: $\{5\}$
(b) Integer: $\{-9, 5\}$
(c) Rational: $\left\{-9, -\frac{7}{2}, 5, \frac{2}{3}, 01\right\}$
(d) Irrational: $\left\{\sqrt{2}\right\}$

3. (a) Natural: $\left\{12, 1, \sqrt{4}\right\}$ $\left(\textit{Note: } \sqrt{4} = 2\right)$
(b) Integer: $\left\{12, -13, 1, \sqrt{4}\right\}$
(c) Rational: $\left\{12, -13, 1, \sqrt{4}, \frac{3}{2}\right\}$
(d) Irrational: $\left\{\sqrt{6}\right\}$

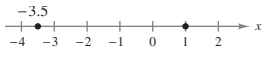
5. (a) Natural: $\left\{\frac{8}{2}, 9\right\}$ $\left(\textit{Note: } \frac{8}{2} = 4\right)$
(b) Integer: $\left\{\frac{8}{2}, -4, 9\right\}$
(c) Rational: $\left\{\frac{8}{2}, -\frac{8}{3}, -4, 9, 142\right\}$
(d) Irrational: $\left\{\sqrt{10}\right\}$

7. 06 **9.** $01\overline{26}$ **11.** $-1 < 25$

13. $\frac{3}{2} < 7$



15. $1 > -35$



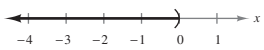
17. $\frac{5}{6} > \frac{2}{3}$



19. $\frac{204}{60}, \frac{31}{9}, 345, 2\sqrt{3}, \frac{7}{2}$ **21.** $\frac{127}{90}, \frac{584}{413}, \frac{7071}{5000}, \sqrt{2}, \frac{47}{33}$

23. $x \leq 4$ **25.** $0 < x \leq 3$

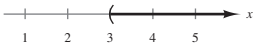
27. $x < 0$ denotes all negative real numbers.



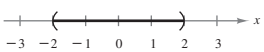
29. $x \leq 5$ denotes all real numbers less than or equal to 5.



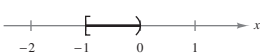
31. $x > 3$ denotes all real numbers greater than 3.



33. $-2 < x < 2$ denotes all real numbers greater than -2 and less than 2.



35. $-1 \leq x < 0$ denotes all real numbers greater than or equal to -1 and less than 0.



37. $x > 0$ **39.** $5 < y \leq 12$ **41.** $A \geq 35$

43. $35\% \leq r \leq 6\%$ **45.** 10 **47.** -6 **49.** -9

51. -1 **53.** $\pi - 3$ **55.** $|-7| = |7|$

57. $|-3| > -|-3|$ **59.** $-|-2| = -|2|$ **61.** 4

63. $\frac{5}{2}$ **65.** $\frac{7}{2}$ **67.** 51 **69.** $\frac{128}{75}$ **71.** $\left|z - \frac{3}{2}\right| > 1$

73. $|x + 10| \geq 6$ **75.** $|y - 0| \geq 6 \Rightarrow |y| \geq 6$

77. $|x - m| > 5$ **79.** 179 miles

81. 129°F

	$\lvert a - b\rvert$	$005b$	*Passes Budget Variance Test*
83.	$87655	$500	No
85.	$36432	$600	es
87.	$93783	$2040	No

	$\lvert a - b\rvert$	$00012b$	*Passes Quality Control Test*
89.	0002	00168	es
91.	0027	00192	No
93.	0027	00216	No

95. (a) No. If $u > 0$ and $v < 0$ or if $u < 0$ and $v > 0$, then $|u + v| \neq |u| + |v|$.
(b) es. If the signs of and are different, then $|u + v| < |u| + |v|$.

97. Answers will vary. Sample answer: The set of natural numbers includes only the integers greater than zero. The set of integers includes all numbers that have no fractional or decimal parts. The set of rational numbers includes all numbers that can be written as the quotient of two integers. Any real number that is not a rational number is in the set of irrational numbers.

SECTION 0.2 *(page 18)*

Skills Review *(page 18)*

1. $-4 < -2$ **2.** $0 > -3$ **3.** $\sqrt{3} > 173$
4. $-\pi < -3$ **5.** $|6 - 4| = 2$ **6.** $|2 - (-2)| = 4$
7. $|0 - (-5)| = 5$ **8.** $|3 - (-1)| = 4$
9. $|-7| + |7| = 7 + 7 = 14$
10. $-|8 - 10| = -|-2| = -2$

1. $7x, 4$ **3.** $x^2, -4x, 8$ **5.** $2x^2, -9x, 13$

7. -6 **9.** 6 **11.** (a) -10 (b) -6

13. (a) 14 (b) 2

15. (a) $\frac{1}{2}$ (b) Undefined. You cannot divide by zero.

17. 1 **19.** 35 **21.** $\frac{1}{4}$ **23.** Commutative (addition)

25. Inverse (addition) **27.** Distributive Property

29. Inverse (multiplication) **31.** Identity (addition)

33. Identity (multiplication) **35.** Associative (addition)

37. $x(3y) = (x \cdot 3)y$ Associative (multiplication)
$= (3x)y$ Commutative (multiplication)

39. $2^4 \cdot 3$ **41.** $2^4 \cdot 3 \cdot 5$ **43.** -14 **45.** $\frac{1}{24}$

47. $\frac{7}{20}$ **49.** $\frac{1}{12}$ **51.** -013 **53.** 156

55. 238 **57.** 4625

59. (a) 212%

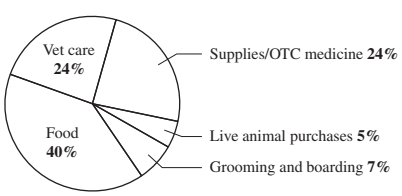
(b) Health and Medicare: ≈ \$49 billion
Education and Veterans' Benefits: ≈ \$68 billion
Income Security: ≈ \$46 billion
Social Security: ≈ \$24 billion
Other: ≈ \$91 billion
National Defense: ≈ \$94 billion

61. ≈ 1695 patients

63. (a) Scientific: 5 [×] [(] 18 [−] 2 [y^x] 3 [)] [÷] 10 [=]
Graphing: 5 [(] 18 [−] 2 [^] 3 [)] [÷] 10 [ENTER]

(b) Scientific: 6 [x^2] [+/−] [−] [(] 7 [+] [(] 2 [+/−] [)] [y^x] 3 [)] [=]
Graphing: [(−)] 6 [x^2] [−] [(] 7 [+] [(] [(−)] 2 [)] [^] 3 [)] [ENTER]

65. Food: 40%
Vet care: 24%
Supplies/OTC medicine: 24%
Live animal purchases: 5%
Grooming and boarding: 7%



SECTION 0.3 *(page 27)*

Skills Review *(page 27)*

1. 1 **2.** 5 **3.** 4 **4.** 4 **5.** $\frac{1}{4}$ **6.** 1
7. $\frac{3}{7}$ **8.** 0 **9.** $-\frac{1}{8}$ **10.** 1

1. 64 **3.** 8 **5.** 729 **7.** -81 **9.** $\frac{1}{2}$

11. 8 **13.** $-\frac{3}{10}$ **15.** 5184 **17.** $-\frac{3}{5}$ **19.** 1

21. 18 **23.** $\frac{7}{16}$ **25.** $-125z^3$ **27.** $16x^7$ **29.** $10x^4$

31. $-3z^7$ **33.** $\frac{5y^4}{2}$ **35.** $\frac{5184}{y^7}$ **37.** $\frac{5x}{3} + 5$

39. $\frac{7}{x}$ **41.** $\frac{1}{x}$ **43.** 27^n **45.** $1, x \neq 0$

47. $\frac{1}{(y+2)^3}$ **49.** $32y^2$ **51.** $\frac{10}{x}$ **53.** $\frac{125x^9}{y^{12}}$

55. 573×10^7 square miles **57.** 946×10^{12} kilometers

59. 1×10^{-7} meter **61.** 350,000,000 air sacs

63. 0000000000000000000001602 coulomb

65. (a) 60×10^4 (b) 20×10^{11}

67. (a) 3071×10^6 (b) 3077×10^{10}

69. (a) 4907×10^{17} (b) 1479

71. (a) $(48 \times 10^{10})(25 \times 10^8) = 12 \times 10^{19}$

(b) $\frac{12 \times 10^{-8}}{64 \times 10^{-6}} = 1875 \times 10^{-3}$

73. (a) \$2,47740 (b) \$2,46728
(c) \$2,42812 (d) \$2,32767
As the number of compoundings per year decreases, the balance in the account also decreases.

75. ≈ 519% **77.** \$10,04859

SECTION 0.4 *(page 36)*

Skills Review *(page 36)*

1. $\frac{4}{27}$ **2.** 48 **3.** $-8x^3$ **4.** $6x^7$
5. $28x^6$ **6.** $\frac{1}{5}x^2$ **7.** $3z^4$ **8.** $\frac{25}{4x^2}$ **9.** 1
10. $(x+2)^{10}$

1. $9^{1/2} = 3$ **3.** $\sqrt[5]{32} = 2$ **5.** $\sqrt{196} = 14$
7. $(-216)^{1/3} = -6$ **9.** $81^{3/4} = 27$
11. $\sqrt[3]{125^2} = 25$ **13.** 3 **15.** 3 **17.** 2
19. -125 **21.** 4 **23.** 216 **25.** $\sqrt{6}$

27. $\frac{27}{8}$ **29.** -4 **31.** $2x\sqrt[3]{2x^2}$ **33.** $\frac{5\sqrt{3}y^2}{|x|}$

35. $\frac{2\sqrt[5]{2}}{y}$ **37.** 90 **39.** 45 **41.** $\frac{\sqrt{5}}{5}$

43. $4\sqrt[3]{4}$ **45.** $\frac{x(5+\sqrt{3})}{11}$ **47.** $3(\sqrt{6}-\sqrt{5})$

49. 25 **51.** $2^{1/2}$ **53.** $x^{3/2}, x \neq 0$ **55.** 5

57. x^3 **59.** $8x^6y^3$ **61.** $2\sqrt[4]{2}$ **63.** $3^{1/2} = \sqrt{3}$

65. $\sqrt[3]{x}$ **67.** $2\sqrt{x}$ **69.** $31\sqrt{2}$ **71.** $-2\sqrt{y}$

73. 3557 **75.** 2006 **77.** 2938 **79.** 0382

81. (4 − √ 7) () ÷ 3

83. $\sqrt{5}+\sqrt{3} > \sqrt{5+3}$ **85.** $5 > \sqrt{3^2+2^2}$

87. $\sqrt{3}\cdot\sqrt[4]{3} > \sqrt[8]{3}$

89. 25 inches × 25 inches × 25 inches **91.** ≈ 1283%

93. No **95.** ≈ 2221 seconds **97.** ≈ 0021 inch

99. ≈ 494 vibrations per second

101. a;Higher notes have higher frequencies.

103. ≈ 402 miles per hour **105.** ≈ 174°F **107.** 1

109. $a^0 = a^{n-n} = \frac{a^n}{a^n} = 1$

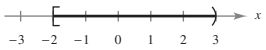
111. Radicals can be added together only if they have the same radicand and index.

MID-CHAPTER QUIZ *(page 39)*

1. $-|-7| < |-7|$ **2.** $-(-3) = |-3|$

3. $x \geq 0$ **4.** $r \geq 95\%$

5. $-2 \leq x < 3$ denotes all numbers greater than or equal to -2 and less than 3.



6. $3x^2, -7x, 2$ **7.** 3 **8.** -13 **9.** $\frac{5}{14}$ **10.** $\frac{11}{9}$

11. $-2x^7$ **12.** $\frac{1}{3}y^4$ **13.** $\frac{27x^6}{y^6}$ **14.** $952779

15. -1 **16.** -64 **17.** 9 **18.** $-\sqrt[3]{3}$ **19.** $2\sqrt{3}$

20. 22 cm × 22 cm × 22 cm

SECTION 0.5 *(page 46)*

Skills Review *(page 46)*

1. $42x^3$ **2.** $-20z^2$ **3.** $-27x^6$ **4.** $-3x^6$

5. $\frac{9}{4}z^3, z \neq 0$ **6.** $4\sqrt{3}$ **7.** $\frac{9}{4x^2}$ **8.** 8

9. $\sqrt{2}$ **10.** $-3x$

1. Degree: 2
Leading coefficient: 2

3. Degree: 5
Leading coefficient:1

5. Degree: 5
Leading coefficient: 3

7. Polynomial, $-3x^3 + 2x + 8$, degree 3

9. Not a polynomial

11. Polynomial, $-w^4 + 2w^3 + w^2$, degree 4

13. (a) -3 (b) 1 (c) 5 (d) 17

15. (a) -10 (b) -1 (c) 4 (d) 5

17. $-2x - 10$ **19.** $2x^3 - 4x - 5$ **21.** $8x^3 + 29x^2 + 11$

23. $3x^3 - 6x^2 + 3x$ **25.** $4x^4 - 12x$

27. $9x^3 - 21x^2$ **29.** $x^2 + 7x + 12$

31. $6x^2 - 7x - 5$ **33.** $x^2 - 25$

35. $x^2 + 12x + 36$ **37.** $4x^2 - 20xy + 25y^2$

39. $x^2 + 2xy + y^2 - 6x - 6y + 9$

41. $x^3 + 3x^2 + 3x + 1$ **43.** $8x^3 - 12x^2y + 6xy^2 - y^3$

45. $9y^4 - 1$ **47.** $m^2 - n^2 - 6m + 9$

49. $x - y$ **51.** $x^4 + x^2 + 1$ **53.** $2x^2 + 2x$

55. The student omitted the middle term when squaring the binomial.

$(x-3)^2 = x^2 - 2(x)(3) + 3^2 = x^2 - 6x + 9$

57. $1000r^3 + 3000r^2 + 3000r + 1000$ **59.** [illegible];No

61. 66,98876; 74,58225; In the years 2004 and 2005, the total amounts of federal student aid disbursed were approximately $66,988,760,000 and $74,582,250,000.

63. $x = 3$ inches: $V = 4968$ cubic inches
$x = 7$ inches: $V = 7448$ cubic inches
$x = 9$ inches: $V = 7344$ cubic inches
$x = 7$ inches produces the greatest volume.

65. $7x^2 + 14x + 4$ **67.** Answers will vary.

SECTION 0.6 *(page 53)*

Skills Review *(page 53)*

1. $15x^2 - 6x$ **2.** $-2y^2 - 2y$ **3.** $4x^2 + 12x + 9$

4. $9x^2 - 48x + 64$ **5.** $2x^2 + 13x - 24$

6. $-5z^2 - z + 4$ **7.** $4y^2 - 1$ **8.** $x^2 - a^2$

9. $x^3 + 12x^2 + 48x + 64$

10. $8x^3 - 36x^2 + 54x - 27$

1. $3(x+2)$ **3.** $3x(x^2-2)$ **5.** $(x-1)(x+5)$

7. $(x+6)(x-6)$ **9.** $(4x+3y)(4x-3y)$

11. $(x+1)(x-3)$ **13.** $(x-2)^2$ **15.** $(2y+3)^2$

17. $\left(y - \frac{1}{3}\right)^2$ **19.** $(x - 2)(x^2 + 2x + 4)$

21. $(y + 5)(y^2 - 5y + 25)$ **23.** $(2t - 1)(4t^2 + 2t + 1)$

25. $(x + 2)(x - 1)$ **27.** $(w - 2)(w - 3)$

29. $(y + 5)(y - 4)$ **31.** $(x - 20)(x - 10)$

33. $(3x - 2)(x - 1)$ **35.** $(3x + 1)(3x - 2)$

37. $(6x + 1)(x + 6)$ **39.** $(x - 1)(x^2 + 2)$

41. $(2x - 1)(x^2 - 3)$ **43.** $(2 - y^3)(3 + y)$

45. $4x(x - 2)$ **47.** $y(y - 3)(y + 3)$

49. $3(x + 4)(x - 4)$ **51.** $(x - 1)^2$ **53.** $(1 - 2x)^2$

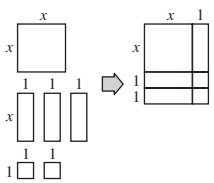
55. $y(2y + 3)(y - 5)$ **57.** $2x(x - 2)(x + 1)$

59. $(3x + 1)(x^2 + 5)$ **61.** $x(x - 4)(x^2 + 1)$

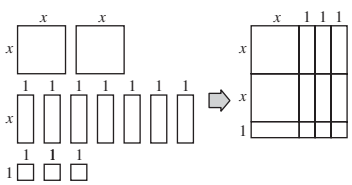
63. $-x(x + 10)$ **65.** $(x + 1)^2(x - 1)^2$

67. $2(t - 2)(t^2 + 2t + 4)$

69.



71.



73. $(2x - 3)$ feet

75. (a) $(x + 8)(x + 3)$; factoring by grouping

(b) $(3x - 5)(x + 4)$; factoring by trial and error

77. $c = \{7, 12, 15, 16\}$; Answers will vary.

79. Answers will vary.Sample answer:

(1) Find a combination of factors of 2 and -15 such that the outer and inner products add up to the middle term $-7x$.

$2x^2 - 7x - 15 = (2x + 3)(x - 5)$

(2) Rewrite -7 as the sum of two factors of the product $2(-15)$. Then factor by grouping.

$$\begin{aligned} 2x^2 - 7x - 15 &= 2x^2 - 10x + 3x - 15 \\ &= 2x(x - 5) + 3(x - 5) \\ &= (x - 5)(2x + 3) \end{aligned}$$

81. Box 1: $V = (a - b)a^2$

Box 2: $V = (a - b)ab$

Box 3: $V = (a - b)b^2$

Multiplying $(a - b)$ by each term of $(a^2 + ab + b^2)$ produces the volumes of the three boxes.

SECTION 0.7 *(page 60)*

Skills Review *(page 60)*

1. $5x^2(1 - 3x)$ **2.** $(4x + 3)(4x - 3)$

3. $(3x - 1)^2$ **4.** $(2y + 3)^2$ **5.** $(z + 3)(z + 1)$

6. $(x - 5)(x - 10)$ **7.** $(3 - x)(1 + 3x)$

8. $(3x - 1)(x - 15)$ **9.** $(s + 1)(s + 2)(s - 2)$

10. $(y + 4)(y^2 - 4y + 16)$

1. (a) No (b) Yes **3.** (a) Yes (b) Yes

5. All real numbers **7.** All real numbers except $x = 2$

9. All real numbers except $x = 0$ and $x = 4$

11. All real numbers greater than or equal to -1

13. $3x,\ x \neq 0$ **15.** $x - 2,\ x \neq 2,\ x \neq 0$

17. $x + 2,\ x \neq -2,\ x \neq 3$

19. $\frac{3x}{2},\ x \neq 0$ **21.** $\frac{x}{2(x + 1)},\ x \neq -1$

23. $-\frac{1}{2},\ x \neq 5$ **25.** $-(x + 5),\ x \neq 5$

27. $\frac{x(x + 3)}{x - 2},\ x \neq -2$ **29.** $\frac{y - 4}{y + 6},\ y \neq 3$

31. $-1 - x^2,\ x \neq 2$ **33.** $z - 3$ **35.** $\frac{1}{5(x - 2)},\ x \neq 1$

37. $-\frac{x(x + 7)}{x + 1},\ x \neq 9$ **39.** $\frac{r + 1}{r},\ r \neq 1$

41. $\frac{t - 3}{(t + 3)(t - 2)},\ t \neq -2$ **43.** $\frac{x - 1}{x(x + 1)^2},\ x \neq -2$

45. $\frac{3}{2},\ x \neq -y$ **47.** $x(x + 1),\ x \neq -1, 0$

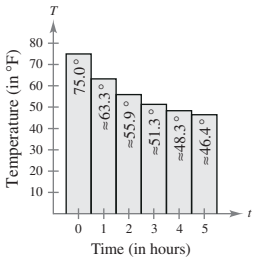
49. $x^2(x - 1)$ **51.** $(x + 5)(x - 7)$

53. $\frac{5x}{x - 2}$ **55.** $\frac{2x}{x - 4}$ **57.** $\frac{4x - 23}{x - 5}$

59. $\frac{x - 4}{(x + 2)(x - 2)(x - 1)}$ **61.** $\frac{2 - x}{x^2 + 1},\ x \neq 0$

63. $\frac{1}{2},\ x \neq 2$ **65.** $\frac{1}{x},\ x \neq -1$ **67.** $\frac{2x - 1}{2x}, x > 0$

69. (a) 12% (b) $\frac{288(NM - P)}{N(12P + NM)}$;12%

71. 

;No

REVIEW EXERCISES *(page 64)*

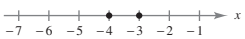
1. (a) Natural: $\{11\}$

(b) Integer: $\{11, -14\}$

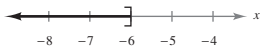
(c) Rational: $\left\{11, -14, -\frac{8}{9}, \frac{5}{2}, 0.4\right\}$

(d) Irrational: $\left\{\sqrt{6}\right\}$

3. $-4 < -3$



5. $x \leq -6$ denotes all real numbers less than or equal to -6.



7. $x \geq 0$ **9.** $2 < x \leq 5$ **11.** -14

13. $|-12| > -|12|$ **15.** 4 **17.** 10 **19.** $|x - 7| \geq 4$

21. $-2x^2, x, 4$ **23.** $3x^3, 7x, -4$ **25.** (a) 2 (b) 0

27. Distributive Property **29.** Commutative (addition)

31. -1 **33.** $\frac{2}{3}$ **35.** 5 **37.** 010 **39.** -2

41. -5 **43.** $8x,\ x \neq 0$ **45.** $\dfrac{5}{x^4}$

47. 3004×10^8 **49.** 864,400 miles

51. (a) 11,414125 (b) 18,380160

53.

Year	5	10	15
Balance	\$2074.23	\$2868.28	\$3966.30

Year	20	25
Balance	\$5484.67	\$7584.30

55. $16^{1/2} = 4$ **57.** 13 **59.** $2x^2$ **61.** $2 + \sqrt{3}$

63. $-3\sqrt{x}$ **65.** $\sqrt{10}$ **67.** $\frac{1}{16}$ **69.** $\sqrt{5}$

71. 11269 **73.** $-6x + 26$ **75.** $x^2 - 8x - 14$

77. $x^2 - x - 2$ **79.** $x^3 + 64$ **81.** $x^2 + 8x + 16$

83. 6155

In 2005, the average sale price for a newly manufactured residential mobile home in the United States was \$61,550.

85. $x = 0$: 1066

$x = 5$: 2025

There were 106,600,000 and 202,500,000 cell phone subscribers in the United States in 2000 and 2005, respectively.

87. $4(x + 3)(x - 3)$ **89.** $3x(x + 1)(x - 2)$

91. $x(x + 4)(x - 4)$ **93.** $(x + 3)(x - 3)(x - 2)$

95. All real numbers except $x = 3$

97. All real numbers **99.** $3x$

101. $\dfrac{x - 2}{2}, x \neq -2$ **103.** $x - 5, x \neq -3$

105. $\dfrac{x + 3}{x - 1}, x \neq 0, 3$ **107.** $\dfrac{x - 1}{x - 3}, x \neq -1, \dfrac{1}{2}$

109. $\dfrac{3x^2 - 4x}{(x - 1)(x - 2)}$ **111.** $\dfrac{6}{x - 1},\ x \neq -1$

113. $\dfrac{x + 1}{x - 1}, x \neq 0$ **115.** $\dfrac{y - x}{y + x}, xy \neq 0$

CHAPTER TEST *(page 67)*

1. -12

2.

Year	5	10	15
Balance	\$5813.18	\$8448.26	\$12,277.81

Year	20	25
Balance	\$17,843.27	\$25,931.52

The longer you leave the \$000 in the account, the more money you earn.

3. $-64x^6$ **4.** $-4\sqrt{x}$ **5.** 25 **6.** $4\left(\sqrt{3} - \sqrt{5}\right)$

7. $2x\sqrt{3x}$ **8.** $\dfrac{5 + \sqrt{7}}{9}$ **9.** $9x^2 + 42x + 49$

10. $-5x^2 + 29x$ **11.** $5(x + 4)(x - 4)$

12. $(2x + 3)^2$ **13.** $(x^2 - 3)(x - 6)$

14. $(x + 2)^2(x - 2)$ **15.** $\frac{1}{3}(x - 4), x \neq -4$

16. $\dfrac{x + 4}{3x + 5}, x \neq -3, \dfrac{5}{3}$ **17.** $\dfrac{4x^2 - 13x}{(x - 3)(x - 4)}$

18. $-\dfrac{x + 26}{(x + 5)(x - 2)}$

19. All real numbers greater than or equal to 2

20. All real numbers except $x = -1$

21. $\dfrac{2x^2 - 5x - 18}{5 + 5x - x^2}, x \neq 1, -2$

22. $x = 5$: 421, $x = 15$: 645

In 1995 and 2005, the average prices of a movie ticket in the United States were \$421 and \$645, respectively.

CHAPTER 1

SECTION 1.1 *(page 76)*

Skills Review *(page 76)*

1. $-3x - 10$ **2.** $5x - 12$ **3.** x **4.** $x + 26$

5. $\frac{8x}{15}$ **6.** $\frac{3x}{4}$ **7.** $-\frac{1}{x(x+1)}$ **8.** $\frac{5}{x}$

9. $\frac{7x-8}{x(x-2)}$ **10.** $-\frac{2}{x^2-1}$

1. Identity **3.** Conditional equation

5. Conditional equation

7. (a) No (b) No (c) Yes (d) No

9. (a) Yes (b) Yes (c) No (d) No

11. (a) Yes (b) No (c) No (d) No

13. (a) No (b) No (c) No (d) Yes

15. (a) Yes (b) No (c) No (d) No

17. 5 **19.** -4 **21.** 3 **23.** 9 **25.** -26

27. -4 **29.** $-\frac{6}{5}$ **31.** 9 **33.** No solution

35. 10 **37.** 4 **39.** 3 **41.** 5 **43.** No solution

45. $\frac{11}{6}$ **47.** No solution **49.** 0

51. All real numbers **53.** No solution

55. Because substituting 2 for x in the equation produces division by zero, $x = 2$ cannot be a solution to the equation.

57. Extraneous solutions may arise when a fractional expression is multiplied by factors involving the variable.

59. Equivalent equations have the same solutions.

Example: $2x - 6 = 0$ and $x - 3 = 0$ both have the solution $x = 3$.

61. $x \approx 138889$ **63.** $x \approx 62372$ **65.** $x \approx 19993$

67. Use the *table* feature in ASK mode or use the scientific calculator part of the graphing utility.

69. (a) 646 (b) 641 Yes **71.** (a) 5609 (b) 5613 Yes

73. 2003 ($t \approx 1281$) **75.** 589 inches

77. 2003 ($t \approx 1297$) **79.** 2001 ($t \approx 1098$)

SECTION 1.2 *(page 87)*

Skills Review *(page 87)*

1. 14 **2.** 4 **3.** -3 **4.** 4 **5.** -2

6. 1 **7.** $\frac{2}{5}$ **8.** $\frac{10}{3}$ **9.** 6 **10.** $-\frac{11}{5}$

1. $x + (x + 1) = 2x + 1$ **3.** $50t$ **5.** $02x$ **7.** $6x$

9. $25x + 1200$ **11.** $5 + x = 8$ **13.** $\frac{r}{2} = 9$

15. $n + 2n = 15$ **17.** $525 = n + (n + 1)$;262, 263

19. $5x - x = 148$;37, 185

21. $n^2 - 5 = n(n + 1)$;$-5, -4$

23. Coworker's check: $800
Your check: $848

25. January: $2,92683; February: $6,07317

27. $\approx$ 3703% decrease **29.** $\approx$ 3942% increase

31. $\approx$ 2240% increase **33.** $\approx$ 12857%

35. $\approx$ 7143% **37.** $\approx$ 5417% decrease

39. (a) $7,800 (b) $10,74840 (c) $14,57875

41. (a) 719 million users (b) $\approx$ 81678 million users
(c) $\approx$ 109285 million users (d) $\approx$ 11857%

43. TV: 15642 hours
Radio, music: 11376 hours
Internet: 2133 hours
Video games: 10665 hours
Print media: 39105 hours
Other: 1422 hours

45. 15 feet $\times$ 225 feet **47.** $\approx$ 57 years

49. 97 or greater **51.** $41176 **53.** $\approx$ 2013%

55. $8 **57.** $6125 **59.** $2\frac{1}{3}$ hours **61.** $\frac{1}{3}$ hour

63. 128 seconds **65.** 625 feet **67.** $81,080

69. $10,500 at 65% and $4500 at 75%

71. Stock A: $3200
Stock B: $3800

73. 1143% **75.** 8571 units per month **77.** $\approx$ 48 feet

79. $\approx$ 321 gallons **81.** $\approx$ 1231 miles per hour

83. $h = \frac{2A}{b}$ **85.** $l = \frac{V}{wh}$ **87.** $h = \frac{V}{\pi r^2}$

89. $C = \frac{S}{1+R}$ **91.** $r = \frac{A-P}{Pt}$ **93.** $b = \frac{2A-ah}{h}$

95. $n = \frac{L+d-a}{d}$ **97.** $h = \frac{A}{2\pi r}$

99. $R_1 = \frac{R_2 f(n-1)}{R_2 + f(n-1)}$

101. Williams: $8,700
Gonzalez: $\approx$ $1,333
Walters: $0,000
Gilbert: $\approx$ $9,933
Hart: $\approx$ $7,833
Team average: January: $7,120, February: $0,100,
March: $1,460

103. Williams: ≈ $5,033
Gonzalez: ≈ $2,867
Walters: $5,400
Gilbert: ≈ $7,467
Hart: $8,100
Reyes: ≈ $4,967
Sanders: ≈ $3,633
Team average: July: ≈ $4,514, August: ≈ $5,157, September: $2,100

105. "takes 30 minutes" "from a depth of 150 feet"

SECTION 1.3 *(page 100)*

Skills Review *(page 100)*

1. $\frac{\sqrt{14}}{10}$ **2.** $4\sqrt{2}$ **3.** 14 **4.** $\frac{\sqrt{10}}{4}$

5. $x(3x + 7)$ **6.** $(2x - 5)(2x + 5)$

7. $-(x - 7)(x - 15)$ **8.** $(x - 2)(x + 9)$

9. $(5x - 1)(2x + 3)$ **10.** $(6x - 1)(x - 12)$

1. $2x^2 + 5x - 3 = 0$ **3.** $x^2 - 25x = 0$

5. $x^2 - 6x + 7 = 0$ **7.** $2x^2 - 2x + 1 = 0$

9. $3x^2 - 60x - 10 = 0$ **11.** $4, -2$ **13.** $0, -\frac{1}{2}$

15. -5 **17.** $3, -\frac{1}{2}$ **19.** $2, -6$ **21.** $-2, -5$

23. ± 4 **25.** $\pm\sqrt{7} \approx \pm 265$ **27.** $\pm 2\sqrt{3} \approx \pm 346$

29. $12 + 3\sqrt{2} \approx 1624$ **31.** $-2 + 2\sqrt{3} \approx 146$
$12 - 3\sqrt{2} \approx 776$ $-2 - 2\sqrt{3} \approx -546$

33. ± 5 **35.** $\pm\sqrt{38} \approx \pm 616$ **37.** $\pm\frac{\sqrt{115}}{5} \approx \pm 214$

39. $\pm\frac{\sqrt{78}}{3} \approx \pm 294$ **41.** ± 8 **43.** 1 **45.** $\pm\frac{3}{4}$

47. $\frac{3}{2}$ **49.** $3, -11$ **51.** $\frac{3}{2}, -\frac{1}{2}$ **53.** $5, -\frac{10}{3}$

55. $9, 3$ **57.** $\frac{1}{5}, 1$ **59.** $-1, -5$ **61.** $-\frac{1}{2}$

63. Algebra argument:

$(x + 2)^2 = (x + 2)(x + 2)$ Definition of exponent
$= x^2 + 2x + 2x + 4$ FOIL
$= x^2 + 4x + 4$ Combine like terms.

So, $(x + 2)^2 \neq x^2 + 4$.

Graphing utility argument:

(1) Let $y_1 = (x + 2)^2$ and $y_2 = x^2 + 4$. Use the *table* feature with an arbitrary value of x (but not $x = 0$). The table will show that the values of y_1 are not the same as the values of y_2.

(2) Use the scientific calculator portion of the graphing utility to show that if $x = 5$, $(5 + 2)^2 = 49$ and $5^2 + 4 = 29$. So, $(x + 2)^2$ is not equal to $x^2 + 4$.

65. 34 feet × 48 feet

67. Base: $2\sqrt{2}$ feet
Height: $2\sqrt{2}$ feet

69. 5 feet **71.** ≈ 354 seconds

73. ≈ 143 seconds **75.** 42 seconds faster

77. ≈ 424 centimeters **79.** 976 miles

81. ≈ 49497 meters **83.** 60,000 units

85. 2012 ($t \approx 116$)

87. (a) 1987 ($t \approx 1874$)

(b) Yes; the model is a good representation through 1890.

(c) Yes; the model is a good representation through 2006.

89. The model in Exercise 88 is *not* valid for the population in 2050 because it predicts 536,526,000 people (not 419,854,000).

91. 1 P.M. ($t \approx 1296$); No; the predicted temperature at 7 P.M. is about 145°F, which is unreasonable.

93. 2003 ($t \approx 308$)

SECTION 1.4 *(page 110)*

Skills Review *(page 110)*

1. $3\sqrt{17}$ **2.** $2\sqrt{3}$ **3.** $4\sqrt{6}$ **4.** $3\sqrt{73}$

5. $2, -1$ **6.** $\frac{3}{2}, -3$ **7.** $5, -1$ **8.** $\frac{1}{2}, -7$

9. $3, 2$ **10.** $4, -1$

1. One real solution **3.** Two real solutions

5. No real solutions **7.** Two real solutions

9. $\frac{1}{2}, -1$ **11.** $\frac{1}{4}, -\frac{3}{4}$ **13.** $1 \pm \sqrt{3}$

15. $-7 \pm \sqrt{5}$ **17.** $-4 \pm 2\sqrt{5}$

19. $\frac{2}{3} \pm \frac{\sqrt{7}}{3}$ **21.** $-\frac{1}{3} \pm \frac{\sqrt{11}}{6}$ **23.** $-\frac{1}{2} \pm \sqrt{2}$

25. $\frac{2}{7}$ **27.** $2 \pm \frac{\sqrt{6}}{2}$ **29.** $6 \pm \sqrt{11}$

31. $x \approx 0976, -0643$ **33.** $x \approx 0561, 0126$

35. No real solution **37.** -11 **39.** $\pm\sqrt{10}$

41. $-\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ **43.** $-2, 4$ **45.** $\frac{-1 \pm \sqrt{37}}{6}$

47. Real-life problems will vary; 50, 50

49. Real-life problems will vary; 7, 8 or $-8, -7$

51. 200 units **53.** 653 units **55.** 9 seats per row

57. 14 inches × 14 inches

59. (a) $s = -16t^2 + \frac{88}{3}t + 984$ (b) $\approx$ 84533 feet

(c) $\approx$ 881 seconds

61. Moon: $\approx$149 seconds **63.** Moon
Earth: $\approx$26 seconds

65. $\approx$ 259 miles; $\approx$541 miles

67. (a) 2003 ($t \approx 1271$) (b) 2005 ($t \approx 1492$)

(c) No.The model's prediction of $79 billion is less than the expected sales.

69. 400 P.M. ($t \approx 39$)

71. Southbound: $\approx$550 miles per hour
Eastbound: $\approx$600 miles per hour

73. (a) $\approx$ 168°C (b) $\approx$ 25

75. 5279 units or 94,721 units **77.** Answers will vary.

MID-CHAPTER QUIZ *(page 114)*

1. $x = -6$ **2.** $x = 6$ **3.** $x = -2$ **4.** No solution

5. Use the *table* feature in ASKmode or the scientific calculator portion of the graphing utility.

6. 328954 **7.** 431398

8. $850x + 30{,}000 = 200{,}000$; 20,000 units

9. $300{,}000 = x(75 - 00002x)$; 4044 units or 370,956 units

10. $x = \frac{2}{3}, -5$ **11.** $x = \pm\sqrt{5}; x \approx \pm 224$

12. $x = -3 \pm \sqrt{17}; x \approx -712, 112$

13. $x = -1 \pm \sqrt{6}$; $x \approx -345, 145$

14. $x = \dfrac{-7 \pm \sqrt{73}}{6}$; $x \approx -259, 026$

15. $x \approx 1568, -0068$ **16.** No real solutions

17. One real solution

18. Answers will vary.*Sample answer:* Use the FOIL method $[(x + 3)^2 = (x + 3)(x + 3) = x^2 + 6x + 9]$, use the *table* feature of your graphing utility, or use the scientific calculator portion of your graphing utility to evaluate the solution.

19. $\approx$ 433 seconds **20.** 8 inches × 8 inches × 6 inches

SECTION 1.5 *(page 123)*

Skills Review *(page 123)*

1. 11 **2.** 20, −3 **3.** 5, −45 **4.** $0, -\frac{1}{5}$

5. $\frac{2}{3}, -2$ **6.** $\frac{11}{6}, -\frac{5}{2}$ **7.** 1, −5 **8.** $\frac{3}{2}, -\frac{5}{2}$

9. $\dfrac{3 \pm \sqrt{5}}{2}$ **10.** $2 \pm \sqrt{2}$

1. 3, −1, 0 **3.** $0, \pm\dfrac{3\sqrt{2}}{2}$ **5.** ±3 **7.** −3, 0

9. ±2, 7 **11.** ±1 **13.** $\pm\sqrt{11}, \pm 1$ **15.** ±2

17. $\pm\frac{1}{2}, \pm 4$ **19.** 1, −2 **21.** 50 **23.** 26

25. −16 **27.** $\frac{1}{4}$ **29.** 6, 5 **31.** 2, −5

33. 0 **35.** −59, 69 **37.** 1 **39.** $\pm\sqrt{69}$

41. $\dfrac{-3 \pm \sqrt{21}}{6}$ **43.** 4, −5 **45.** −1 **47.** 1, −3

49. 1, −3 **51.** 3, −2 **53.** $\sqrt{3}, -3$ **55.** 10, −1

57. The quadratic equation was not written in general form before the values of a, b, and c were substituted in the Quadratic Formula.The general form for this equation is $3x^2 - 7x - 4 = 0$ ($a = 3$, $b = -7$, and $c = -4$), and the correct solution is

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-4)}}{2(3)}.$$

59. $x \approx \pm 1038$ **61.** $x \approx 16756$ **63.** 34 **65.** 6%

67. $\approx$ 1298% **69.** 45,000 passengers **71.** 63 years old

73. 67,760 units;It does not make sense for demand xor price p to be less than zero.

75. $\approx$ 1212 feet

77. Least acceptable weight: 788 ounces
Greatest acceptable weight: 812 ounces

79. $8\frac{2}{11}$ hours

SECTION 1.6 *(page 134)*

Skills Review *(page 134)*

1. $-\frac{1}{2}$ **2.** $-\frac{1}{6}$ **3.** −3 **4.** −6 **5.** $x \geq 0$

6. $-3 < z < 10$ **7.** $P \leq 2$ **8.** $W \geq 200$

9. 2, 7 **10.** 0, 1

1. $-1 \leq x \leq 5$; Bounded **3.** $x > 11$; Unbounded

5. $x < -2$; Unbounded

7. c **8.** h **9.** f **10.** e

11. g **12.** a **13.** b **14.** d

15. (a) Yes (b) No (c) Yes (d) No

17. (a) Yes (b) No (c) No (d) Yes

19. (a) Yes (b) Yes (c) Yes (d) No

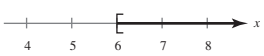
21. (a) No (b) Yes (c) Yes (d) Yes

23. If $2x > 6$, then $x > 3$. **25.** If $2x \leq -8$, then $x \leq -4$.

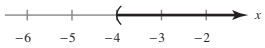
27. If $2 - 4x > -10$, then $x < 3$.

29. If $-\frac{2}{3}x \geq -6$, then $x \leq 9$.

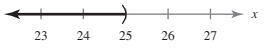
31. $x \geq 6$



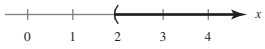
33. $x > -4$



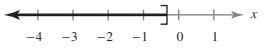
35. $x < 25$



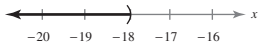
37. $x > 2$



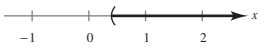
39. $x \leq -\frac{1}{3}$



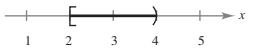
41. $x < -18$



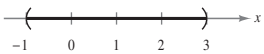
43. $x > \frac{2}{5}$



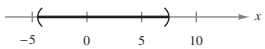
45. $2 \leq x < 4$



47. $-1 < x < 3$



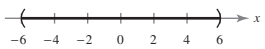
49. $-\frac{9}{2} < x < \frac{15}{2}$



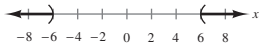
51. $-\frac{3}{4} < x < -\frac{1}{4}$



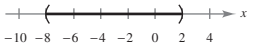
53. $-6 < x < 6$



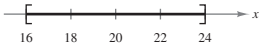
55. $x < -6$ or $x > 6$



57. $-8 < x < 2$



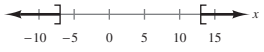
59. $16 \leq x \leq 24$



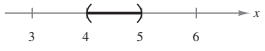
61. $x < -\frac{1}{2}$ or $x > \frac{11}{2}$



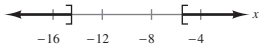
63. $x \leq -7$ or $x \geq 13$



65. $4 < x < 5$



67. $x \leq -\frac{29}{2}$ or $x \geq -\frac{11}{2}$



69. No solution

71. $|x| \leq 2$ **73.** $|x - 9| \geq 3$ **75.** $|x - 12| \leq 10$

77. $|x + 3| > 5$ **79.** More than 250 miles

81. Greater than 867% **83.** $33\frac{1}{3}$ weeks **85.** 16 inches

87. (a)

x	10	20	30
R	\$1399.50	\$2799.00	\$4198.50
C	\$1820.00	\$2790.00	\$3760.00

x	40	50	60
R	\$5598.00	\$6997.50	\$8397.00
C	\$4730.00	\$5700.00	\$6670.00

(b) $x \geq 20$ units

89. Less than 29,6875 miles **91.** $x \geq 11401$

93. 2008 ($t > 1772$) **95.** [≈ 106864, ≈ 109464]

97. Undercharged or overcharged by as much as \$25

99. [658, 712]

101. Minimum = 20% Maximum = 80%

103. 2001 ($t > 1132$)

105. (a) 1995 ($t > 486$) (b) 2009 ($t > 1933$)

SECTION 1.7 *(page 145)*

Skills Review *(page 145)*

1. $y < -6$ **2.** $z > -\frac{9}{2}$ **3.** $-3 \leq x < 1$

4. $x \leq -5$ **5.** $-3 < x$ **6.** $5 < x < 7$

7. $-\frac{7}{2} \leq x \leq \frac{7}{2}$ **8.** $x < 2, x > 4$

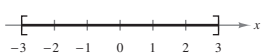
9. $x < -6, x > -2$ **10.** $-2 \leq x \leq 6$

1. $(-\infty, -5), (-5, 5), (5, \infty)$

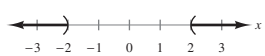
3. $(-\infty, -4), \left(-4, \frac{1}{2}\right), \left(\frac{1}{2}, \infty\right)$

5. $(-\infty, -1), (-1, 1), (1, \infty)$

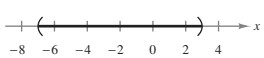
7. $[-3, 3]$



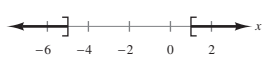
9. $(-\infty, -2) \cup (2, \infty)$



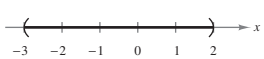
11. $(-7, 3)$



13. $(-\infty, -5] \cup [1, \infty)$



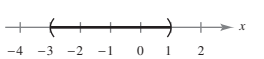
15. $(-3, 2)$



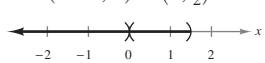
17. $(-\infty, -1) \cup (1, \infty)$



19. $(-3, 1)$



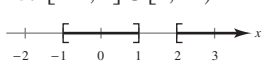
21. $(-\infty, 0) \cup \left(0, \frac{3}{2}\right)$



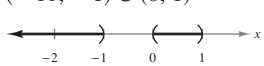
23. $[-2, 0] \cup [2, \infty)$



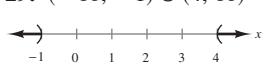
25. $[-1, 1] \cup [2, \infty)$



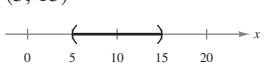
27. $(-\infty, -1) \cup (0, 1)$



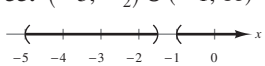
29. $(-\infty, -1) \cup (4, \infty)$



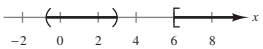
31. $(5, 15)$



33. $\left(-5, -\frac{3}{2}\right) \cup (-1, \infty)$



35. $\left(-\frac{3}{4}, 3\right) \cup [6, \infty)$



37. $(-\infty, -3] \cup [3, \infty)$ **39.** All real numbers

41. $\left[-\frac{9}{2}, \frac{9}{2}\right]$ **43.** $(-\infty, 2) \cup (5, \infty)$

45. All real numbers

47. The cube root of any real number is a real number.

49. $\left(\frac{5}{3}, \infty\right)$ **51.** $(-\infty, -3] \cup [0, 3]$

53. $[-2, \infty)$ **55.** $(-351, 351)$ **57.** $(-013, 2513)$

59. $(226, 239)$ **61.** Between 25 and 10 seconds

63. Between about 138 meters and about 362 meters

65. (a) $90{,}000 \le x \le 100{,}000$ (b) $\$0 \le p \le \$2$

(c) about 185,967 units; The company should produce a maximum of about 185,967 units.

67. 145% **69.** 2012 $(t > 2245)$

71. 20082009 $(t > 1894)$ **73.** $R_1 \ge 2$ ohms

REVIEW EXERCISES *(page 150)*

1. Conditional equation

3. (a) No (b) Yes (c) Yes (d) No

5. 13 **7.** $-\frac{1}{2}$ **9.** $-\frac{5}{3}$ **11.** $-\frac{2}{3}$ **13.** 377778

15. 0033 **17.** 12 **19.** $150 - x = 120$; 30 pounds

21. 295 feet $\times$ 59 feet **23.** $\$0$ **25.** $\$6353$

27. 2 hours **29.** $\$51{,}664$ **31.** $2\frac{2}{9}$ quarts

33. $-\frac{1}{2}, \frac{4}{3}$ **35.** 3, 8 **37.** $\pm\sqrt{11}, \approx \pm 332$

39. $-4 + 3\sqrt{2} \approx 024$

$-4 - 3\sqrt{2} \approx -824$

41. (1) Use the *table* feature in ASK mode with the variable equal to a solution.

(2) Use the scientific calculator portion of the graphing utility to evaluate the quadratic equation at a particular solution.

43. 15 feet $\times$ 27 feet **45.** 200,000 units or 400,000 units

47. Two real solutions **49.** $6 \pm \sqrt{6}$

51. $\dfrac{-19 \pm \sqrt{165}}{2}$ **53.** $-3 \pm 2\sqrt{3}$

55. 1866, − 0283 **57.** 8544, 0162

59. Moon: ≈ 861 seconds

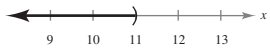
Earth: ≈ 354 seconds

61. $0, -1, 4$ **63.** $\pm 2, \pm 1$ **65.** $\frac{25}{4}$

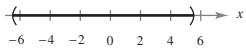
67. No solution **69.** $\pm 4\sqrt{2}$ **71.** $-3, \frac{7}{5}$

73. $2 \pm \sqrt{19}$ **75.** $\$900$ **77.** $\approx 2795\%$

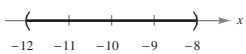
79. $x < 11$



81. $-\frac{13}{2} < x < \frac{11}{2}$

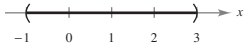


83. $-12 < x < -8$

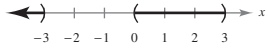


85. $x > 45$ units

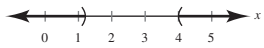
87. $(-1, 3)$



89. $(-\infty, -3) \cup (0, 3)$



91. $\left(-\infty, \frac{6}{5}\right) \cup (4, \infty)$



93. $(-169, 169)$ **95.** $(165, 174)$ **97.** $[10, \infty)$

99. All real numbers **101.** $(-\infty, 6] \cup [9, \infty)$

103. Between 365 and 472 seconds

105. (a) $-0054x^2 + 143x < 8$

(b) $x < 803$ or $x > 1845$

(c) No, 15 is not a solution of the inequality.

107. Between about 63 feet and about 237 feet.

109. Greater than 95% **111.** $\$134 \le p \le \$866$

113. (a)

t	6	10	13	15
R	\$1.58	\$3.16	\$5.18	\$6.93

(b) Yes. The model predicts the revenue per share in 2007 to be \$99.

(c) Yes. The model predicts that revenue per share will exceed \$110 by 2009 $(t > 1878)$.

CHAPTER TEST *(page 154)*

1. $\frac{17}{23}$ **2.** (a) All real numbers (b) $-3 \le x \le 3$

3. April: \$25,78600 May: \$99,72312 **4.** $-\frac{5}{3}, \frac{1}{2}$ **5.** $4, -\frac{3}{2}$

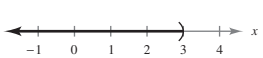
6. $\pm\sqrt{15}$ **7.** $\dfrac{-13 \pm \sqrt{69}}{2}$ **8.** $\dfrac{11 \pm \sqrt{145}}{6}$

9. 1038, − 0446 **10.** $-\frac{7}{2}, \frac{13}{2}$

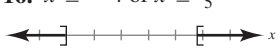
11. 4 (7 is extraneous) **12.** $-1, 1, -3, 3$ **13.** $-6, 6$

14. Selling either 341,421 units or 58,579 units will produce a revenue of \$3,000,000.

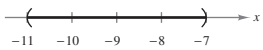
15. $x < 3$



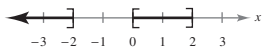
16. $x \le -4$ or $x \ge \frac{28}{5}$



17. $(-11, -7)$



18. $(-\infty, -2] \cup [0, 2]$



19. Between 19,189 units and 143,311 units

20. 2008 ($t > 792$)

CUMULATIVE TEST: CHAPTERS 0–1

(page 155)

1. $-32x^6$ **2.** $3x^2\sqrt{2x}$ **3.** $\dfrac{3+\sqrt{5}}{2}$

4. $(x+\sqrt{3})(x-\sqrt{3})(x-6)$

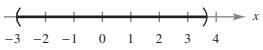
5. $\dfrac{x+4}{5}, x \neq 4$ **6.** $\dfrac{y-x}{x+y}, x \neq 0, y \neq 0$

7. (a) \$025 billion (b) 2010 ($t \approx 983$)

8. $5, \frac{1}{2}$ **9.** 0734, − 1022 **10.** $\frac{8}{3}, -\frac{10}{3}$

11. $5 - 2\sqrt{2}$ **12.** $\pm 1, \pm 4$ **13.** $\pm 3\sqrt{2}$

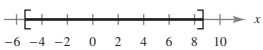
14. $-3 < x < \frac{11}{3}$



15. $[-2\sqrt{2}, 0] \cup [2\sqrt{2}, \infty)$



16. $-\frac{16}{3} \leq x \leq \frac{26}{3}$



17. Between 10,263 units and 389,737 units

18. 2007 ($t > 748$)

CHAPTER 2

SECTION 2.1 *(page 167)*

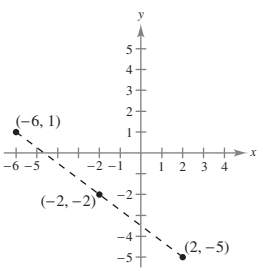
Skills Review *(page 167)*

1. 5 **2.** $3\sqrt{2}$ **3.** 1 **4.** -2

5. $3(\sqrt{2}+\sqrt{5})$ **6.** $2(\sqrt{3}+\sqrt{11})$ **7.** $-3, 11$

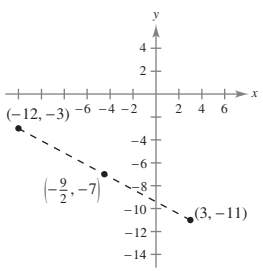
8. 9, 1 **9.** $0, \pm 3$ **10.** ± 2

1. (a)



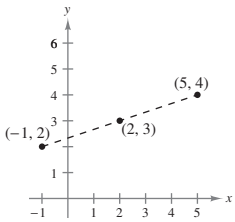
(b) 10 (c) $(-2, -2)$

3. (a)



(b) 17 (c) $\left(-\frac{9}{2}, -7\right)$

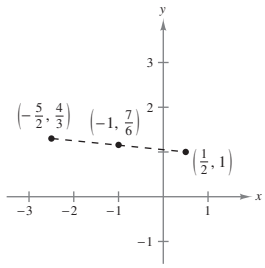
5. (a)



(b) $2\sqrt{10}$

(c) (2, 3)

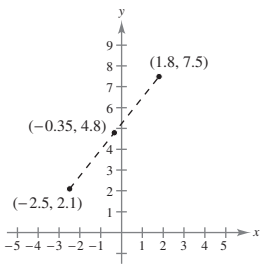
7. (a)



(b) $\dfrac{\sqrt{82}}{3}$

(c) $\left(-1, \dfrac{7}{6}\right)$

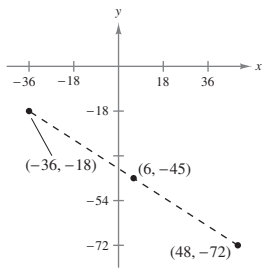
9. (a)



(b) $\sqrt{4765}$

(c) (−035, 48)

11. (a)



(b) $6\sqrt{277}$

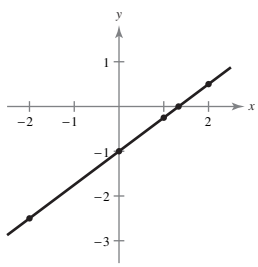
(c) (6, −45)

13. 5 **15.** $\sqrt{109}$ **17.** $x = 15, -9$ **19.** $y = 9, -23$

21. (a) es (b) No **23.** (a) es (b) es

25.

x	y
-2	-25
0	-1
1	-025
$\frac{4}{3}$	0
2	05



27. $(\frac{1}{2}, 0), (0, -1)$ **29.** $(-2, 0), (1, 0), (0, -2)$

31. $(0, 0), (-2, 0)$ **33.** $(\frac{4}{3}, 0), (0, 2)$

35. Every ordered pair on the x-axis has a y-coordinate of zero $[(x, 0)]$, so to find an x-intercept we let $y = 0$. Similarly, every ordered pair on the y-axis has an x-coordinate of zero $[(0, y)]$, so to find a y-intercept we let $x = 0$.

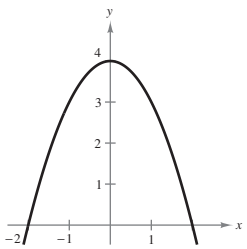
37. y-axis symmetry **39.** x-axis symmetry

41. y-axis symmetry **43.** Origin symmetry

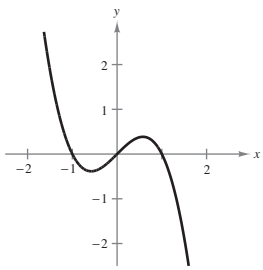
45. Origin symmetry

47. x-axis, y-axis, and origin symmetry

49.

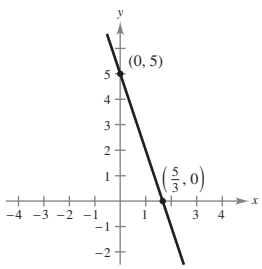


51.



53. c **54.** d **55.** f **56.** a **57.** e **58.** b

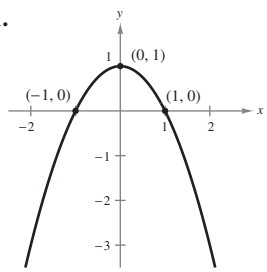
59.



Intercepts:
$\left(\frac{5}{3}, 0\right)$, $(0, 5)$

Symmetry:none

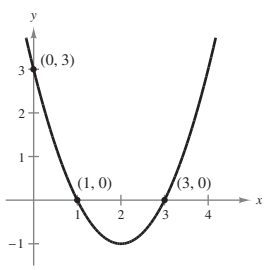
61.



Intercepts:
$(-1, 0)$, $(1, 0)$, $(0, 1)$

Symmetry: y-axis

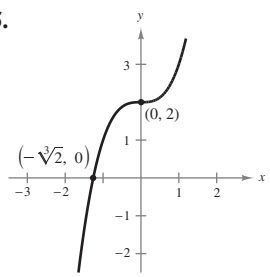
63.



Intercepts:
$(3, 0)$, $(1, 0)$, $(0, 3)$

Symmetry:none

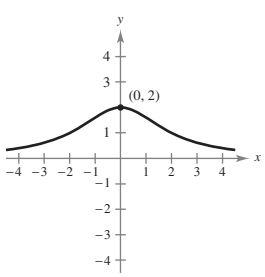
65.



Intercepts:
$\left(-\sqrt[3]{2}, 0\right)$, $(0, 2)$

Symmetry:none

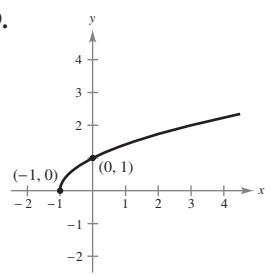
67.



Intercept: $(0, 2)$

Symmetry: y-axis

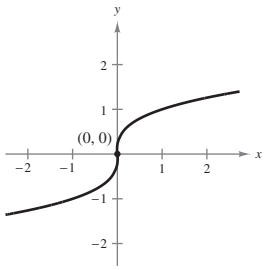
69.



Intercepts: $(-1, 0)$, $(0, 1)$

Symmetry:none

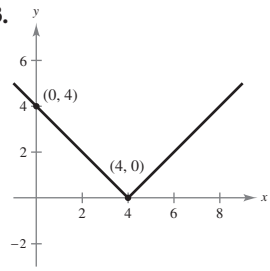
71.



Intercept: $(0, 0)$

Symmetry:origin

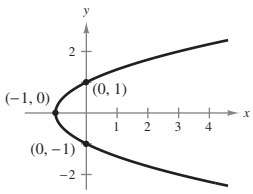
73.



Intercepts: $(4, 0)$, $(0, 4)$

Symmetry:none

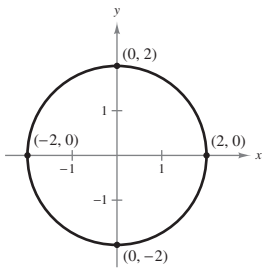
75.



Intercepts: $(-1, 0)$, $(0, 1)$, $(0, -1)$

Symmetry: x-axis

77.



Intercepts: $(-2, 0)$, $(2, 0)$, $(0, 2)$, $(0, -2)$

Symmetry: x-axis, y-axis, origin

79. Radius: 2 **81.** Radius: $\sqrt{5}$

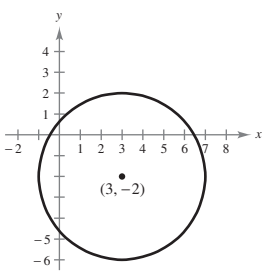
83. Center: $(3, 2)$;Radius: $\sqrt{5}$

85. $x^2 + y^2 = 9$ **87.** $(x + 4)^2 + (y - 1)^2 = 2$

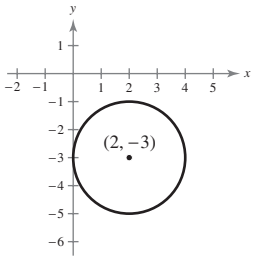
89. $(x + 1)^2 + (y - 2)^2 = 5$

91. $(x - 1)^2 + (y - 1)^2 = 25$

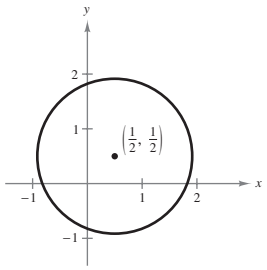
93. $(x - 3)^2 + (y + 2)^2 = 16$



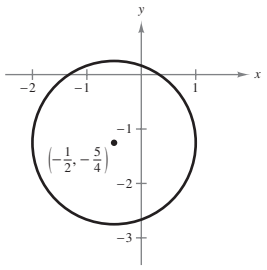
95. $(x-2)^2+(y+3)^2=4$



97. $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=2$



99. $\left(x+\frac{1}{2}\right)^2+\left(y+\frac{5}{4}\right)^2=\frac{9}{4}$



101. Center: $(3, -1)$; Radius: 5

$x^2+y^2-6x+2y-15=0$

103. 610 dollars per fine ounce; 1980 **105.** $\approx 17\%$

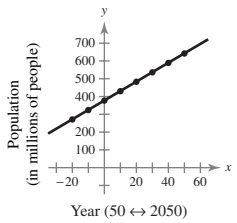
107. (a) (0, 377); It represents the population (in millions of people) of North America in 2000.

(b)

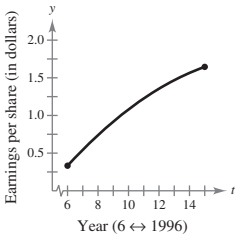
x	-20	-10	0	10	20
y	271	324	377	430	483

x	30	40	50
y	536	589	642

(c)



109. (a)



(b) 2006: \$1.71; 2007: \$1.76

The model's prediction for 2006 is close to Dollar Tree's prediction, but the model's prediction for 2007 is not close to Dollar Tree's prediction.

(c) The model does not support Dollar Tree's prediction. The model predicts an earnings per share of \$1.80 in 2009. After 2009, the predicted values begin to decrease.

111. $x^2+(y-67.5)^2=4556.25$

SECTION 2.2 *(page 179)*

Skills Review *(page 179)*

1. $-\frac{9}{2}$ **2.** $-\frac{13}{3}$ **3.** $-\frac{5}{4}$ **4.** $\frac{1}{2}$

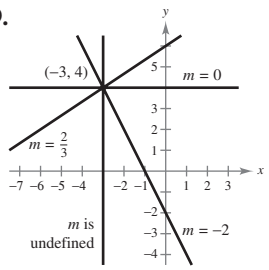
5. $y=\frac{2}{3}x-\frac{5}{3}$ **6.** $y=-2x$

7. $y=3x-1$ **8.** $y=\frac{2}{3}x+5$

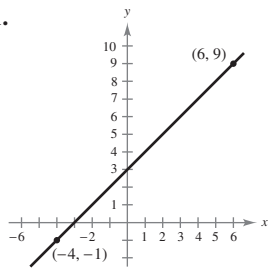
9. $y=-2x+7$ **10.** $y=x+3$

1. 1 **3.** 0 **5.** -1 **7.** Positive

9.

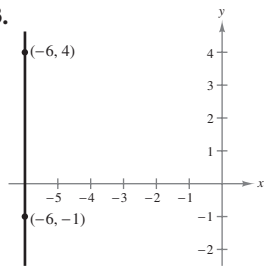


11.



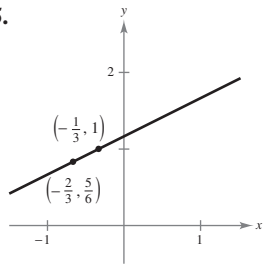
$m=1$

13.



m is undefined.

15.



$m=\frac{1}{2}$

CHAPTER 2

17. Answers will vary.

Sample answer: $(3, -2), (-1, -2), (0, -2)$

19. Answers will vary.

Sample answer: $(2, -3), (2, -7), (2, 9)$

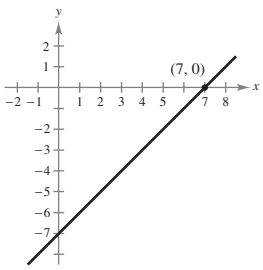
21. Answers will vary.

Sample answer: $(6, -5), (7, -4), (8, -3)$

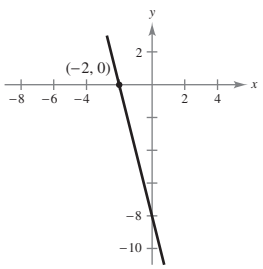
23. Answers will vary.

Sample answer: $(2, 3), (-4, 0), (4, 4)$

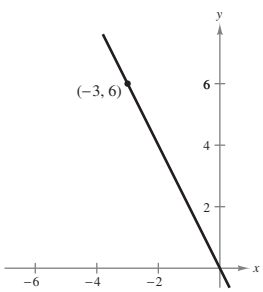
25. $x - y - 7 = 0$



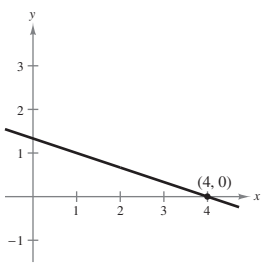
27. $4x + y + 8 = 0$



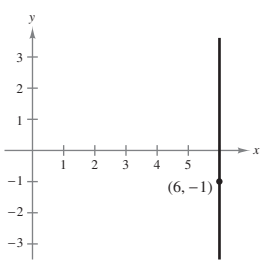
29. $2x + y = 0$



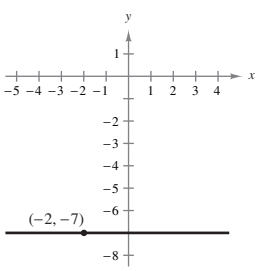
31. $x + 3y - 4 = 0$



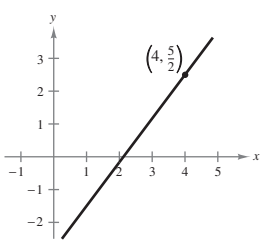
33. $x - 6 = 0$



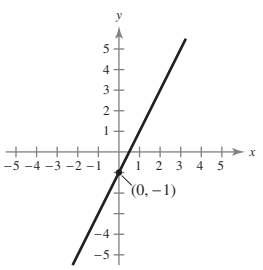
35. $y + 7 = 0$



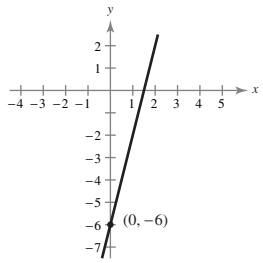
37. $8x - 6y - 17 = 0$



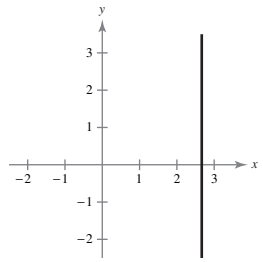
39. $m = 2, (0, -1)$



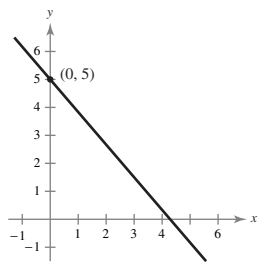
41. $m = 4, (0, -6)$



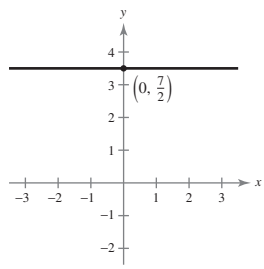
43. m is undefined; no y-intercept



45. $m = -\frac{7}{6}, (0, 5)$



47. $m = 0, \left(0, \frac{7}{2}\right)$



49. $3x - y - 1 = 0$ **51.** $x + 2y + 1 = 0$

53. $x + 9 = 0$ **55.** $y - 7 = 0$ **57.** $x + 2y - 3 = 0$

59. $2x - 5y + 1 = 0$

61. Answers will vary. Sample answer: You could graph a vertical line and pick two convenient points on the line to find the slope. Regardless of the points selected, the slope will have zero in the denominator. Division by zero is not possible, so the slope does not exist.

63. $4x - y - 4 = 0$ **65.** $x + y + 2 = 0$

67. $12x + 3y + 2 = 0$ **69.** Neither

71. Perpendicular **73.** Parallel **75.** Perpendicular

77. Parallel **79.** Perpendicular **81.** Neither

83. Parallel

85. (a) $2x - y - 10 = 0$

(b) $x + 2y - 10 = 0$

87. (a) $4x - 6y - 5 = 0$

(b) $36x + 24y + 7 = 0$

89. (a) $y = 0$

(b) $x + 1 = 0$

91. $F = \frac{9}{5}C + 32$ **93.** $A = 65t + 800$

95. $\left(m = \frac{17}{180}\right)$ **97.** 4191 students

99. No; \$79,000

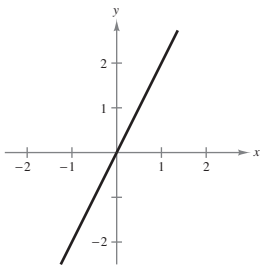
101. $\approx$ \$9684 million; No. If the actual yearly revenue followed a linear trend, then the yearly revenue in 2005 would be close to \$9684 million.

103. $p = \frac{1}{33}d + 1$; $\frac{1}{33}$ atmosphere per foot

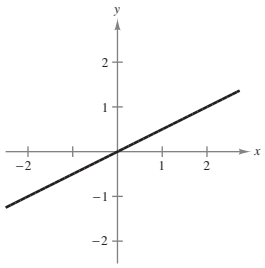
SECTION 2.3 *(page 189)*

Skills Review *(page 189)*

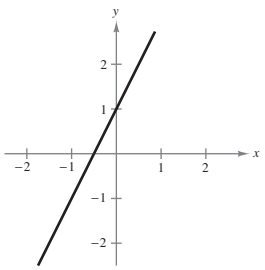
1.



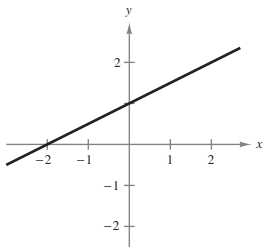
2.



3.



4.

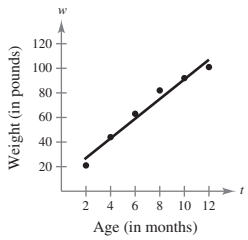


5. $y = x + 2$ **6.** $y = \frac{3}{2}x + 3$ **7.** $y = x + 2$

8. $y = \frac{6}{7}x + 4$ **9.** $5x + 40y - 213 = 0$

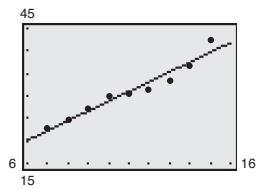
10. $29x + 60y - 448 = 0$

1.



The model is a good fit for the actual data.

3.



$y = 217x + 71$

The model is a good fit for the actual data.

5. $y = \frac{3}{8}x$ **7.** $y = 20x$ **9.** $y = \frac{32}{7}x$ or $y = \frac{16}{35}x$

11. $H = 3p$ **13.** $c = \frac{3}{5}d$ **15.** $I = 0075P$

17. (a) $y = 00368x$ (b) $6808

19. (a) $C = \frac{33}{13}I$

(b)

Inches	5	10	20	25	30
Centimeters	1269	2538	5077	6346	7615

21. $V = 125t + 1915, \quad 5 \le t \le 10$

23. $V = 30{,}400 - 2000t, \quad 5 \le t \le 10$

25. $V = 12{,}500t + 91{,}500, \quad 5 \le t \le 10$

27. (a) $h = 7000 - 20t$ (b) 21350 P.M.

29. $V = 875 - 175t, \quad 0 \le t \le 5$ **31.** $S = 085L$

33. $W = 075x + 1150$

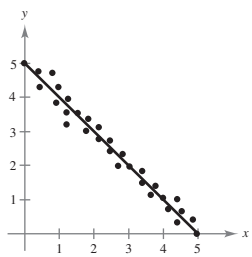
35. (a) $P = 60t + 1300$ (b) 2020 deer

37. b; Slope $= -10$; The amount owed *decreases* by $10 per week.

39. a; Slope $= 048$; The amount received *increases* by $0.48 per mile driven.

41. No;Earning 10 points per coin would result in a positive slope.

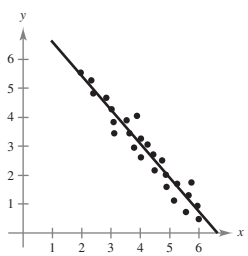
43. Yes;Answers will vary.
Sample answer:



$y = 5 - x$

45. No

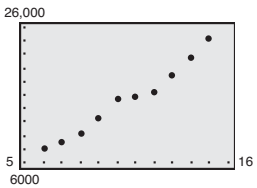
47. Yes;Answers will vary.
Sample answer:



$y = -118x + 776$

CHAPTER 2

49. (a)



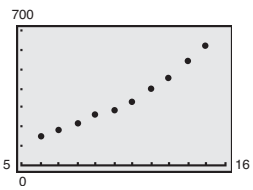
The data appear to be approximately linear.

(b) $A = 17793t - 3525$

(c) 17793;The amount spent on advertising increased by $7793 million each year.

(d) 2006: $84,9438 million
2007: $86,7231 million

51. (a)



(b) Answers will vary.
Sample answer: $y = 5288t - 2039$

(c) $y = 5049t - 1872$

Using linear regression equation:

2006: $62064 million
2007: $67113 million

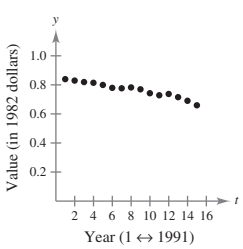
Using equation from part (b):

2006: $64218 million
2007: $69506 million

(d) The projections made by Sonic are higher than the predictions given by the models.

(e) No. Using the linear regression equation, the yearly revenue is expected to reach only about $873 million by 2011. Using the equation from part (b), the yearly revenue is expected to reach only about $907 million by 2011.

53. (a)

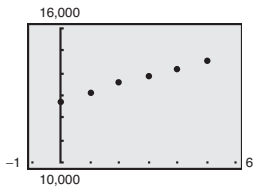


The data appear to be approximately linear.

(b) $y = -00115t + 0840$

(c) 2007: 0645, 2008: 0633;Because the data followed a linear pattern from 1991 to 2005, you can assume that the estimates for 2007 and 2008 are reliable.

55. (a)

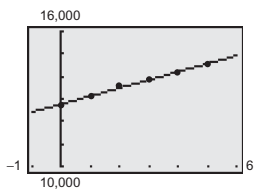


The data appear to be approximately linear.

(b) $E = 3580t + 12,774$

(c) 2007: 15,280,000 employees
2009: 15,996,000 employees

(d)



The predictions are most likely just about right because the model is a good fit for the actual data.

57. The model for population, because population tends to change at a consistent rate, whereas snowfall can be quite different year to year. You should use more than three data points to ensure the data can be represented accurately by a linear model.

SECTION 2.4 *(page 202)*

Skills Review *(page 202)*

1. -73 **2.** 13 **3.** $2(x + 2)$ **4.** $-8(x - 2)$

5. $y = \frac{7}{5} - \frac{2}{5}x$ **6.** $y = \pm x$ **7.** $x \leq -2, x \geq 2$

8. $-3 \leq x \leq 3$ **9.** All real numbers

10. $x \leq 1, x \geq 2$

1. This is a function from A to B, because each element of A is matched with an element of B.

3. Not a function; The relationship does not match the element b of A with an element of B.

5. This is a function from A to B, because each element of A is matched with an element of B.

7. This is a function from A to B, because each element of A is matched with an element of B.

9. Not a function;The relationship assigns two elements of B to the element c of A.

11. Not a function from A to B;The relationship defines a function from B to A.

13. $\{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$

15. $\{(-2, 0), (-1, 1), (0, \sqrt{2}), (1, \sqrt{3}), (2, 2)\}$

17. Not a function **19.** Function **21.** Function

23. Not a function **25.** Function

27. (a) -6 (b) 34 (c) $6 - 4t$ (d) $2 - 4c$

29. (a) -1 (b) $\frac{1}{15}$ (c) $\frac{1}{t^2 - 2t}$ (d) $\frac{1}{t^2 - 1}$

31. (a) -1 (b) -9 (c) $2x - 5$ (d) $-\frac{5}{2}$

33. (a) 0 (b) 3 (c) $x^2 + 2x$ (d) -0.75

35. (a) 36π (b) 0 (c) $\frac{9\pi}{2}$ (d) $\frac{32\pi r^3}{3}$

37. (a) 1 (b) -7 (c) $3 - 2|x|$ (d) 2.5

39. (a) Undefined (b) $-\frac{1}{16}$

(c) $\frac{1}{y^2 + 4y - 12}$, $y \neq -6, 2$

(d) $\frac{1}{y^2 - 4y - 12}$, $y \neq -2, 6$

41. (a) 1 (b) -1 (c) 1 (d) $\frac{|x - 1|}{x - 1}$

43. (a) -4 (b) 3 (c) -7 (d) 7

45. 5 **47.** ± 3 **49.** $0, 1, -1$

51. $\frac{10}{7}$ **53.** All real numbers x

55. All real numbers except $t = 0$

57. All real numbers y **59.** $-1 \leq x \leq 1$

61. All real numbers except $x = 0, -2$

63. All real numbers $x \geq -1$ except $x = 2$ **65.** $x > 0$

67. The domain of $f(x) = \sqrt{x - 2}$ is all real numbers $x \geq 2$, because an even root of a negative number is not a real number. The domain of $g(x) = \sqrt[3]{x - 2}$ is all real numbers. f and g have different domains because an odd root of a negative number is a real number, but an even root of a negative number is not a real number.

69. (a) $V = x(18 - 2x)^2$ (b) Domain: $0 < x < 9$

(c) 400 cubic inches

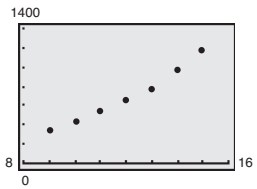
71. (a) $C = 11.75x + 112{,}000$ (b) $R = 21.95x$

(c) $10.2x - 112{,}000$

73. Yes $[y(30) = 6]$

75. 1995: \$37.55 billion
2005: \$68.375 billion

77. (a)



(b) Linear: $S = 135.24t - 923.3$

Quadratic: $S = 12.843t^2 - 172.99t + 874.7$

(c)

1999	347.5	293.9	358.1
2000	438.3	429.1	429.1
2001	539.1	564.3	525.8
2002	652.0	699.6	648.2
2003	773.8	834.8	796.3
2004	969.2	970.1	970.1
2005	1177.6	1105.3	1169.5

The quadratic model is a better fit because its values for S are closest to the actual values of S.

79. (a) $C = 8000 + 2.95x$ (b) $\overline{C} = \frac{8000}{x} + 2.95$

(c)

	100	1000	10,000	100,000
$\overline{C}$	82.95	10.95	3.75	3.03

(d) Answers will vary. Sample answer: The average cost per unit decreases as x gets larger.

81. (a) $R = 12.00n - 0.05n^2, \quad n \geq 80$

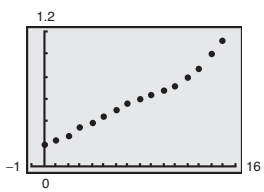
(b)

	90	100	110	120
	\$675	\$700	\$715	\$720

	130	140	150
	\$715	\$700	\$675

(c) Answers will vary. Sample answer: Maximum revenue of \$720 occurs when a group of 120 people charter a bus.

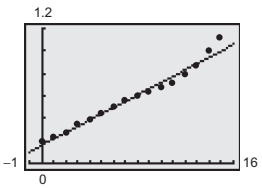
83. (a)

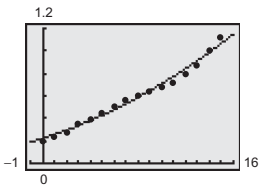


(b) Linear: $D = 0.056t + 0.16$

Quadratic: $D = 0.0015t^2 + 0.034t + 0.22$

(c) Linear Quadratic





(d) The quadratic model is a better fit because its graph represents the actual values of D more closely than the linear model.

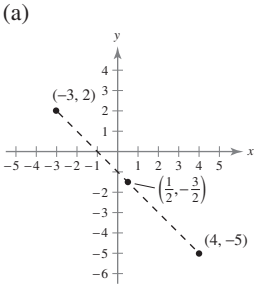
(e) 2006: \$15, 2007: \$23; The predictions given by the model are a little lower than the estimates given by Coca-Cola.

85. (a) Incorrect

(b) Correct

MID-CHAPTER QUIZ *(page 207)*

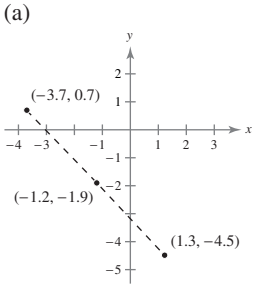
1. (a)



(b) $7\sqrt{2}$

(c) $\left(\frac{1}{2}, -\frac{3}{2}\right)$

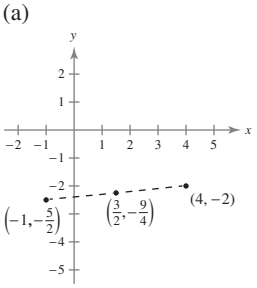
2. (a)



(b) $\sqrt{5204}$

(c) $(-12, -19)$

3. (a)

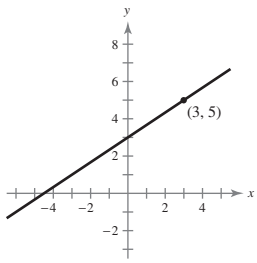


(b) $\frac{\sqrt{101}}{2}$

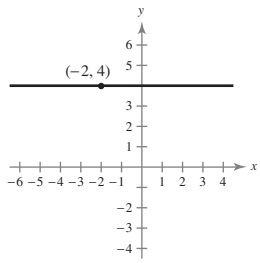
(c) $\left(\frac{3}{2}, -\frac{9}{4}\right)$

4. If the population follows a linear growth pattern, then the population will be 251,724 in 2009.

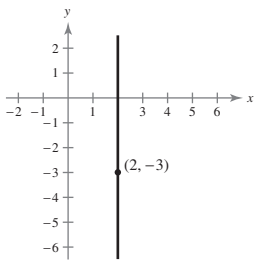
5. $2x - 3y + 9 = 0$



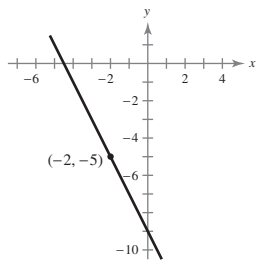
6. $y - 4 = 0$



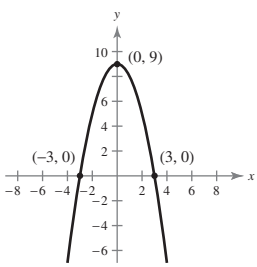
7. $x - 2 = 0$



8. $2x + y + 9 = 0$



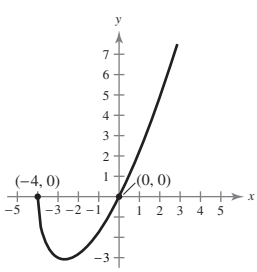
9.



Intercepts: $(-3, 0), (3, 0), (0, 9)$

Symmetry: y-axis

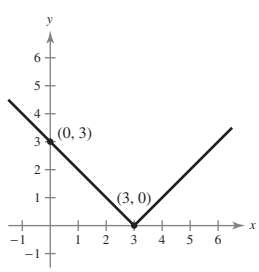
10.



Intercepts: $(-4, 0), (0, 0)$

Symmetry: none

11.



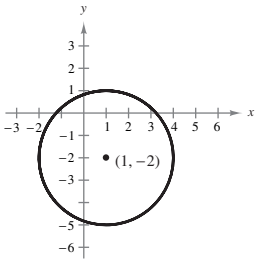
Intercepts: $(3, 0), (0, 3)$

Symmetry: none

12. $(x - 2)^2 + (y + 3)^2 = 16$

13. $x^2 + \left(y + \frac{1}{2}\right)^2 = 5$

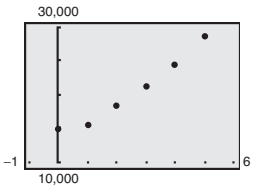
14. $(x - 1)^2 + (y + 2)^2 = 9$



15. (a) 2 (b) -7 **16.** (a) 1 (b) -20

17. $x \geq 4$ **18.** All real numbers except $x = -2$

19.



Linear: $C = 28084t + 13{,}542$

Quadratic: $C = 33823t^2 + 11172t + 14{,}670$

20. Linear

2006: \$0,392,400,000

2007: \$3,200,800,000

Quadratic

2006: \$3,549,480,000

2007: \$9,063,670,000

21. $A = \dfrac{C^2}{4\pi}$

SECTION 2.5 *(page 215)*

Skills Review *(page 215)*

1. 2 **2.** 0 **3.** $-\dfrac{3}{x}$ **4.** $x^2 + 3$ **5.** $0, \pm 4$

6. $\frac{1}{2}$, 1 **7.** All real numbers except $x = 4$

8. All real numbers except $x = 4, 5$ **9.** $t \leq \frac{5}{3}$

10. All real numbers

1. Domain: $[1, \infty)$; Range: $[0, \infty)$0

3. Domain: $(-\infty, \infty)$; Range: $(-\infty, 4]$;4

5. Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$; -1

7. Domain: $[-5, 5]$; Range: $[0, 5]$5

9. Function **11.** Not a function

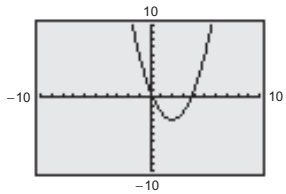
13. Increasing on $(-\infty, \infty)$; No change

15. Increasing on $(-\infty, 0)$ and $(2, \infty)$, decreasing on $(0, 2)$; behavior changes at $(0, 0)$ and $(2, -4)$.

17. Increasing on $(-1, 0)$ and $(1, \infty)$, decreasing on $(-\infty, -1)$ and $(0, 1)$; behavior changes at $(-1, -3)$, $(0, 0)$, and $(1, -3)$.

19. Increasing on $(-2, \infty)$, decreasing on $(-3, -2)$; behavior changes at $(-2, -2)$.

21.

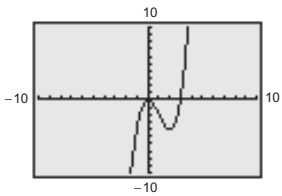


Minimum: $(2, -3)$

Increasing: $(2, \infty)$

Decreasing: $(-\infty, 2)$

23.



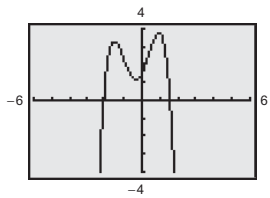
Relative maximum: $(0, 0)$

Relative minimum: $(2, -4)$

Increasing: $(-\infty, 0), (2, \infty)$

Decreasing: $(0, 2)$

25.



Relative maxima: $(-154, 329)$, $(095, 377)$

Relative minimum: $(-034, 114)$

Increasing: $(-\infty, -154)$, $(-034, 095)$

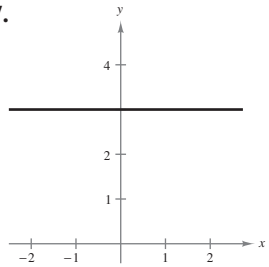
Decreasing: $(-154, -034)$, $(095, \infty)$

27. Even **29.** Odd **31.** Odd

33. (a) 2 (b) 2 (c) -3 (d) -4

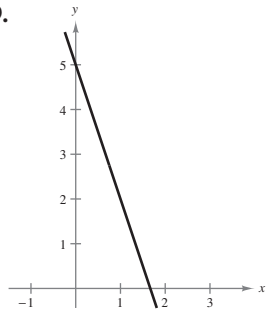
35. (a) 2 (b) 1 (c) -8 (d) -9

37.



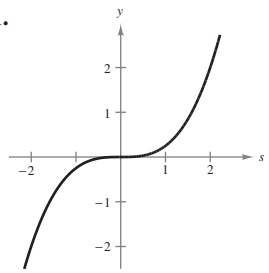
Even

39.



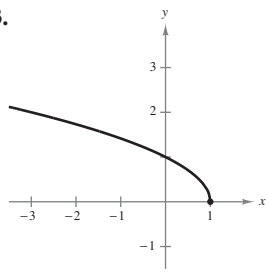
Neither even nor odd

41.



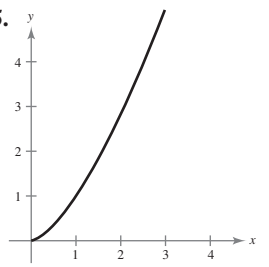
Odd

43.



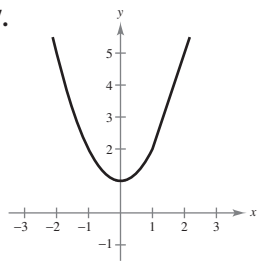
Neither even nor odd

45.



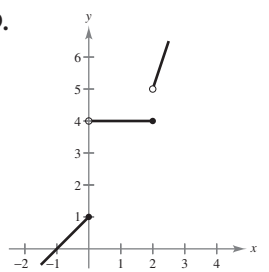
Neither even nor odd

47.



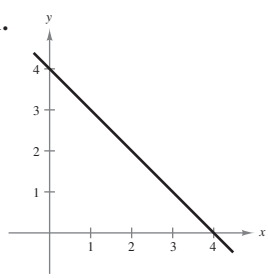
Neither even nor odd

49.

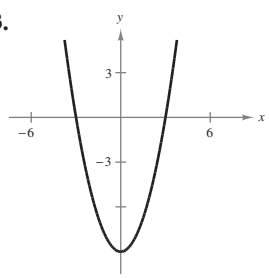


Neither even nor odd

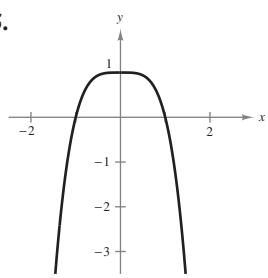
51.



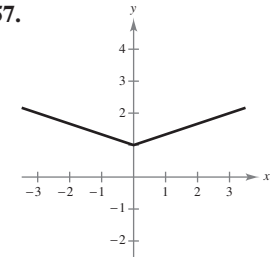
53.



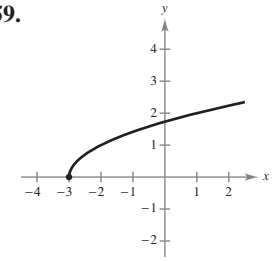
55.



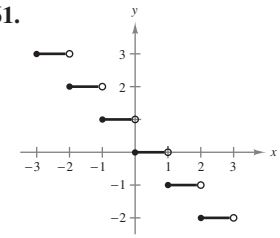
57.



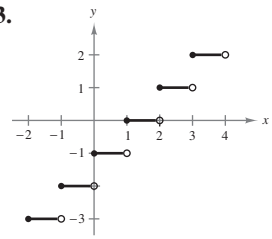
59.



61.



63.



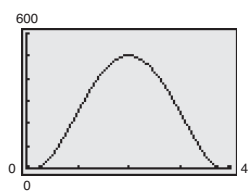
65. Maximum: $444.53 per ounce

Decreasing: 1995–2000

Increasing: 2000–2005

It is not realistic to assume that the price of gold will continue to follow this model, because the function decreases after the year 2005, and eventually yields negative values.

67.



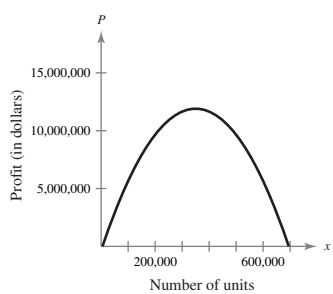
Increasing: 0 seconds to 2 seconds

Decreasing: 2 seconds to 4 seconds

Maximum change in volume: $\approx$ 5019 milliliters

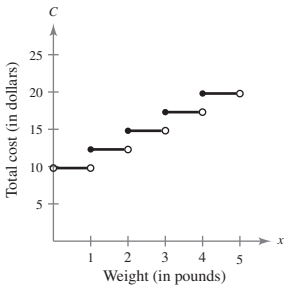
69. Maximum or minimum values may occur at endpoints.

71.

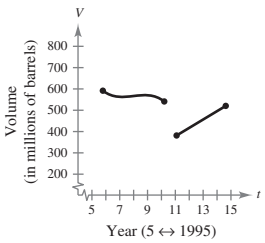


Approximately 350,000 units

73.



75.



77. (a) $y = 25215t^2 - 30704t + 62638, \quad 5 \le t \le 11$

(b) $y = -301450t^2 + 876399t - 56{,}4903, \quad 12 \le t \le 15$

(c)

$$y = \begin{cases} 25215t^2 - 30704t + 62638, & 5 \le t \le 11 \\ -301450t^2 + 876399t - 56{,}4903, & 12 \le t \le 15 \end{cases}$$

79. (a) Even;The graph is a reflection in the x-axis.

(b) Even;The graph is a reflection in the y-axis.

(c) Even;The graph is a vertical translation of f.

(d) Neither;The graph is a horizontal translation of f.

81. (a) $\left(\frac{5}{3}, -7\right)$

(b) $\left(\frac{5}{3}, 7\right)$

83. (a) $(-5, -1)$

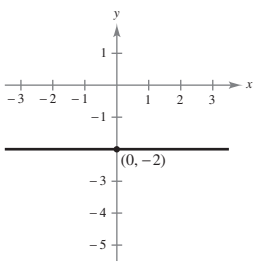
(b) $(-5, 1)$

SECTION 2.6 *(page 225)*

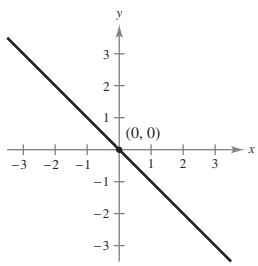
Skills Review *(page 225)*

1. 12 **2.** $\frac{-2x}{-x-3}$ **3.** $0, \pm\sqrt{10}$ **4.** $\frac{4}{3}, -2$

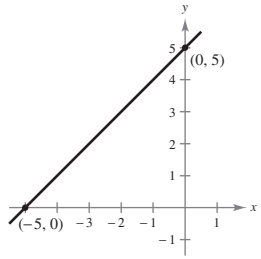
5. $f(x) = -2$



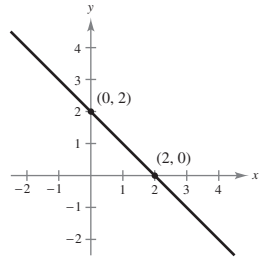
6. $f(x) = -x$



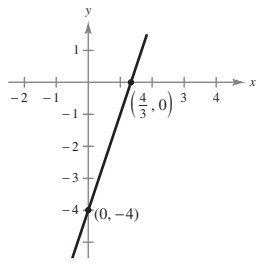
7. $f(x) = x + 5$



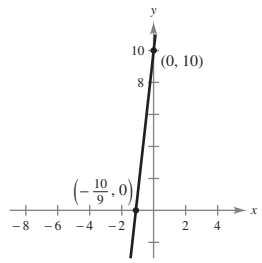
8. $f(x) = 2 - x$



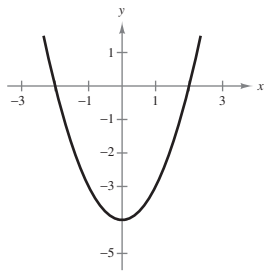
9. $f(x) = 3x - 4$



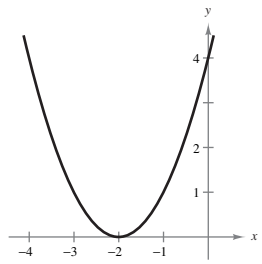
10. $f(x) = 9x + 10$



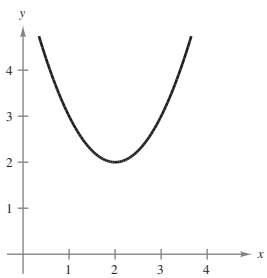
1. Shifted four units downward



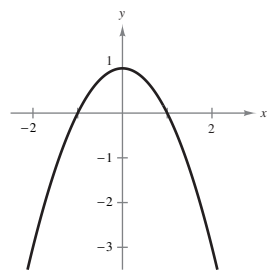
3. Shifted two units to the left



5. Shifted two units upward and two units to the right

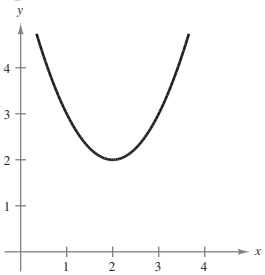


7. Reflected about the x-axis and shifted one unit upward

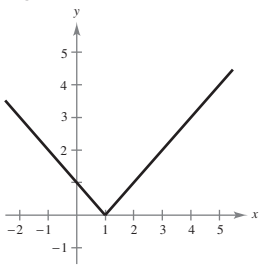


CHAPTER 2

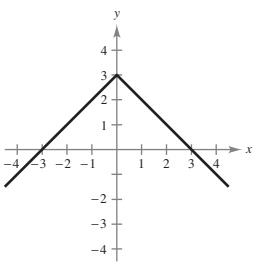
9. Shifted two units upward



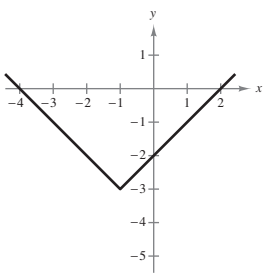
11. Shifted one unit to the right



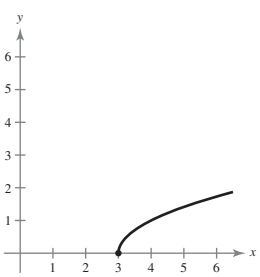
13. Reflected about the x-axis and shifted three units upward



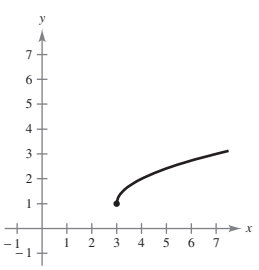
15. Shifted one unit to the left and three units downward



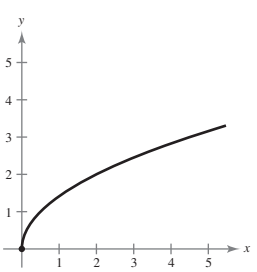
17. Shifted three units to the right



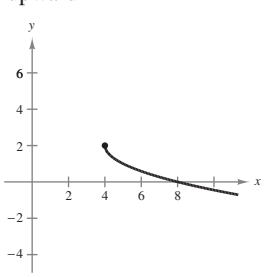
19. Shifted three units to the right and one unit upward



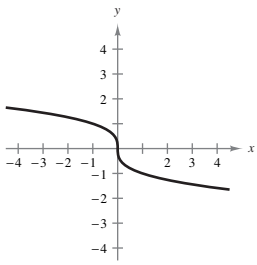
21. Vertically stretched by a factor of $\sqrt{2}$



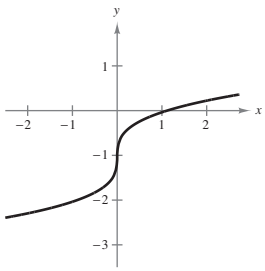
23. Reflected about the x-axis and shifted four units to the right and two units upward



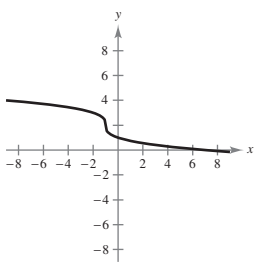
25. Reflected about the y-axis



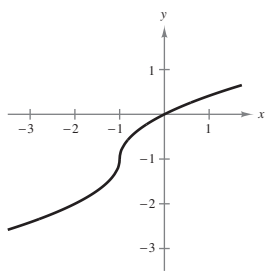
27. Shifted one unit downward



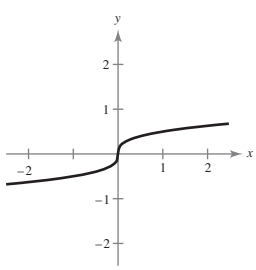
29. Reflected about the x-axis, shifted one unit to the left, and two units upward



31. Shifted one unit to the left and one unit downward



33. Vertically shrunk by a factor of $\frac{1}{2}$



35. Common function: $y = x^3$

Transformation: shifted two units to the right

Equation: $y = (x - 2)^3$

37. Common function: $y = x^2$

Transformation: reflected about the x-axis

Equation: $y = -x^2$

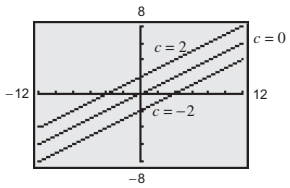
39. Common function: $y = \sqrt{x}$

Transformation: reflected about the x-axis and shifted one unit upward

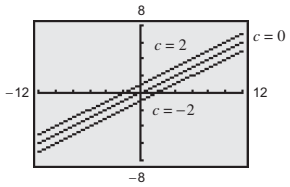
Equation: $y = -\sqrt{x} + 1$

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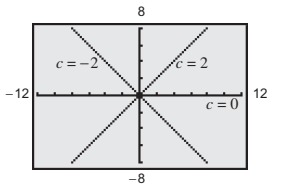
41. (a) Vertical shift of two units



(b) Horizontal shift of two units



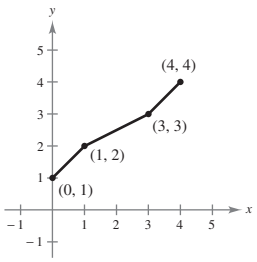
(c) Slope of the function changes



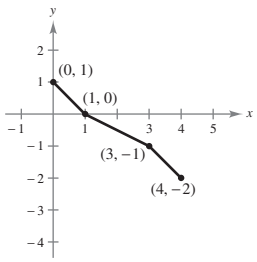
43. (a) $g(x) = (x - 1)^2 + 1$

(b) $g(x) = -(x + 1)^2$

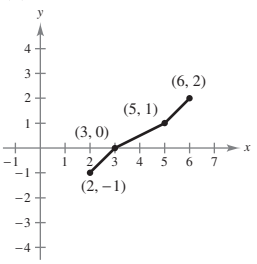
45. (a)



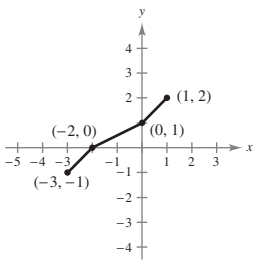
(b)



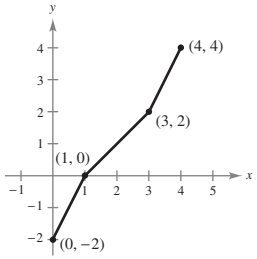
(c)



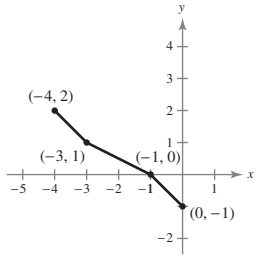
(d)



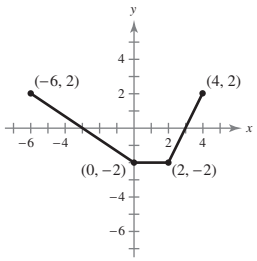
(e)



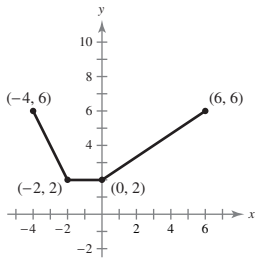
(f)



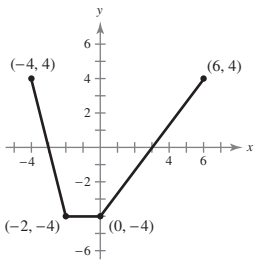
47. (a)



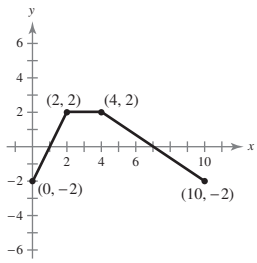
(b)



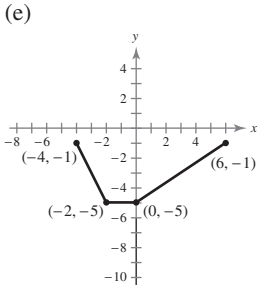
(c)



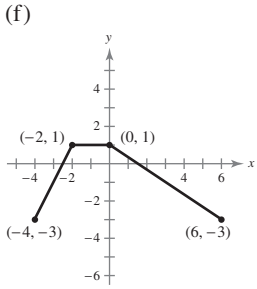
(d)



(e)



(f)



49. $y = x^3 - 2$ **51.** $y = 4x^3$

53. $g(x) = |x - 3| + 2$ **55.** $g(x) = -4|x|$

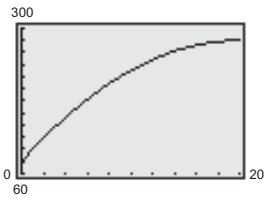
57. $h(x) = \sqrt{x - 4} - 3$ **59.** $h(x) = \frac{1}{2}\sqrt{x - 3}$

61. $g(x) = -x^3 + 3x^2 + 1$

63. Shifted one unit to the right and two units downward

$g(x) = (x - 1)^2 - 2$

65. (a)



(b) $P(x) = 80 + 20x - 05x^2 - 25$

$P(x) = 55 + 20x - 05x^2, \quad 0 \le x \le 20$

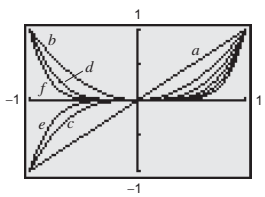
Shifted 25 units down

(c) $P\left(\frac{x}{100}\right) = 80 + \frac{x}{5} - 000005x^2,$

$0 \le x \le 2000$ (x in dollars)

Horizontal stretch

67.



(a), (c), and (e) are odd functions;(b), (d), and (f) are even. Also, (a), (c), and (e) are increasing for all real numbers; (b), (d), and (f) are decreasing for all $x < 0$ and increasing for all $x > 0$.

SECTION 2.7 *(page 234)*

Skills Review *(page 234)*

1. $\frac{1}{x(1-x)}$ **2.** $-\frac{12}{(x+3)(x-3)}$ **3.** $\frac{3x-2}{x(x-2)}$

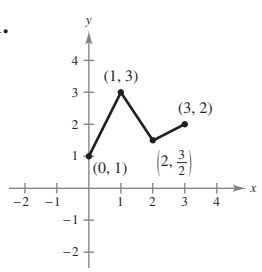
4. $\frac{4x-5}{3(x-5)}$ **5.** $\frac{\sqrt{x^2-1}}{x+1}, \ x \ne 1$

6. $\frac{x+1}{x(x+2)}, \ x \ne 2$ **7.** $5(x-2), \quad x \ne -2$

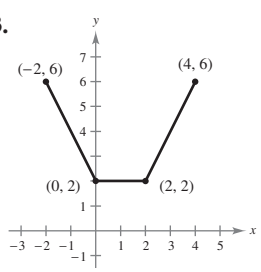
8. $\frac{x+1}{(x-2)(x+3)}, \ x \ne -5, -1, 0$

9. $\frac{1+5x}{3x-1}, \ x \ne 0$ **10.** $\frac{x+4}{4x}, \ x \ne 4$

1.



3.



5. (a) $2x$ (b) 2 (c) $x^2 - 1$

(d) $\frac{x+1}{x-1}$; Domain: $(-\infty, 1) \cup (1, \infty)$

7. (a) $x^2 - x + 1$ (b) $x^2 + x - 1$ (c) $x^2 - x^3$

(d) $\frac{x^2}{1-x}$; Domain: $(-\infty, 1) \cup (1, \infty)$

9. (a) $x^2 + \sqrt{1-x} + 5$ (b) $x^2 - \sqrt{1-x} + 5$

(c) $x^2\sqrt{1-x} + 5\sqrt{1-x}$

(d) $\frac{x^2+5}{\sqrt{1-x}}$; Domain: $(-\infty, 1)$

11. (a) $\frac{x+1}{x^2}$ (b) $\frac{x-1}{x^2}$ (c) $\frac{1}{x^3}$

(d) $x, x \ne 0$; Domain: $(-\infty, 0) \cup (0, \infty)$

13. 14 **15.** $-4t^2 + 4t + 3$ **17.** -6

19. $\frac{11}{23}$ **21.** 3 **23.** -6

25. (a) $6x + 15$ (b) $6x + 5$ (c) $9x$

27. (a) $9x^2 + 6x + 1$ (b) $3x^2 + 1$ (c) x^4

29. (a) $\sqrt{x^2+4}$ (b) $x + 4, x \ge -4$

31. (a) $x - \frac{8}{3}$ (b) $x - 8$ **33.** (a) $\sqrt[4]{x}$ (b) $\sqrt[4]{x}$

35. (a) $|x + 6|$ (b) $|x| + 6$

37. (a) All real numbers, or $(-\infty, \infty)$

(b) $x \ge 0$, or $[0, \infty)$ (c) $x \ge 0$, or $[0, \infty)$

39. (a) All real numbers except $x = 0$, or $(-\infty, 0) \cup (0, \infty)$

(b) All real numbers, or $(-\infty, \infty)$

(c) All real numbers except $x = 2$, or $(-\infty, 2) \cup (2, \infty)$

41. (a) 3 (b) 0 **43.** (a) 0 (b) 4

45. Answers will vary.

Sample answer: $f(x) = x^2, g(x) = 2x + 1$

47. Answers will vary.

Sample answer: $f(x) = \sqrt[3]{x}, g(x) = x^2 - 4$

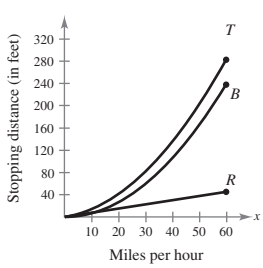
49. Answers will vary.

Sample answer: $f(x) = \frac{1}{x}, g(x) = x + 2$

51. Answers will vary.

Sample answer: $f(x) = x^2 + 2x, g(x) = x + 4$

53. $T = \frac{3}{4}x + \frac{1}{15}x^2$



55. $(C \circ x)(t) = 1500t + 495$

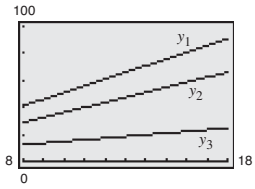
$C \circ x$ represents the cost of producing x units in t hours.

57. $R_1 + R_2 = 917 - 67t$,
$t = 0, 1, 2, 3, 4, 5, 6, 7, 8$

Total sales were decreasing.

59. (a) $y_1 = 495t + 16,\ y_2 = 372t - 09$

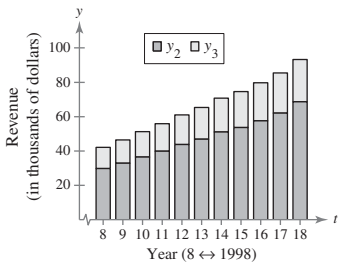
(b)



$y_3 = 123t + 25;$

y_3 represents the profits for 1998 through 2008; $7,100

(c)



The heights of the bars represent the revenues for 1998 through 2008.

61. (a) $N(T(t)) = 100t^2 + 275$

(b) Approximately 218 hours

63. (a) $R(p) = p - 2000$ (b) $S(p) = 09p$

(c) $(R \circ S)(p) = 09p - 2000$; $(R \circ S)(p)$ represents the factory rebate after the dealership discount.

$(S \circ R)(p) = 09\,(p - 2000)$; $(S \circ R)(p)$ represents the dealership rebate after the factory discount.

(d) $(R \circ S)(20{,}500) = \$6{,}450$

$(S \circ R)(20{,}500) = \$6{,}650$

$6,450 is the lower cost because 10%f the price of the car is larger than $000.

65.

	2001	2002	2003	2004	2005
	129	121	95	111	146

67. False. $(f \circ g)(x) = 6x + 1$ and $(g \circ f)(x) = 6x + 6$

69. Answers will vary.

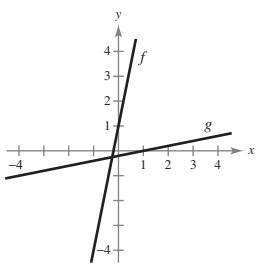
SECTION 2.8 *(page 245)*

Skills Review *(page 245)*

1. All real numbers 2. $[-1, \infty)$
3. All real numbers except $x = 0, 2$
4. All real numbers except $x = -\frac{5}{3}$ 5. x
6. x 7. x 8. x 9. $x = \frac{3}{2}y + 3$
10. $x = \dfrac{y^3}{2} + 2$

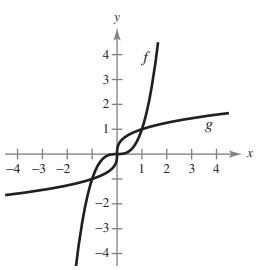
1. $f^{-1} = \{(4, 1), (5, 2), (6, 3), (7, 4)\}$
3. $f^{-1} = \{(1, -1), (2, -2), (3, -3), (4, -4)\}$
5. $f^{-1}(x) = \frac{1}{2}x$ 7. $f^{-1}(x) = x + 5$
9. (a) $5\left(\dfrac{x-1}{5}\right) + 1 = x;\ \dfrac{(5x+1)-1}{5} = x$

(b)



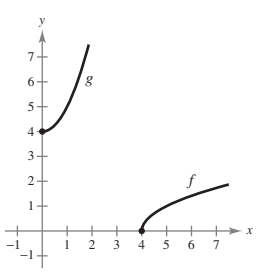
11. (a) $\left(\sqrt[3]{x}\right)^3 = x;\ \sqrt[3]{x^3} = x$

(b)



13. (a) $\sqrt{(x^2 + 4) - 4} = x;\ \left(\sqrt{x - 4}\right)^2 + 4 = x$

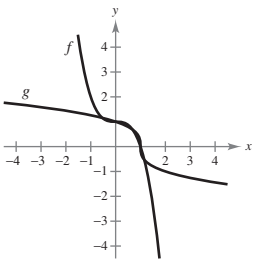
(b)



CHAPTER 2

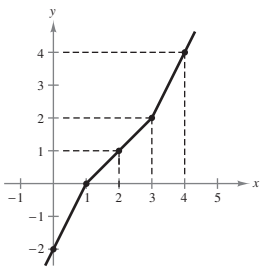
15. (a) $1 - \left(\sqrt[3]{1-x}\right)^3 = x;\ \sqrt[3]{1-(1-x^3)} = x$

(b)



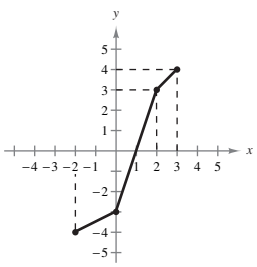
17.

	0	1	2	3	4
	−2	0	1	2	4



19.

	−2	0	2	3
	−4	−3	3	4



21. f doesn't have an inverse function.

23. $g^{-1}(x) = 8x$

25. p doesn't have an inverse function.

27. $f^{-1}(x) = \sqrt{x} - 3,\ x \geq 0$

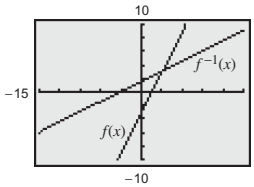
29. $h^{-1}(x) = \dfrac{1}{x}$ **31.** $f^{-1}(x) = \dfrac{x^2 - 3}{2},\ x \geq 0$

33. g doesn't have an inverse function.

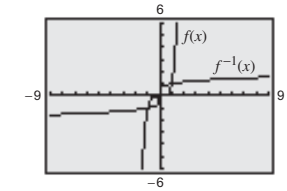
35. $f^{-1}(x) = -\sqrt{25 - x},\ x \leq 25$

37. Error: f^{-1} does not mean to take the reciprocal of $f(x)$.

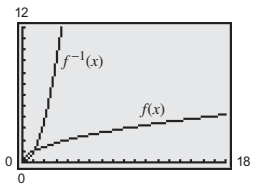
39. $f^{-1}(x) = \dfrac{x+3}{2}$



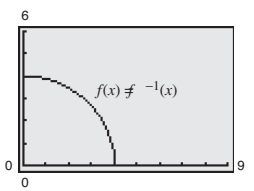
41. $f^{-1}(x) = \sqrt[5]{x}$



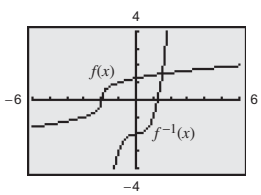
43. $f^{-1}(x) = x^2,\ x \geq 0$



45. $f^{-1}(x) = \sqrt{16 - x^2},\ 0 \leq x \leq 4$



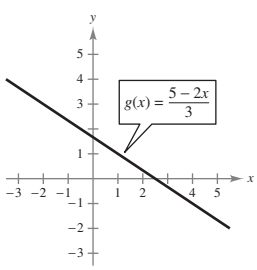
47. $f^{-1}(x) = x^3 - 2$



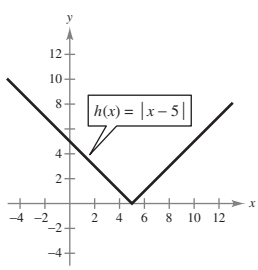
49. Because f is one-to-one, f has an inverse function.

51. f doesn't have an inverse function because two x-values share the same y-value.

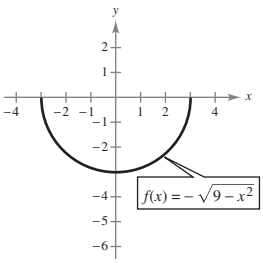
53. g has an inverse function.



55. h doesn't have an inverse function.



57. f doesn't have an inverse function.



59. 32 **61.** 600

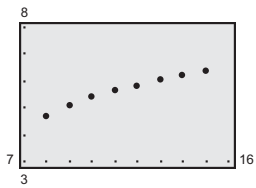
63. $(g^{-1} \circ f^{-1})(x) = \dfrac{x+1}{2}$ **65.** $(f \circ g)^{-1}(x) = \dfrac{x+1}{2}$

67. $C^{-1}(x) = \dfrac{x - 1500}{75}$; C^{-1} computes the number of T-shirts that can be made for a cost of x.

Domain of C: $[0, \infty)$

Domain of C^{-1}: $[1500, \infty)$

69. (a)



(b) $y = 0235t + 295$

(c) $y^{-1} = \dfrac{t - 295}{0235}$; y^{-1} represents the year in which the average admission price is t dollars.

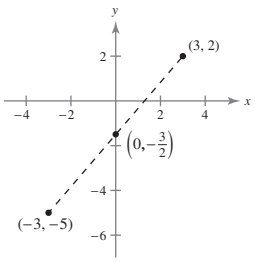
(d) 2012

71. After graphing $f(x) = x^2 + 1$, $x \geq 0$, and $f^{-1}(x) = \sqrt{x - 1}$, it is observed that $f(x)$ and $f^{-1}(x)$ are reflections of each other about the line $y = x$. Because of this reflection, interchanging the roles of x and y seems reasonable.

73. $f^{-1}(x) = \sqrt[3]{\dfrac{x^2 - 0008}{00161}}$; $840

REVIEW EXERCISES *(page 250)*

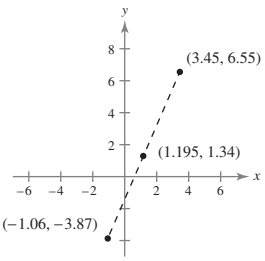
1. (a)



(b) $\sqrt{85} \approx 922$

(c) $\left(0, -\frac{3}{2}\right)$

3. (a)



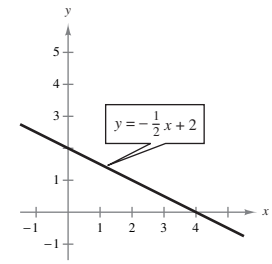
(b) $\sqrt{1289165} \approx 1135$

(c) (1195, 134)

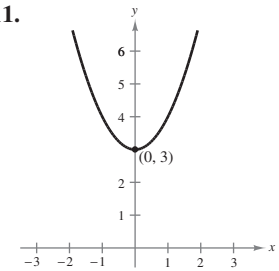
5. $x = -10$ or 30 **7.** (a) ¥s (b) ¥s

9.

−2	3
0	2
2	1
3	$\frac{1}{2}$
4	0



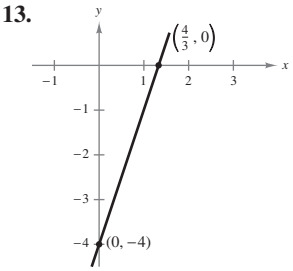
11.



y-intercept: $(0, 3)$

Symmetry: y-axis

13.

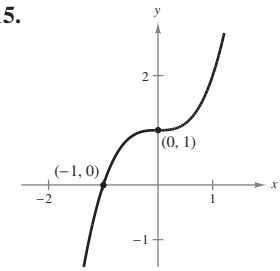


x-intercept: $\left(\frac{4}{3}, 0\right)$

y-intercept: $(0, -4)$

Symmetry:none

15.



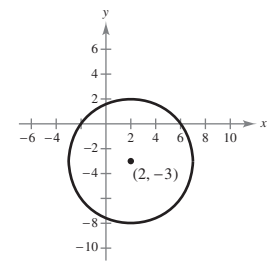
x-intercept: $(-1, 0)$

y-intercept: $(0, 1)$

Symmetry: none

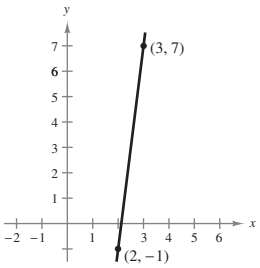
17. $(x + 1)^2 + (y - 2)^2 = 36$

19. $(x - 2)^2 + (y + 3)^2 = 25$



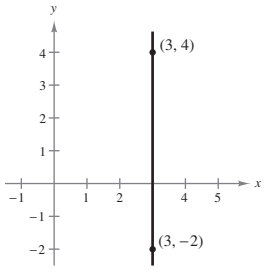
CHAPTER 2

21.



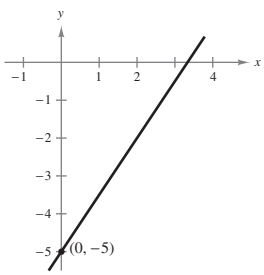
$m = 8$

23.

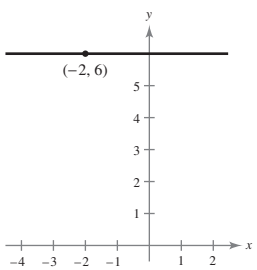


m is undefined.

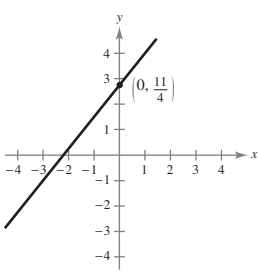
25. $3x - 2y - 10 = 0$



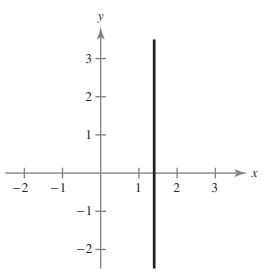
27. $y = 6$



29. Slope: $\frac{5}{4}$;
y-intercept: $\left(0, \frac{11}{4}\right)$



31. Slope: undefined;
y-intercept: none



33. Parallel **35.** Neither

37. (a) $5x - 4y - 23 = 0$ (b) $4x + 5y - 2 = 0$

39. (a) $y = -2$ (b) $x = -1$

41. $y = \frac{7}{3}x$ **43.** $y = 348x$ **45.** $A = 5r$

47. $a = \frac{3}{4}b$ **49.** $y = 00365x$; \$723 **51.** \$36,000

53. $V = -12{,}950t + 135{,}000,\ 0 \le t \le 10$

55. y is a function of x. **57.** y is a function of x.

59. Function;Every element of Ais assigned to an element of B.

61. (a) -2 (b) -3 (c) -5 (d) $\sqrt{x+7} - 5$

63. All real numbers x **65.** $x \ge -5$

67. $1 \le t < 4, t > 4$

69. The domain of $h(x)$ is all real numbers except $x = 0$, because division by zero is undefined.The domain of $k(x)$ is all real numbers except $x = -2$ and $x = 2$, because if $x = 2$ or $x = -2$, then $x^2 - 4$ equals zero, and division by zero is undefined. When a graphing utility and the *table* feature are used, $h(0)$ results in an error and $k(-2)$ or $k(2)$ also result in an error.

71. (a) $B = 6500\left(1 + \frac{00685}{4}\right)^{4t}$ (b) $t \ge 0$

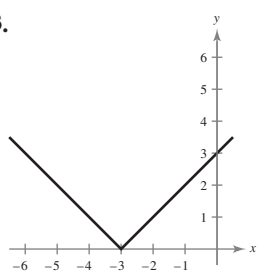
73. (a) Domain: all real numbers
Range: $[1, \infty)$
(b) Decreasing: $(-\infty, 0)$
Increasing: $(0, \infty)$
(c) Even (d) Minimum: $(0, 1)$

75. (a) Domain: all real numbers
Range: all real numbers
(b) Decreasing: $\left(0, \frac{8}{3}\right)$
Increasing: $(-\infty, 0) \cup \left(\frac{8}{3}, \infty\right)$
(c) Neither
(d) Relative minimum: $\left(\frac{8}{3}, -\frac{256}{27}\right)$
Relative maximum: $(0, 0)$

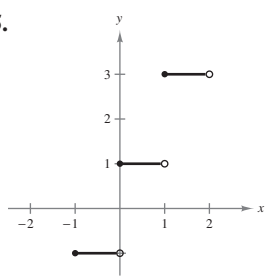
77. y is a function of x. **79.** y is not a function of x.

81. y is a function of x.

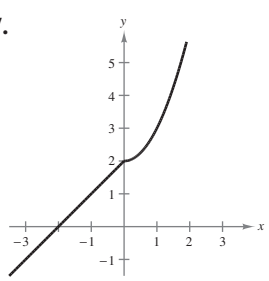
83.



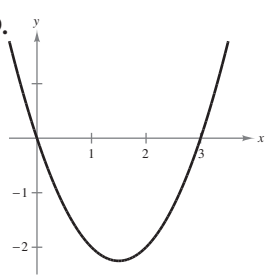
85.



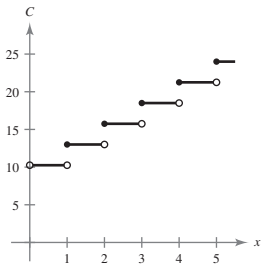
87.



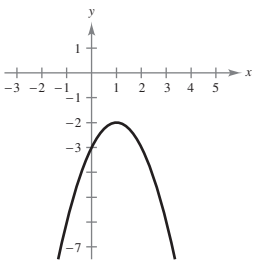
89.



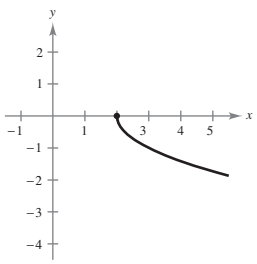
91.



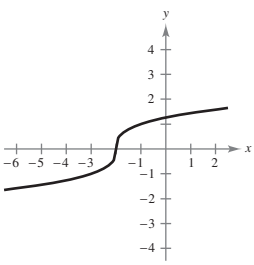
93. Reflected about the x-axis and shifted two units downward and one unit to the right



95. Reflected about the x-axis and shifted two units to the right



97. Shifted two units to the left



99. Common function: $y = \sqrt{x}$
Transformation: reflected about the y-axis and shifted three units to the right
Equation: $y = \sqrt{3 - x}$ or $y = \sqrt{-(x - 3)}$

101. $(f + g)(x) = x^2 + 5x - 1$

$(f - g)(x) = -x^2 + x - 1$

$(fg)(x) = 3x^3 + 5x^2 - 2x$

$(f/g)(x) = \dfrac{3x - 1}{x^2 + 2x}$

Domain of f/g: $x < -2$, $-2 < x < 0$, $x > 0$, or $(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$

103. 9 **105.** 18

107. (a) $x^2 + 6x + 9$
Domain: All real numbers

(b) $x^2 + 3$
Domain: All real numbers

109. (a) $\dfrac{1}{3x + x^2}$
Domain: All real numbers except $x = 0$and $x = -3$

(b) $\dfrac{3}{x} + \dfrac{1}{x^2}$, or $\dfrac{3x + 1}{x^2}$

Domain: All real numbers except $x = 0$

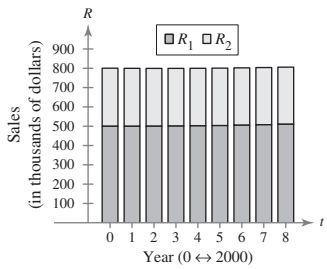
111. Answers will vary.

Sample answer: $f(x) = x^2$, $g(x) = 6x - 5$

113. Answers will vary.

Sample answer: $f(x) = \dfrac{1}{x^2}$, $g(x) = x - 1$

115. $R_1 + R_2 = 8005 - 092t + 02t^2$, $t = 0, 1, 2, \ldots, 8$



Total sales are increasing.

117. $g(f(x))$; The bonus is based on sales over \$00,000, and is calculated by multiplying $x - 500{,}000$ by 003.

119. $f(g(x)) = 3\left(\dfrac{x - 5}{3}\right) + 5 = x$

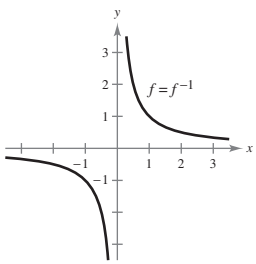
$g(f(x)) = \dfrac{3x + 5 - 5}{3} = x$

$f(g(x)) = x = g(f(x))$, so f and g are inverse functions of each other.

121. $f(x)$ does not have an inverse function.

123. $f^{-1}(x) = \dfrac{1}{x}$

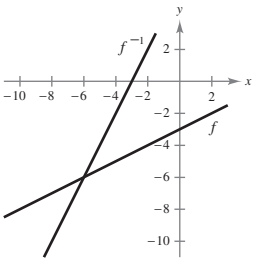
(f is its own inverse function)



CHAPTER 2

125. (a) $f^{-1}(x) = 2(x + 3) = 2x + 6$

(b)



(c) $f^{-1}(f(x)) = 2\left(\frac{1}{2}x - 3\right) + 6$

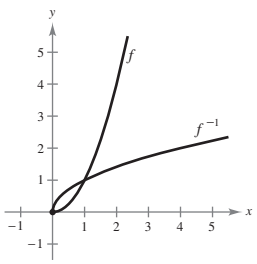
$= x$

$f(f^{-1}(x)) = \frac{1}{2}(2x + 6) - 3$

$= x$

127. (a) $f^{-1}(x) = \sqrt{x}, x \geq 0$

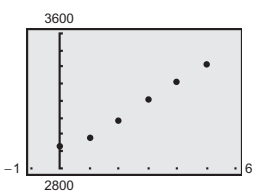
(b)



(c) $f^{-1}(f(x)) = \sqrt{x^2} = x, x \geq 0$

$f(f^{-1}(x)) = \left(\sqrt{x}\right)^2 = x, x \geq 0$

129. (a)



(b) $A = 1035t + 2899$

(c) $A^{-1} = \dfrac{t - 2899}{1035}$; 2007

CHAPTER TEST *(page 255)*

1. Distance: $4\sqrt{5}$

Midpoint: $(1, 0)$

2. Distance: ≈ 781

Midpoint: $(044, 4335)$

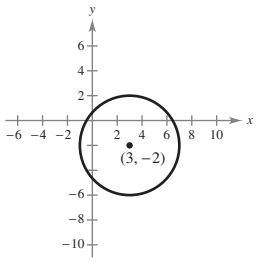
3. x-intercepts: $(-5, 0), (3, 0)$

y-intercept: $(0, -15)$

4. Symmetric with respect to the origin

5. $2x - 3y + 21 = 0$

6. $(x - 3)^2 + (y + 2)^2 = 16$



7. True.Each value of xcorresponds to exactly one value of y.

8. False.The element -9is not included in set B.

9. (a) Domain: All real numbers

Range: $(-\infty, 2]$

(b) Decreasing: $(0, \infty)$

Increasing: $(-\infty, 0)$

(c) Even

(d) Maximum: $(0, 2)$

10. (a) Domain: $(-\infty, -2] \cup [2, \infty)$

Range: $[0, \infty)$

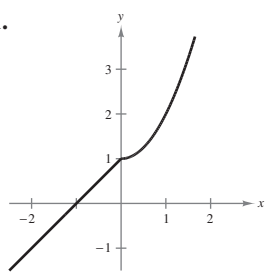
(b) Decreasing: $(-\infty, -2)$

Increasing: $(2, \infty)$

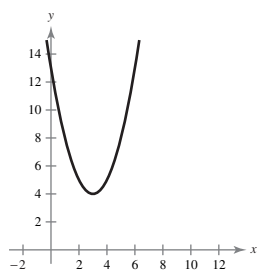
(c) Even

(d) Minima: $(-2, 0), (2, 0)$

11.



12.



13. $(f - g)(x) = x^2 - 2x + 3$

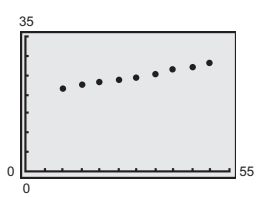
14. $(fg)(x) = 2x^3 - x^2 + 4x - 2$

15. $(f \circ g)(x) = 4x^2 - 4x + 3$

16. $g^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$

17. $V = 30{,}000 - 5200t$

18.



$P = 017\ t + 196$

CHAPTER 3

SECTION 3.1 *(page 265)*

Skills Review *(page 265)*

1. $\frac{1}{2}, -6$ **2.** $-\frac{3}{5}, 3$ **3.** $\frac{3}{2}, -1$ **4.** -10

5. $3 \pm \sqrt{5}$ **6.** $-2 \pm \sqrt{3}$ **7.** $4 \pm \frac{\sqrt{14}}{2}$

8. $-5 \pm \frac{\sqrt{3}}{3}$ **9.** $-\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ **10.** $-\frac{3}{2} \pm \frac{\sqrt{21}}{2}$

1. g **2.** e **3.** c **4.** f

5. b **6.** a **7.** h **8.** d

9. $y = -(x + 2)^2$ **11.** $y = (x - 3)^2 - 9$

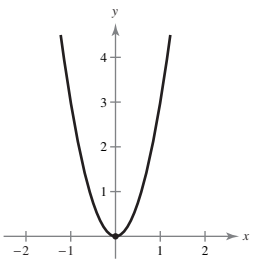
13. $y = -2(x + 3)^2 + 3$

15. Compared with the graph of $y = x^2$, each output of $f(x) = 5x^2$ vertically stretches the graph by a factor of 5.

17. The graph of f is the graph of $y = x^2$ reflected in the x-axis, shifted to the left 1 unit and shifted upward 1 unit.

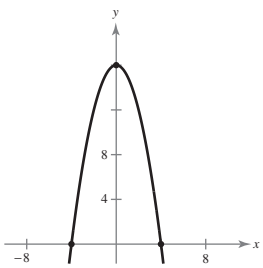
19. Intercept: $(0, 0)$

Vertex: $(0, 0)$



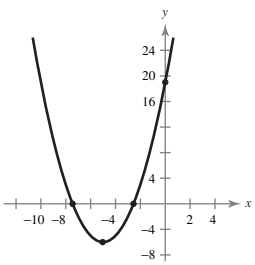
21. Intercepts: $(\pm 4, 0), (0, 16)$

Vertex: $(0, 16)$



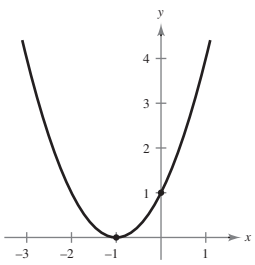
23. Intercepts: $\left(-5 \pm \sqrt{6}, 0\right), (0, 19)$

Vertex: $(-5, -6)$



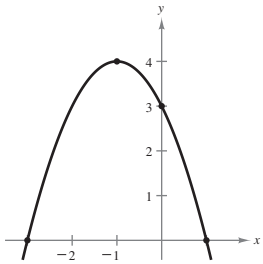
25. Intercepts: $(-1, 0), (0, 1)$

Vertex: $(-1, 0)$



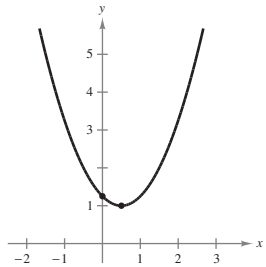
27. Intercepts: $(1, 0), (-3, 0), (0, 3)$

Vertex: $(-1, 4)$



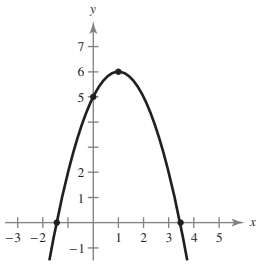
29. Intercept: $\left(0, \frac{5}{4}\right)$

Vertex: $\left(\frac{1}{2}, 1\right)$



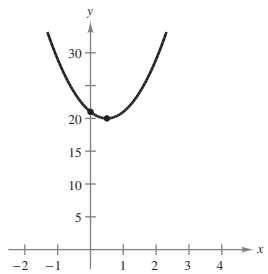
31. Intercepts: $\left(1 \pm \sqrt{6}, 0\right), (0, 5)$

Vertex: $(1, 6)$



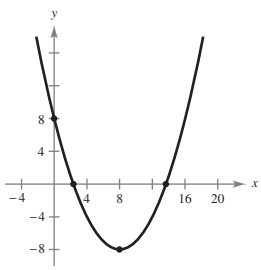
33. Intercept: $(0, 21)$

Vertex: $\left(\frac{1}{2}, 20\right)$



35. Intercepts: $\left(8 \pm 4\sqrt{2}, 0\right), (0, 8)$

Vertex: $(8, -8)$



37. $y = -\frac{1}{2}(x - 2)^2 - 1$

39. $y = \frac{3}{4}(x - 5)^2 + 12$

41. Answers will vary. Sample answer:

$f(x) = x^2 - x - 2$

$g(x) = -x^2 + x + 2$

43. Answers will vary. Sample answer:

$f(x) = x^2 - 10x$

$g(x) = -x^2 + 10x$

45. Answers will vary. Sample answer:

$f(x) = 2x^2 + 7x + 3$

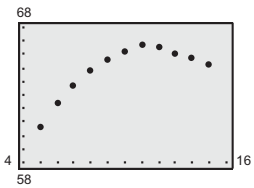
$g(x) = -2x^2 - 7x - 3$

47. $A = 100x - x^2$; (50, 2500); The rectangle has the greatest area ($A = 2500$ square feet) when its width is 50 feet.

49. $x = 150$ feet, $y = 200$ feet; 300 feet $\times$ 200 feet

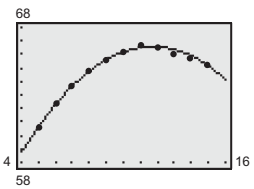
51. 25,000 units **53.** 20 fixtures **55.** 14 feet

57. (a)



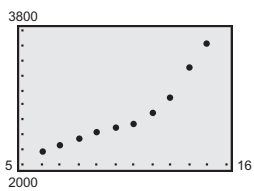
(b) $S = -0.132t^2 + 3.09t + 48.5$

(c)



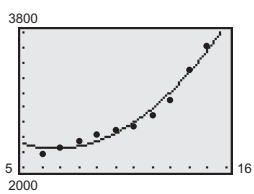
(d) $2002(t \approx 11.70)$; No, according to the actual data, the year in which the number of basic cable subscribers was the greatest was 2001.

59. (a)



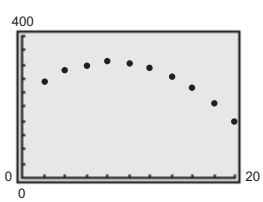
(b) $F = 19.53t^2 - 264.5t + 3149$

(c)



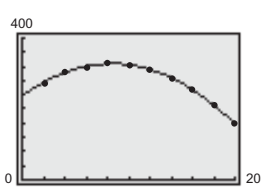
(d) \$4715.72

61. (a)



(b) $p(x) = -1.276x^2 + 21.63x + 236.5$

(c)



(d) $\approx (8.5, 328.2)$; Producing about 85,000 units yields the maximum profit, about \$32,820,000.

(e) Sample answer: Production costs may be growing faster than revenue, so profit decreases.

63. $f(x) = a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a}$

SECTION 3.2 *(page 276)*

Skills Review *(page 276)*

1. $(3x - 2)(4x + 5)$ **2.** $x(5x - 6)^2$

3. $z^2(12z + 5)(z + 1)$ **4.** $(y + 5)(y^2 - 5y + 25)$

5. $(x + 3)(x + 2)(x - 2)$ **6.** $(x + 2)(x^2 + 3)$

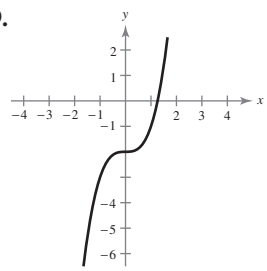
7. No real solution **8.** $3 \pm \sqrt{5}$

9. $-\frac{1}{2} \pm \sqrt{3}$ **10.** ± 3

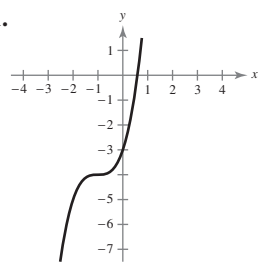
1. e **2.** c **3.** g **4.** d **5.** f **6.** h

7. a **8.** b

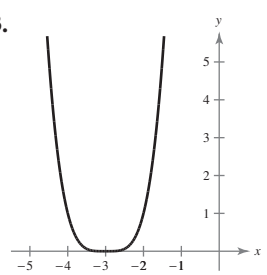
9.



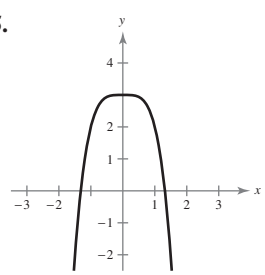
11.



13.



15.



17. Rises to the left
Falls to the right

19. Rises to the left
Falls to the right

21. Rises to the left
Rises to the right

23. Rises to the left
Rises to the right

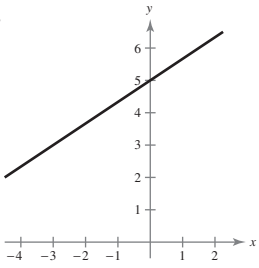
25. Falls to the left
Falls to the right

27. (a) 1 (b) 2 **29.** (a) 4 (b) 5

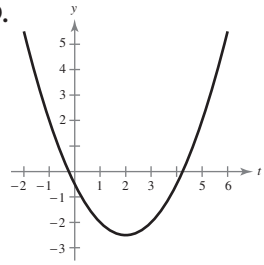
31. ± 3 **33.** -4 **35.** $1, -2$ **37.** No real zeros

39. $2, 0$ **41.** ± 1 **43.** $\pm\sqrt{5}$ **45.** 3

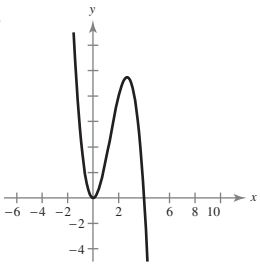
47.



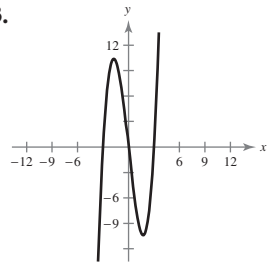
49.



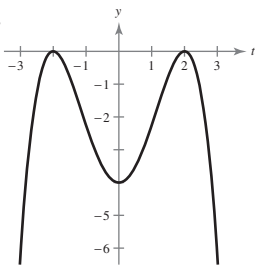
51.



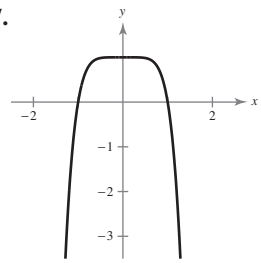
53.



55.

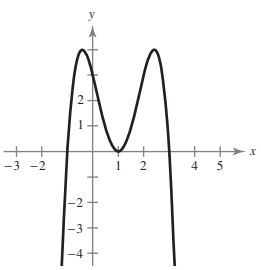


57.

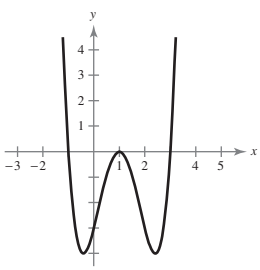


59. Answers will vary. Sample answers:

$a_n < 0$

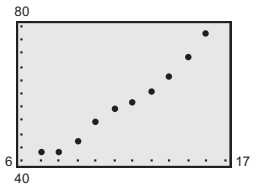


$a_n > 0$



61. $f(x) = \frac{1}{2}(x^4 - 11x^3 + 28x^2)$

63. (a)

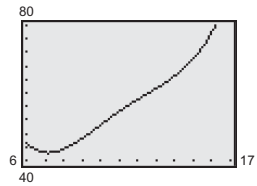


(b) Positive

(c) $C = 0.02025t^4 - 0.9103t^3 + 15.124t^2 - 106.31t + 309.0$

The model agrees with the prediction from part (b).

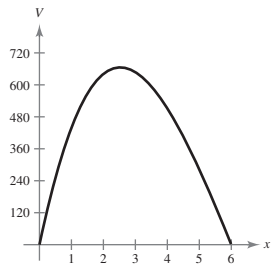
(d)



$2007 (t \approx 17.03)$; The prediction does not seem reasonable. Through 2006, the number increased by at most 7 million in a year. To reach 92 million in 2007, the number would have had to increase by 14 million from 2006.

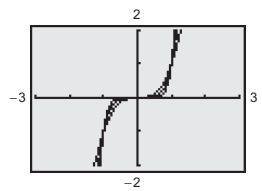
65. (200, 320)

67. Answers will vary; Domain: $0 < x < 6$



$x \approx 2.5$

69.



The functions have a common shape because their degrees are odd, but their graphs are not identical because they have different degrees.

SECTION 3.3 *(page 286)*

Skills Review *(page 286)*

1. $x^3 - x^2 + 2x + 3$ **2.** $2x^3 + 4x^2 - 6x - 4$

3. $x^4 - 2x^3 + 4x^2 - 2x - 7$

4. $2x^4 + 12x^3 - 3x^2 - 18x - 5$

5. $(x - 3)(x - 1)$ **6.** $8(x + 2)(x - 5)$

7. $(3x + 5)(x - 1)$ **8.** $(3x - 4)^2$

9. $2x(x - 1)(2x - 3)$ **10.** $x(3x + 2)(2x + 1)$

1. $3x - 4$ **3.** $2x + 4$ **5.** $x + 3$

7. $x^3 + 3x^2 - 1$ **9.** $7 - \dfrac{25}{x + 4}$

11. $3x + 5 - \dfrac{2x - 3}{2x^2 + 1}$ **13.** $x + \dfrac{x - 27}{x^2 - 1}$

CHAPTER 3

15. $x^2 - 6x + 17 - \dfrac{36}{x + 2}$

17. $2x^3 + 4x^2 - 2x - 8 - \dfrac{10x - 7}{x^2 - 2x + 1}$

19. $2x^2 - 3x + 5$ **21.** $4x^2 - 9$ **23.** $-x^2 + 10x - 25$

25. $x^3 - 7x^2 + 14x - 20 + \dfrac{84}{x + 3}$

27. $10x^3 + 10x^2 + 60x + 360 + \dfrac{1360}{x - 6}$

29. $2x^4 + 8x^3 + 2x^2 + 8x - 5 - \dfrac{7}{x - 4}$

31. $-3x^3 - 6x^2 - 12x - 24 - \dfrac{48}{x - 2}$

33. $-x^2 + 3x - 6 + \dfrac{11}{x + 1}$ **35.** $4x^2 + 14x - 30$

37. $f(x) = (x - 3)(x^2 + 4x) + 20;\ f(3) = 20$

39. $f(x) = \left(x - \frac{1}{3}\right)(3x^2 + 3x + 6);\ f\left(\frac{1}{3}\right) = 0$

41. $f(x) = \left(x - \sqrt{3}\right)\left[x^2 + \left(2 + \sqrt{3}\right)x + 2\sqrt{3}\right] - 6;$
$f\left(\sqrt{3}\right) = -6$

43. $f(x) = \left(x - 1 - \sqrt{3}\right)\left[2x^2 + \left(3 + 2\sqrt{3}\right)x + (-5 + 5\sqrt{3})\right];$
$f\left(1 + \sqrt{3}\right) = 0$

45. (a) -69 (b) -2081 (c) -6 (d) 446

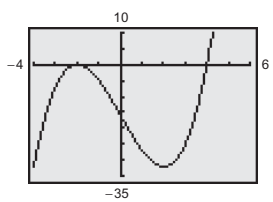
47. (a) 6 (b) -27 (c) -6.168 (d) 37

49. (a) 1 (b) -267 (c) $-\frac{11}{3}$ (d) -3.8

51. (a) Proof (b) $x + 2$

(c) $f(x) = (x - 4)(x + 2)^2$ (d) $4, -2$

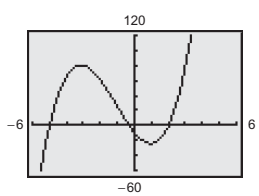
(e)



53. (a) Proof (b) $x + 5$

(c) $f(x) = (3x + 1)(x - 2)(x + 5)$ (d) $-\frac{1}{3}, 2, -5$

(e)

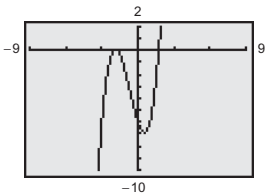


55. (a) Proof (b) $x + \sqrt{3}$

(c) $f(x) = \left(x - \sqrt{3}\right)(x + 2)\left(x + \sqrt{3}\right)$

(d) $\sqrt{3}, -2, -\sqrt{3}$

(e)



57. The second polynomial is a factor of the first polynomial.

59. e; $3, \dfrac{-1 \pm \sqrt{17}}{2}$ **61.** a; $-1, -2 \pm \sqrt{2}$

63. b; $-3, \pm\sqrt{5}$

65. Answers will vary. Sample answer:

$f(x) = 3x^3 - 13x^2 + 4x + 20$

$f(x) = -3x^3 + 13x^2 - 4x - 20$

Infinitely many polynomial functions

67. $x^2 - 7x + 10$ **69.** $3x^2 - x - 10$ **71.** $x^2 - 6x$

73. $x^2 + 4x + 3$ **75.** $x^2 + 10x + 24$ square feet

77. (a) \$199,978 (b) Proof

79. The remainder is 0. **81.** $c = -210$

SECTION 3.4 *(page 298)*

Skills Review *(page 298)*

1. $f(x) = 3x^3 - 8x^2 - 5x + 6$

2. $f(x) = 4x^4 - 3x^3 - 16x^2 + 12x$

3. $x^4 - 3x^3 + 5 + \dfrac{3}{x + 3}$

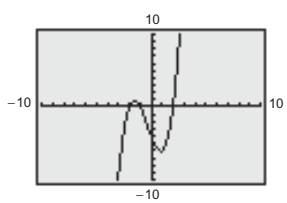
4. $3x^3 + 15x^2 - 9 - \dfrac{2}{x + (2/3)}$

5. $\frac{1}{2}, -3 \pm \sqrt{5}$ **6.** $10, -\frac{2}{3}, -\frac{3}{2}$ **7.** $-\frac{3}{4}, 2 \pm \sqrt{2}$

8. $\frac{2}{5}, -\frac{7}{2}, -2$ **9.** $\pm\sqrt{2}, \pm 1$ **10.** $\pm 2, \pm\sqrt{3}$

1. Possible: $\pm 1, \pm 2, \pm 4$

Actual: $-1, \pm 2$



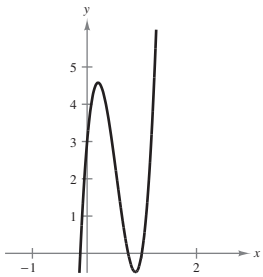
3. $-3, \frac{1}{2}, 4$ **5.** $\pm\frac{1}{2}, \pm 2$ **7.** 1, 2, 3 **9.** $-1, -10$

11. $\frac{1}{2}, -1$ **13.** $\pm 3, \pm\sqrt{2}$ **15.** $-1, 2$ **17.** $-6, \frac{1}{2}, 1$

19. $-2, 0, 1$

21. (a) $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{1}{16}, \pm\frac{3}{16}, \pm\frac{1}{32}, \pm\frac{3}{32}$

(b)



(c) $1, \frac{3}{4}, -\frac{1}{8}$

23. $f(1) = -2, f(2) = 7$ **25.** $f(2) = -6, f(3) = 44$

27. Real zero ≈ 0.7 **29.** Real zero ≈ 3.3

31. e; -1.769 **33.** d; 0.206 **35.** f; 2.769

37. $-1.164, 1.453$ **39.** 0.900, 1.100, 1.900

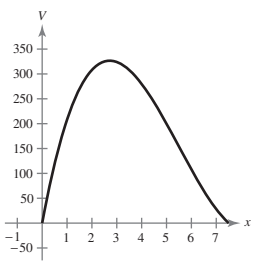
41. $-1.453, 1.164$ **43.** $-2.177, 1.563$

45. d **46.** a **47.** b **48.** c

49. (a) $V = x(18 - 2x)(15 - 2x)$

Domain: $0 < x < 7.5$

(b)



Approximate measurements:
2.72 inches × 12.56 inches × 9.56 inches

(c) $x \approx 0.448, 6, \approx 10.052$

A value of $x \approx 10.052$ inches is impossible because it would yield a negative length and width.

(d) $x = 6$

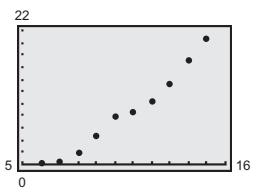
51. 18 inches × 18 inches × 36 inches

53. (a) $V = x^3 + 9x^2 + 26x + 24 = 120$

(b) 4 feet by 5 feet by 6 feet

55. 4.49 hours

57. (a)



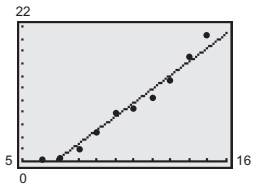
(b) Linear: $R = 2.255t - 15.27$

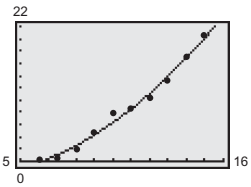
Quadratic: $R = 0.1295t^2 - 0.464t - 2.06$

Cubic: $R = 0.01105t^3 - 0.2184t^2 + 3.027t - 13.15$

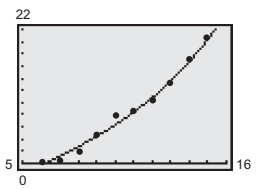
Quartic: $R = 0.008724t^4 - 0.35535t^3 + 5.3735t^2 - 33.612t + 73.59$

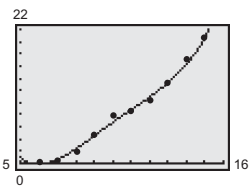
(c) Linear Quadratic





Cubic Quartic





All four models fit the data well. The nonlinear models seem to fit a little better than the linear model because the curve of each graph tends to fit the points at the far left and far right better.

(d) Linear: 2013 ($t \approx 23.18$)

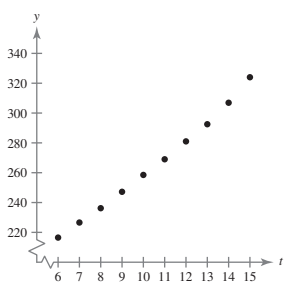
Quadratic: 2009 ($t \approx 19.25$)

Cubic: 2008 ($t \approx 18.33$)

Quartic: 2007 ($t \approx 16.92$)

Answers will vary. Sample answer: The higher the degree of the model, the faster the graph climbs after 2005, so the quartic model makes the earliest prediction of when the revenue per share will reach \$37. Based on the increasing trend of the last few data points, the year (2013) predicted by the linear model seems too late.

59. (a)



(b) Linear: $y = 11.65t + 143.5$

Quadratic: $y = 0.328t^2 + 4.76t + 177.0$

Cubic: $y = 0.0479t^3 - 1.180t^2 + 19.89t + 128.9$

Quartic: $y = 0.01219t^4 - 0.4643t^3 + 6.636t^2 - 31.32t + 250.1$

(c) Linear: 2012 ($t \approx 22.02$)

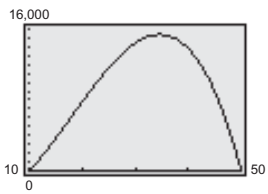
Quadratic: 2010 ($t \approx 19.81$)

Cubic: 2009 ($t \approx 18.64$)

Quartic: 2008 ($t \approx 17.79$)

Answers will vary. Sample answer: The long-term trend of the data points appears to be nearly linear, although there is an upward trend in the last three data points. If prices change according to the long-term trend, then the linear or quadratic model will give good predictions. If prices start rising more quickly as reflected in the last three data points, however, the cubic or quartic model may give better future predictions.

61. $\approx$ \$399,890 or \$744,400

63. (a) 

30 and $\approx$ 38.91

(b) You can solve $-x^3 + 54x^2 - 140x - 3000 = 14{,}400$ by rewriting the equation as $y = -x^3 + 54x^2 - 140x - 17{,}400$. Using the *table* feature of a graphing utility, you can approximate the solutions to be $x \approx -14.91$, $x = 30$, and $x \approx 38.91$. The company should charge \$38.91 to generate greater revenue.

65. No; setting $h = 64$ and solving the resulting equation yields imaginary roots.

67. No

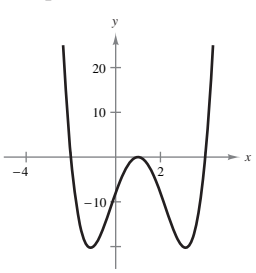
69. (a) $x = -2, 1, 4$

(b) It touches the x-axis at $x = 1$ but does not cross the x-axis.

(c) f is at least fourth degree. The degree cannot be less than 3, because there are three zeros. The degree of f cannot be odd because its left-hand behavior matches its right-hand behavior. So, its degree cannot be 3.

(d) It is positive because f increases to the right and left.

(e) Answers will vary.
Sample answer: $f(x) = (x + 2)(x - 1)^2(x - 4)$

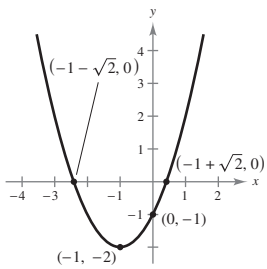
(f) 

71. Answers will vary.

MID-CHAPTER QUIZ *(page 303)*

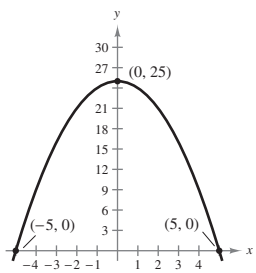
1. Vertex: $(-1, -2)$

Intercepts: $\left(-1 \pm \sqrt{2}, 0\right), (0, -1)$



2. Vertex: $(0, 25)$

Intercepts: $(\pm 5, 0), (0, 25)$



3. Rises to the left
Falls to the right

4. Rises to the left
Rises to the right

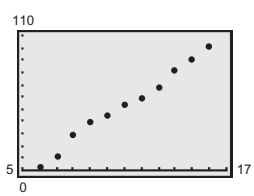
5. 293

6. $f(x) = (x - 1)(x^3 + x^2 - 4x - 4) + 0; f(1) = 0$

7. $f(x) = (x + 3)(x^2 + 2x - 8) + 0; f(-3) = 0$

8. $2x^2 + 5x - 12$ **9.** $\pm\sqrt{5}, -\frac{7}{2}$ **10.** $\pm 3, \pm\frac{1}{2}$

11. $1, -\frac{4}{3}$ **12.** $\frac{3}{2}$ **13.** P = \$2,534,375; \$337,600

14. (a) 

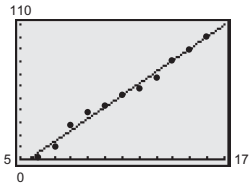
(b) Linear: $A(t) = 9.52t - 52.3$

Quadratic: $A(t) = -0.052t^2 + 10.67t - 58.1$

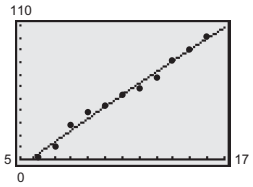
Cubic: $A(t) = 0.1014t^3 - 3.400t^2 + 45.69t - 173.3$

Quartic: $A(t) = -0.00495t^4 + 0.3194t^3 - 6.872t^2 + 69.34t - 231.2$

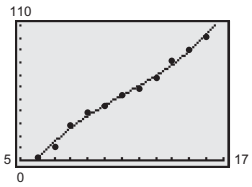
(c) Linear



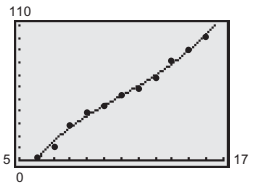
Quadratic



Cubic



Quartic



Each model could be considered a good fit, but the cubic and quartic models appear to fit the data a little better than the linear and quadratic models.

(d) Linear: 2011 ($t \approx 21.25$)

Quadratic: 2012 ($t \approx 21.82$)

Cubic: 2008 ($t \approx 18.49$)

Quartic: 2009 ($t \approx 18.94$)

Answers will vary. Sample answer: The linear and quadratic predictions are close to each other and the cubic and quartic predictions are close to each other. The cubic and quartic models give earlier predictions because their graphs rise faster in the years after 2006 than the graphs of the linear and quadratic models.

SECTION 3.5 *(page 312)*

Skills Review *(page 312)*

1. $2\sqrt{3}$ **2.** $10\sqrt{5}$ **3.** $\sqrt{5}$ **4.** $-6\sqrt{3}$

5. 12 **6.** 48 **7.** $\frac{\sqrt{3}}{3}$ **8.** $\sqrt{2}$

9. $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ **10.** $-1 \pm \sqrt{2}$

1. $i, -1, -i, 1, i, -1, -i, 1,\ i, -1, -i, 1, i, -1, -i, 1$
$i^{4n} = 1,\ i^{4n+1} = i,\ i^{4n+2} = -1,\ i^{4n+3} = -i,$
n is an integer.

3. $a = 7, b = 12$ **5.** $a = 4, b = -3$

7. $9 + 4i; 9 - 4i$ **9.** $-3 - 2\sqrt{3}i; -3 + 2\sqrt{3}i$

11. $-21; -21$ **13.** $-1 - 6i; -1 + 6i$

15. $-5i; 5i$ **17.** $-3; -3$ **19.** $2 + i$

21. $5 + 6i$ **23.** $3 - 3\sqrt{2}i$ **25.** $\frac{1}{6} + \frac{7}{6}i$

27. -14 **29.** $-2\sqrt{6}$ **31.** -10 **33.** $5 + i$

35. 25 **37.** $30 + 20i$ **39.** $-11 + 60i$ **41.** 8

43. $\left(16 + 4\sqrt{3}\right) + \left(-16\sqrt{2} + 2\sqrt{6}\right)i$ **45.** $\frac{4}{5} - \frac{3}{5}i$

47. $1 + \frac{1}{2}i$ **49.** $10 - 7i$ **51.** $\frac{1}{8}i$

53. $-\frac{44}{125} - \frac{8}{125}i$ **55.** $\frac{35}{29} + \frac{595}{29}i$

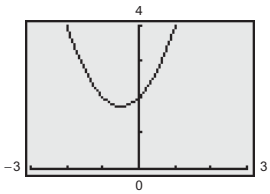
57. Error: $(3 - 2i)(3 + 2i) = 9 - 4i^2 = 9 + 4 = 13$
(not $9 - 4 = 5$)

59. $1 \pm i$ **61.** $-2 \pm \frac{1}{2}i$

63. $-\frac{3}{2}, -\frac{5}{2}$ **65.** $\frac{1}{8} \pm \frac{\sqrt{11}}{8}i$

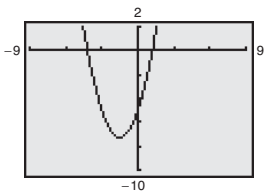
67. $x = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

The graph has no x-intercepts, so the solutions must be complex conjugate pairs.

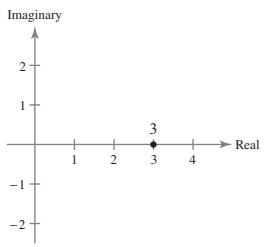


69. $x = -\frac{3}{2} \pm \frac{\sqrt{29}}{2}$

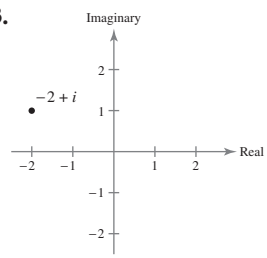
The x-intercepts of the graph correspond to the zeros of the function.



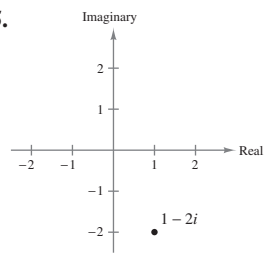
71.



73.



75.



77. The complex number 0 is in the Mandelbrot Set because, for $c = 0$, the corresponding Mandelbrot sequence is 0, 0, 0, 0, 0, 0, which is bounded.

79. The complex number 1 is not in the Mandelbrot Set because, for $c = 1$, the corresponding Mandelbrot sequence is 1, 2, 5, 26, 677, 458,330, . . . , which is unbounded.

81. The complex number $\frac{1}{2}i$ is in the Mandelbrot Set because, for $c = \frac{1}{2}i$, the corresponding Mandelbrot sequence is $\frac{1}{2}i$, $-\frac{1}{4} + \frac{1}{2}i$, $-\frac{3}{16} + \frac{1}{4}i$, $-\frac{7}{256} + \frac{13}{32}i$, $-\frac{10,767}{65,536} + \frac{1957}{4096}i$, $-\frac{864,513,055}{4,294,967,296} + \frac{46,037,845}{134,217,728}i$, which is bounded.

83. False. If the complex number is real, it equals its conjugate.

85. Answers will vary.

SECTION 3.6 *(page 320)*

Skills Review *(page 320)*

1. $4 - \sqrt{29}i, 4 + \sqrt{29}i$ **2.** $-5 - 12i, -5 + 12i$

3. $-1 + 4\sqrt{2}i, -1 - 4\sqrt{2}i$ **4.** $6 + \frac{1}{2}i, 6 - \frac{1}{2}i$

5. $-13 + 9i$ **6.** $12 + 16i$ **7.** $26 + 22i$

8. 29 **9.** i **10.** $-9 + 46i$

1. 1 **3.** 3 **5.** 4 **7.** $\pm 4i; (x + 4i)(x - 4i)$

9. $\frac{5 \pm \sqrt{5}}{2}; \left(x - \frac{5 + \sqrt{5}}{2}\right)\left(x - \frac{5 - \sqrt{5}}{2}\right)$

11. $\pm 3, \pm 3i; (x - 3)(x + 3)(x - 3i)(x + 3i)$

13. $0, \pm\sqrt{5}i; x\left(x - \sqrt{5}i\right)\left(x + \sqrt{5}i\right)$

15. $-5, 8 \pm i; (x + 5)(x - 8 + i)(x - 8 - i)$

17. $2, 2 \pm i; (x - 2)(x - 2 + i)(x - 2 - i)$

19. $-5, 4 \pm 3i; (t + 5)(t - 4 + 3i)(t - 4 - 3i)$

21. $-10, -7 \pm 5i; (x + 10)(x + 7 - 5i)(x + 7 + 5i)$

23. $-5, -2 \pm \sqrt{3}i; (x + 5)\left(x + 2 - \sqrt{3}i\right)\left(x + 2 + \sqrt{3}i\right)$

25. $-\frac{3}{4}, 1 \pm \frac{1}{2}i; (4x + 3)(2x - 2 + i)(2x - 2 - i)$

27. $-\frac{1}{5}, 1 \pm \sqrt{5}i; (5x + 1)\left(x - 1 + \sqrt{5}i\right)\left(x - 1 - \sqrt{5}i\right)$

29. $2, \pm 2i; (x - 2)^2(x + 2i)(x - 2i)$

31. $\pm i, \pm 3i; (x + i)(x - i)(x + 3i)(x - 3i)$

33. $-4, 3, \pm i; (t + 4)^2(t - 3)(t + i)(t - i)$

35. Answers will vary.

Sample answer: $x^3 + 2x^2 + 9x + 18$

37. Answers will vary. Sample answer: $x^3 - 5x^2 + 9x - 5$

39. Answers will vary.

Sample answer: $x^5 + 4x^4 + 13x^3 + 52x^2 + 36x + 144$

41. Answers will vary.

Sample answer: $x^4 + 8x^3 + 9x^2 - 10x + 100$

43. Answers will vary.

Sample answer: $3x^4 - 17x^3 + 25x^2 + 23x - 22$

45. (a) $(x^2 - 8)(x^2 + 1)$

(b) $\left(x - 2\sqrt{2}\right)\left(x + 2\sqrt{2}\right)(x^2 + 1)$

(c) $\left(x - 2\sqrt{2}\right)\left(x + 2\sqrt{2}\right)(x - i)(x + i)$

47. (a) $(x^2 - 2x + 3)(x^2 - 3x - 5)$

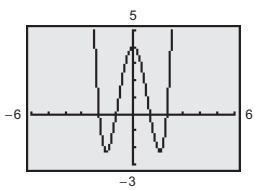
(b) $(x^2 - 2x + 3)\left(x - \frac{3 + \sqrt{29}}{2}\right)\left(x - \frac{3 - \sqrt{29}}{2}\right)$

(c) $\left(x - 1 + \sqrt{2}i\right)\left(x - 1 - \sqrt{2}i\right)$
$\left(x - \frac{3 + \sqrt{29}}{2}\right)\left(x - \frac{3 - \sqrt{29}}{2}\right)$

49. $\pm 2i, \frac{7}{3}$ **51.** $\pm 6i, 1$ **53.** $-3 \pm i, \frac{1}{4}$

55. $1, 2, -3 \pm \sqrt{2}i$ **57.** $\frac{3}{4}, \frac{1}{2} \pm \frac{\sqrt{5}}{2}i$

59. $\pm 1, \pm 2$. The x-intercepts occur at the solutions of the equation.

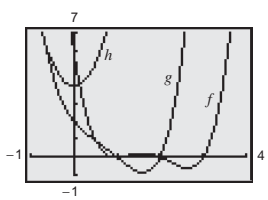


61. Answers will vary. Sample answer:

(a) $f(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$
Zeros: 1, 1, 2, 3

(b) $g(x) = x^4 - 3x^3 + 3x^2 - 3x + 2$
Zeros: $1, 2, \pm i$

(c) $h(x) = x^4 + 5x^2 + 4$
Zeros: $\pm i, \pm 2i$

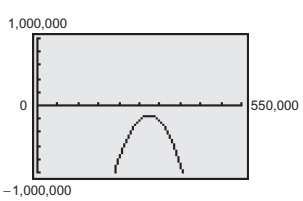


Similarities: f, g, and h all rise to the left and rise to the right.

Differences: f, g, and h have different numbers of x-intercepts.

63. There is no price p that would yield a profit of $9 million.

The graph of $y = P - 9{,}000{,}000$ has no intercepts, so there is no solution.



65. No. The conjugate pairs statement specifies polynomials with *real* coefficients. $f(x)$ has imaginary coefficients.

67. The imaginary zeros of f can only occur in conjugate pairs. f has only one unknown zero and no unpaired complex zeros, so the unknown zero must be real.

69. Imaginary zeros can only occur in conjugate pairs. Because f has three zeros, one or all of them must be real numbers.

71. Polynomials of odd degree eventually rise to one side and fall to the other side. So, f must cross the x-axis, which means f must have a real zero. You can show this by graphing several third-degree polynomials.

SECTION 3.7 *(page 330)*

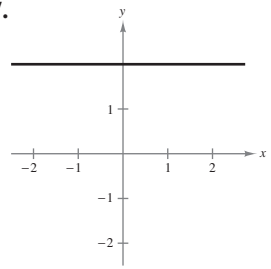
Skills Review *(page 330)*

1. $x(x - 4)$ **2.** $2x(x^2 - 3)$ **3.** $(x - 5)(x + 2)$

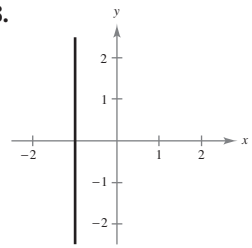
4. $(x - 5)(x - 2)$ **5.** $x(x + 1)(x + 3)$

6. $(x^2 - 2)(x - 4)$

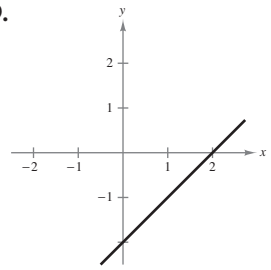
7.



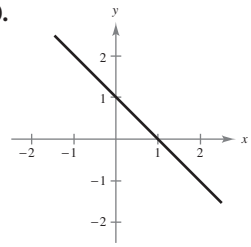
8.



9.



10.



1. Domain: All $x \neq -1$; Horizontal asymptote: $y = 3$; Vertical asymptote: $x = -1$

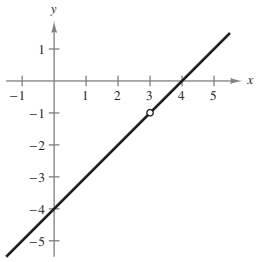
3. Domain: All $x \neq 5$; Horizontal asymptote: $y = -1$; Vertical asymptote: $x = 5$

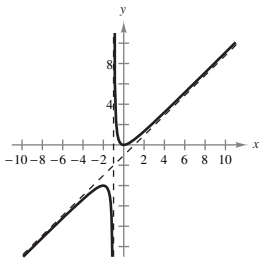
5. Domain: All reals; Horizontal asymptote: $y = 3$; Vertical asymptote: None

7. Domain: All $x \neq -4$; Horizontal asymptote: $y = 0$; Vertical asymptote: $x = -4$

9. (a) None
(b) None
(c) None

11. (a) $x = -1$
(b) None
(c) $y = x - 1$





13. f **14.** e **15.** a **16.** b **17.** c **18.** d

19. $g(x)$ shifts downward two units.

21. $g(x)$ is a reflection about the x-axis.

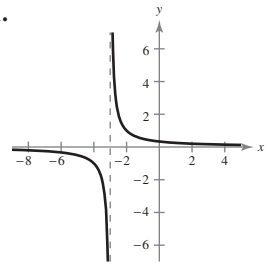
23. $g(x)$ shifts upward three units.

25. $g(x)$ is a reflection about the x-axis.

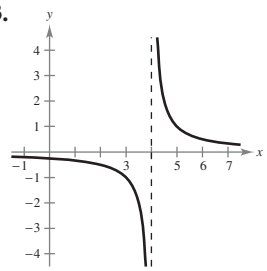
27. $g(x)$ shifts upward five units.

29. $g(x)$ is a reflection about the x-axis.

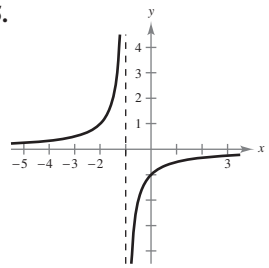
31.



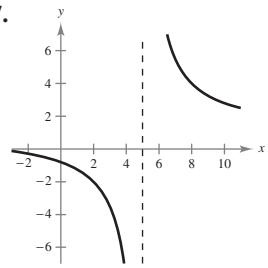
33.



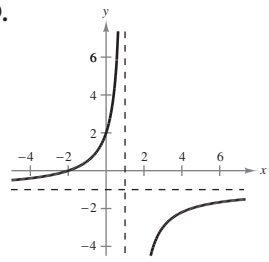
35.



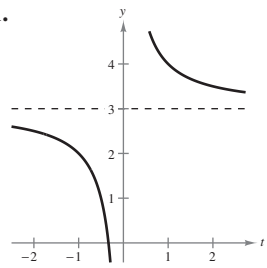
37.



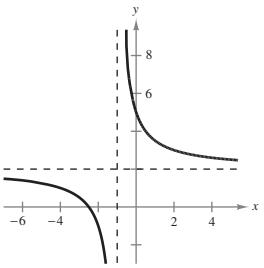
39.



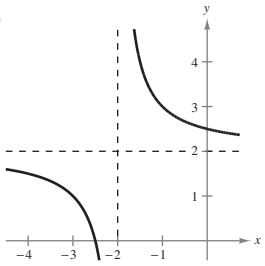
41.



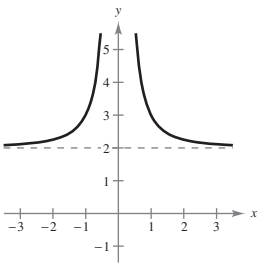
43.



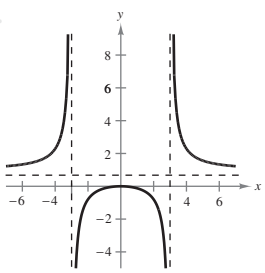
45.



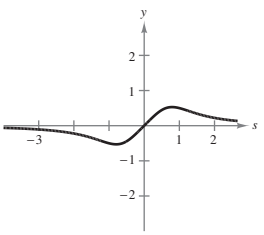
47.



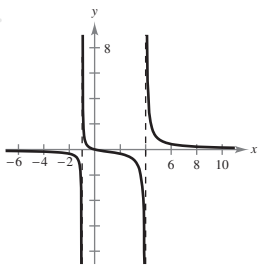
49.



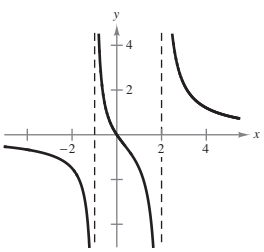
51.



53.



55.



57. Answers will vary. Sample answer: $f(x) = \dfrac{2x^2}{x^2 + 1}$

59. Answers will vary. Sample answer: $f(x) = \dfrac{x^3}{x^2 + x - 2}$

61. Answers will vary. Sample answer: $f(x) = \dfrac{1}{x^2 + 1}$

63. Answers will vary. Sample answer: $f(x) = \dfrac{x}{x^2 - 2x - 3}$

65. No. Given

$$f(x) = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$$

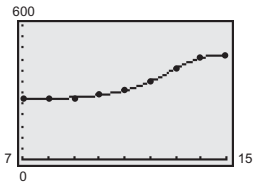
if $n > m$, there is no horizontal asymptote and n must be greater than m for a slant asymptote to occur.

67. (a) \$176 million (b) \$528 million (c) \$1584 million

(d) No. The model has a vertical asymptote at $p = 100$.

69. (a) 318 deer; 500 deer; 900 deer (b) 2500 deer

71. (a)



The model fits the data very well.

(b) \$366.8 billion; \$332.3 billion; \$319.1 billion; Answers will vary.

(c) $y \approx 292.7$; as time passes, the national defense outlays approach \$292.7 billion.

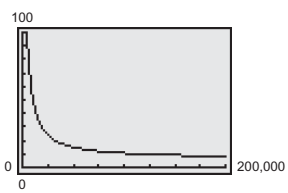
73. (a)

	1	2	3	4	5
	0.60	0.79	0.86	0.90	0.92

	6	7	8	9	10
	0.93	0.94	0.95	0.95	0.96

(b) 100%

75. (a)



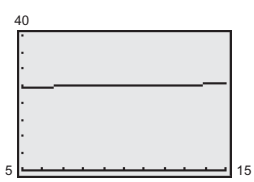
(b)

	10,000	100,000	1,000,000	10,000,000
	\$51	\$10.50	\$6.45	\$6.05

Eventually, the average recycling cost per pound will approach the horizontal asymptote of \$6.

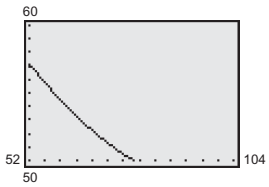
77. (a) $B = \dfrac{100.9708t + 6083.999}{3.0195t + 251.817}, \quad 5 \le t \le 15$

(b)



(c) ≈ 25.96 barrels per person

79. (a)

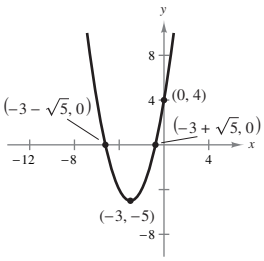


(b) 2008: 48.18 seconds; 2012: 48.43 seconds

(c) This model does not have a horizontal asymptote. A model with a horizontal asymptote would be reasonable for this type of data because the winning times should continue to improve, but it is not humanly possible for the winning times to decrease without bound.

REVIEW EXERCISES *(page 336)*

1.

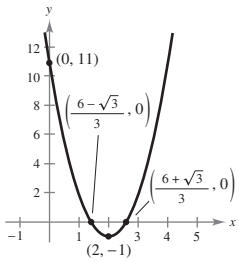


Vertex: $(-3, -5)$

Intercepts:

$(0, 4), \left(-3 \pm \sqrt{5}, 0\right)$

3.



Vertex: $(2, -1)$

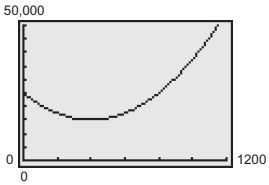
Intercepts:

$(0, 11), \left(\frac{6 \pm \sqrt{3}}{3}, 0\right)$

5. $f(x) = \frac{7}{9}(x + 5)^2 - 1$

7. $A(x) = 250x - x^2$; 125 feet × 125 feet

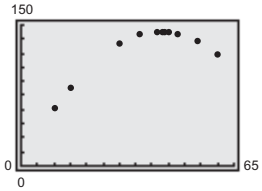
9. (a)



(b) About 385 units

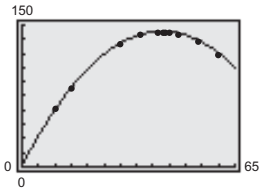
(c) Write the equation of the quadratic function in standard form. The vertex is the minimum cost.

11. (a)



(b) $d(x) = -0.077x^2 + 6.59x + 2.4$

(c)



(d) (42.8, 143.4); The vertex gives the angle 42.8° which results in the greatest distance of 143.4 meters.

13. Falls to the left
Rises to the right

15. Falls to the left
Falls to the right

17. ±4 **19.** 0, 2, 5

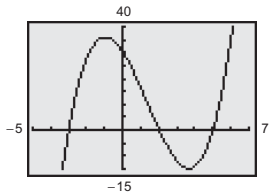
21. $x^2 - 3x + 1 - \frac{1}{2x + 1}$ **23.** $x^2 + 11x + 24$

25. $x^2 - 3x + 3$ **27.** (a) −10 (b) 11

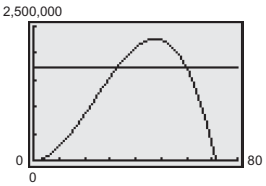
29. (a) Proof (b) $x - 2$

(c) $(x - 5)(x + 3)(x - 2)$ (d) $-3, 2, 5$

(e)



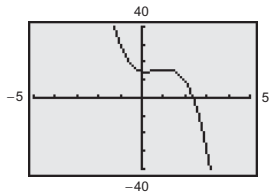
31. (a)



(b) Answers will vary.

≈ $325,167

33. $\pm 1, \pm 3, \pm 5, \pm 15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}, \pm\frac{15}{4}$

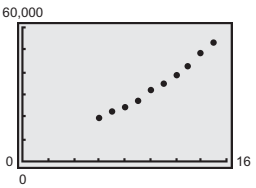


From the graph: $x \approx 2.357$, so the zero is not rational.

35. $-3, -1, 2$ **37.** $\pm 2, \pm\sqrt{5}$ **39.** $-\frac{1}{3}, \frac{3}{2}, 2$

41. $-1, 2$ **43.** $x \approx -2.3$ **45.** $-1.321, -0.283, 1.604$

47. (a)



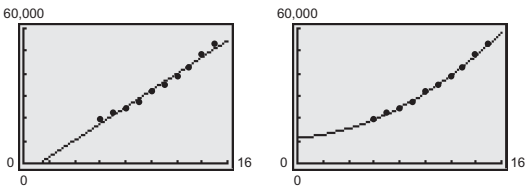
(b) Linear: $R(t) = 3708.4t - 4690$

Quadratic: $R(t) = 157.78t^2 + 395.0t + 11{,}403$

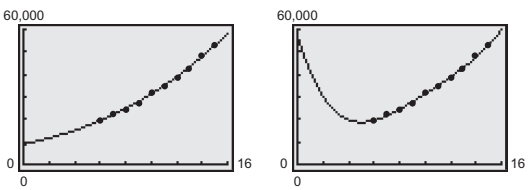
Cubic: $R(t) = 2.427t^3 + 81.33t^2 + 1162.2t + 8967$

Quartic: $R(t) = 4.8829t^4 - 202.653t^3 + 3211.25t^2 - 19{,}345.9t + 57{,}518$

(c) Linear Quadratic



Cubic Quartic



Each model could be considered a good fit for the data.

(d) Linear: 2009 ($t \approx 18.79$)

Quadratic: 2007 ($t \approx 17.22$)

Cubic: 2007 ($t \approx 17.14$)

Quartic: 2007 ($t \approx 16.59$)

Answers will vary. Sample answer: The quadratic, cubic, and quartic predictions are all around 2007. It appears that the curves of the graphs of these models fit the data points of the last few years better than the graph of the linear model. So, the linear prediction (2009) may not be as good.

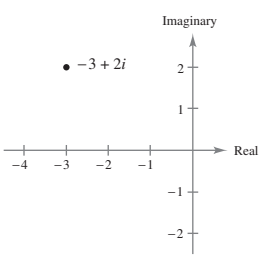
49. $4\sqrt{2}i; -4\sqrt{2}i$ **51.** $-3 + 4i; -3 - 4i$

53. $5 + i$ **55.** $11 + 9\sqrt{3}i$ **57.** 89 **59.** $-10 - 8i$

61. $-7 + 24i$ **63.** 10 **65.** $3 - 2i$

67. $-3 - 4i$ **69.** $\dfrac{1 \pm \sqrt{23}i}{4}$ **71.** $\dfrac{-11 \pm \sqrt{73}}{8}$

73.



75. $\pm 3, \pm 3i; (x - 3)(x + 3)(x - 3i)(x + 3i)$

77. $-5, \pm\sqrt{3}i; (t + 5)\left(t - \sqrt{3}i\right)\left(t + \sqrt{3}i\right)$

79. $2, \pm\dfrac{3i}{2}; (x - 2)(2x - 3i)(2x + 3i)$

81. Answers will vary. Sample answer: $x^3 - x^2 + 9x - 9$

83. (a) $(x^2 + 8)(x^2 - 3)$

(b) $(x^2 + 8)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$

(c) $\left(x + 2\sqrt{2}i\right)\left(x - 2\sqrt{2}i\right)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$

85. $\pm 4i, \frac{1}{4}$ **87.** $-1 \pm 3i, -1, -4$

89. Domain: All $x \neq -2$

Vertical asymptote: $x = -2$

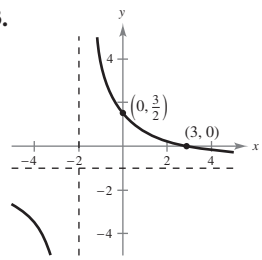
Horizontal asymptote: $y = 0$

91. Domain: All $x \neq \pm 3$

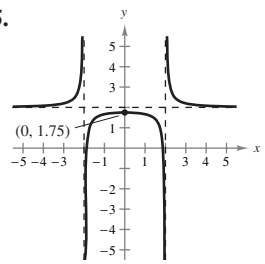
Vertical asymptotes: $x = -3, x = 3$

Horizontal asymptote: $y = 2$

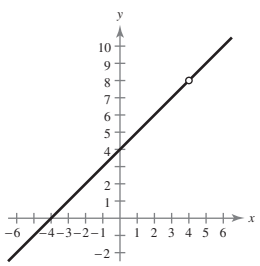
93.



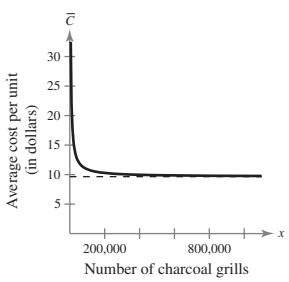
95.



97. None



99. (a)



(b)

x	1000	10,000	100,000	1,000,000
$\overline{C}$	\$134.65	\$22.15	\$10.90	\$9.78

Eventually, the average cost per charcoal grill will approach the horizontal asymptote of \$9.65.

101. (a) 430,769 fish; 662,500 fish; 1,024,000 fish

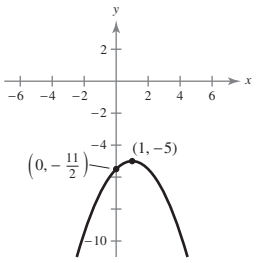
(b) 1,666,667 fish

103. (a) \$35,000 (b) \$157,500 (c) \$10,395,000

(d) No. The model has a vertical asymptote at $p = 100$.

CHAPTER TEST *(page 340)*

1.



Vertex: $(1, -5)$

Intercept: $\left(0, -\frac{11}{2}\right)$

2. (a) Falls to the left
Rises to the right

(b) Rises to the left
Rises to the right

3. $x^2 + 7x + 12$

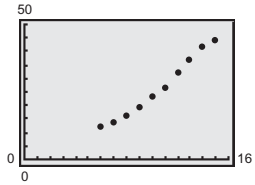
4. $\pm 1, \pm 3, \pm 9, \pm 27, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm\frac{3}{2}, \pm\frac{9}{4}, \pm\frac{9}{2}, \pm\frac{27}{4}, \pm\frac{27}{2}$

$f\left(\frac{3}{2}\right) = 0$

$f\left(-\frac{3}{2}\right) = 0$

$f(x) = \left(x - \frac{3}{2}\right)\left(x + \frac{3}{2}\right)(x - 1)(x - 3)$

5. (a)

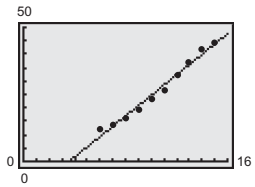


(b) Linear: $S = 3836t - 1379$

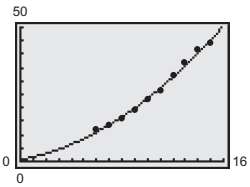
Quadratic: $S = 01415t^2 + 0864t + 065$

Cubic: $S = -003857t^3 + 13564t^2 - 11327t + 3936$

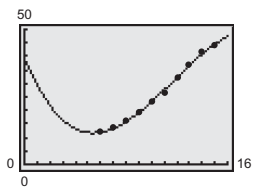
(c) Linear



Quadratic



Cubic



Each model could be considered a good fit for the data.

(d) Linear: 2007 ($t \approx 1663$)

Quadratic: 2006 ($t \approx 1587$)

Cubic: 2008 ($t \approx 1791$)

Answers will vary. Sample answer: All three models give predictions that reflect the increasing data. Beyond 2008, however, the sales per share predicted by the cubic model begin to decrease, so the cubic model is probably not appropriate for future predictions. The linear and quadratic models show increases beyond 2008, but neither is certain to be a good predictor far into the future, because the data do not show a steady trend.

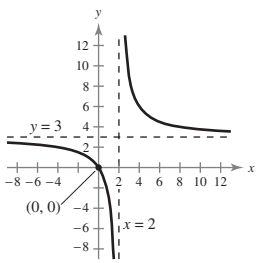
6. $16 - 3i$ **7.** $7 - 9i$ **8.** $27 - 4\sqrt{3}i$

9. $23 - 14i$ **10.** i

11. $x = \dfrac{-5 \pm \sqrt{3}i}{2}$ **12.** $x = \dfrac{5 \pm 3\sqrt{7}i}{4}$

13. $x^4 - 7x^3 + 19x^2 - 63x + 90$ **14.** $\pm\sqrt{5}i, -2$

15.



Domain: All $x \neq 2$

CHAPTER 4

SECTION 4.1 *(page 350)*

Skills Review *(page 350)*

1. 5^x **2.** 3^{2x} **3.** 4^{3x} **4.** 10^x **5.** 4^{2x}

6. 4^{10x} **7.** $\left(\frac{3}{2}\right)^x$ **8.** 4^{3x} **9.** $\frac{1}{2^x}$ **10.** $\frac{3^x}{5^x}$

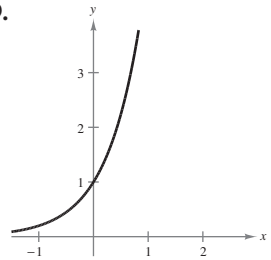
11. 2^x **12.** 3^x

1. 3463 **3.** 95946 **5.** 0079 **7.** 54598

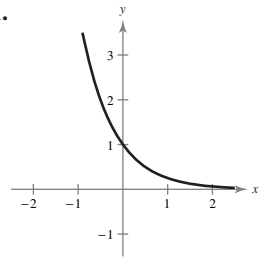
9. 1948 **11.** g **12.** e **13.** b

14. h **15.** d **16.** a **17.** f **18.** c

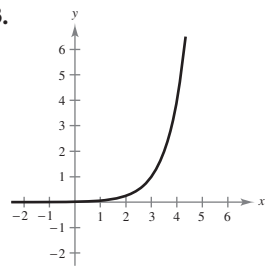
19.



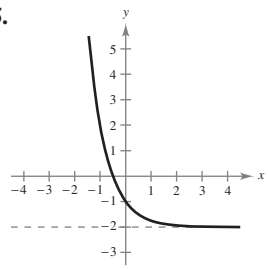
21.



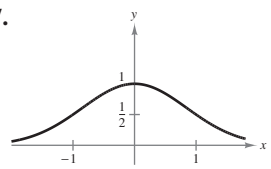
23.



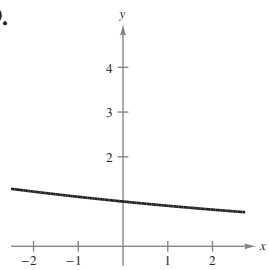
25.



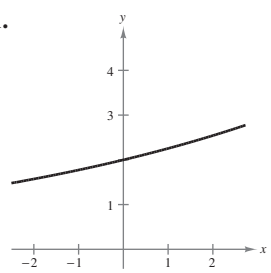
27.



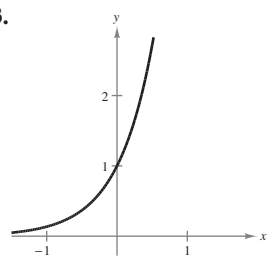
29.



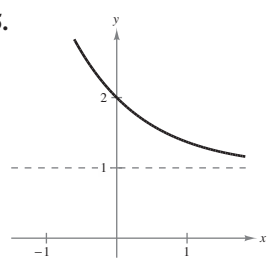
31.



33.



35.



37.

	1	2	4
	$34664	$40122	$42974

	12	365	Continuous
	$44923	$45880	$45912

39.

	1	2	4
	$4,11573	$5,71429	$6,60223

	12	365	Continuous
	$7,23138	$7,54707	$7,55794

41.

	1	10	20
	$1,39312	$0,65697	$0,52989

	30	40	50
	$72055	$73237	$1090

43.

	1	10	20
	$0,52124	$6,94070	$3,64615

	30	40	50
	$04098	$86217	$8790

45. The account paying 5% interest compounded quarterly earns more money. Even though the interest is compounded less frequently, the higher interest rate yields a higher return.

47. You should choose the account with the online access fee because it yields a higher return.

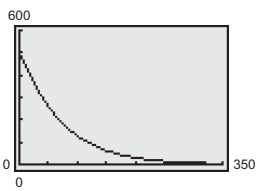
49. $9,69117 **51.** $47,68376

53. $0,70076 **55.** $55,25566

57. (a) $8291

(b) $958

(c)



(d) $1719

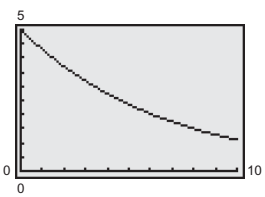
59. (a) 100 (b) ≈ 110 (c) ≈ 121 (d) ≈ 158

61. $42470 **63.** $2,43443

65. (a) 5907 (b) 6237 (c) 6767 (d) 7753

67. ≈ 3042 kilograms

69. (a)



(b) ≈ 111 pounds

(c) On the graph, when $P = 25$, $t \approx 46$ months.

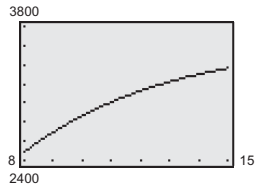
71. (a) $810 million, $650 million, $100 million

(b) $0821 million, $4949 million, $07390 million

73. (a) 22, 24, 25, 255 (b) 221, 240, 251, 254

75. Women tend to marry about 2 years younger than men do. The median ages of both have been rising, and the age difference is decreasing.

77. (a)



(b) 2696 billion prescriptions, 3127 billion prescriptions, 3363 billion prescriptions

79. Answers will vary.

SECTION 4.2 *(page 361)*

Skills Review *(page 361)*

1. 3 **2.** 0 **3.** -1 **4.** 1 **5.** 7389

5. 7389 **6.** 0368

7. The graph of g is the graph of f shifted two units to the left.

8. The graph of g is the graph of f reflected about the x-axis.

9. The graph of g is the graph of f shifted downward one unit.

10. The graph of g is the graph of f reflected about the y-axis.

1. c **2.** f **3.** b **4.** d **5.** a **6.** e

7. $\log_4 256 = 4$ **9.** $\log_{81} 3 = \frac{1}{4}$

11. $\log_6 \frac{1}{36} = -2$ **13.** $\ln e = 1$ **15.** $\ln 4 = x$

17. $4^2 = 16$ **19.** $2^{-1} = \frac{1}{2}$ **21.** $e^1 = e$

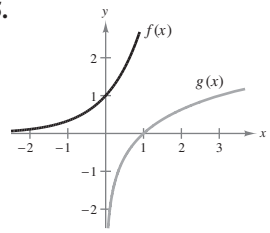
23. $5^{-1} = 02$ **25.** $27^{1/3} = 3$ **27.** 2

29. -4 **31.** $\frac{1}{3}$ **33.** 1 **35.** -4 **37.** 1

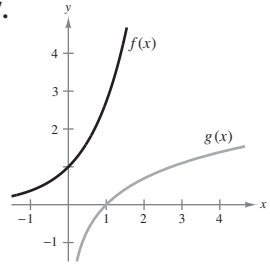
39. -4 **41.** 5 **43.** 2538 **45.** -0097

47. 0452 **49.** 1946 **51.** 2913 **53.** 0896

55.



57.

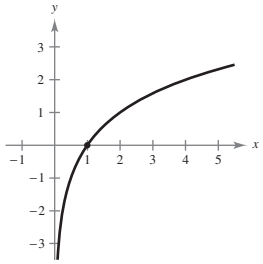


59. d **60.** e **61.** a **62.** c **63.** f **64.** b

65. Domain: $(0, \infty)$

Asymptote: $x = 0$

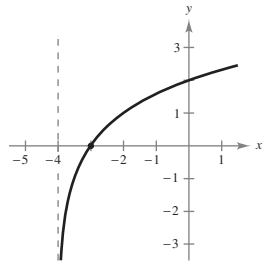
x-intercept: $(1, 0)$



67. Domain: $(-4, \infty)$

Asymptote: $x = -4$

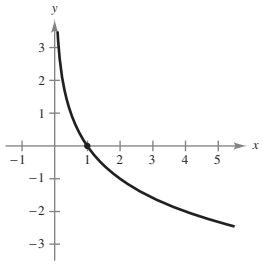
x-intercept: $(-3, 0)$



69. Domain: $(0, \infty)$

Asymptote: $x = 0$

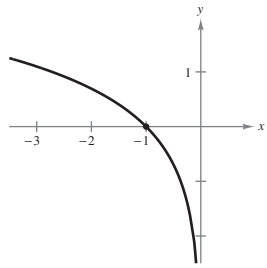
x-intercept: $(1, 0)$



71. Domain: $(-\infty, 0)$

Asymptote: $x = 0$

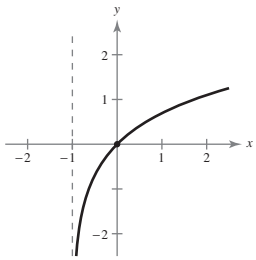
x-intercept: $(-1, 0)$



73. Domain: $(-1, \infty)$

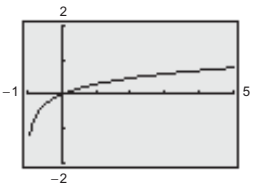
Asymptote: $x = -1$

x-intercept: $(0, 0)$

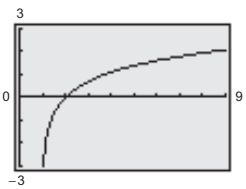


CHAPTER 4

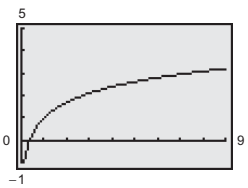
75.



77.



79.



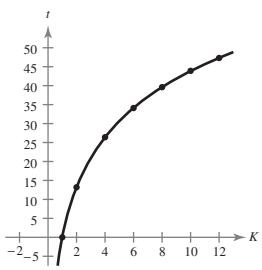
81. $t \approx 261$ years

83. (a) 78 (b) 682 (c) In 3 months ($t \approx 27$)

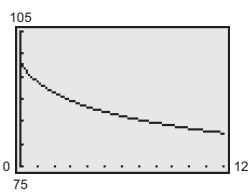
85. (a)

	1	2	4	6	8	10	12
	0	132	264	341	396	439	473

(b)

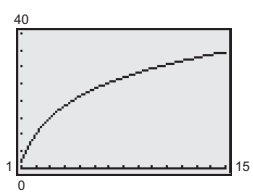


87.



The domain $0 \le t \le 12$ covers the period of 12 months or 1 year.The scores range from the average score on the original exam to the average score after 12 months, so the range is $8240 \le f(t) \le 98$

89.



The domain $1 \le t \le 15$ represents the time period of 1 day to 15 days.The range $2 \le g(t) \le 345$ represents the possible units produced by the employees over the given time period.

91. (a) $\approx$ 294 years (b) $\approx$ 333 years
(c) $\approx$ 6 years (d) $\approx$ 102 years

93. (a) False (b) True (c) True (d) False

SECTION 4.3 *(page 369)*

Skills Review *(page 369)*

1. 2 **2.** -5 **3.** -2 **4.** -3 **5.** e^5

6. $\frac{1}{e}$ **7.** e^6 **8.** 1 **9.** $y = x^{-2}$

10. $y = x^{1/2}$ **11.** $4^3 = 64$ **12.** $16^{1/2} = 4$

1. $\dfrac{\log_{10} 8}{\log_{10} 5}$ **3.** $\dfrac{\log_{10} 30}{\log_{10} e}$ **5.** $\dfrac{\log_{10} n}{\log_{10} 3}$ **7.** $\dfrac{\log_{10} x}{\log_{10} \frac{1}{5}}$

9. $\dfrac{\log_{10} \frac{3}{10}}{\log_{10} x}$ **11.** $\dfrac{\log_{10} x}{\log_{10} 26}$ **13.** $\dfrac{\ln 8}{\ln 5}$ **15.** $\dfrac{\ln 5}{\ln 10}$

17. $\dfrac{\ln n}{\ln 3}$ **19.** $\dfrac{\ln x}{\ln \frac{1}{5}}$ **21.** $\dfrac{\ln \frac{3}{10}}{\ln x}$ **23.** $\dfrac{\ln x}{\ln 26}$

25. 2585 **27.** 1079 **29.** 2633 **31.** -0683

33. -1661 **35.** 2322 **37.** 11833 **39.** -02084

41. 10686 **43.** 01781 **45.** 18957 **47.** -27124

49. 05708 **51.** $\frac{1}{3}$ **53.** $-\frac{1}{2}$ **55.** -3

57. $-1 - \log_9 2$ **59.** $\frac{1}{2} + \frac{1}{2}\log_7 10$

61. $-3 - \log_5 2$ **63.** $6 + \ln 5$ **65.** $6 + 5\log_2 3$

67. $\log_3 4 + \log_3 n$ **69.** $\log_5 x - 2$ **71.** $4\log_2 x$

73. $\frac{1}{2}\ln z$ **75.** $\ln x + \ln y + \ln z$ **77.** $\frac{1}{2}\ln(a - 1)$

79. $2\ln(z - 1) - \ln z$ **81.** $\ln z - \frac{1}{3}\ln(z + 3)$

83. $\frac{1}{3}(\ln x - \ln y)$ **85.** $\frac{3}{4}\ln x + \frac{1}{4}\ln(x^2 + 3)$

87. $\log_3 5x$ **89.** $\log_4 \dfrac{8}{x}$ **91.** $\log_{10}(x + 4)^2$

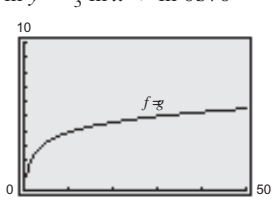
93. $\ln \dfrac{1}{216x}$ **95.** $\ln \dfrac{\sqrt[3]{5x}}{x + 1}$ **97.** $\log_8 \dfrac{x - 2}{x + 2}$

99. $\ln \dfrac{9}{\sqrt{x^2 + 1}}$ **101.** $\ln \dfrac{x}{(x + 2)(x - 2)}$

103. $\ln y = \frac{1}{4}\ln x$ **105.** $\ln y = -\frac{1}{4}\ln x + \ln \frac{5}{2}$

107. $\ln y = \frac{2}{3}\ln x + \ln 0070$ **109.** $\approx$ 26 decibels

111.



The two graphs are the same.The property is $\log_a(uv) = \log_a u + \log_a v$.

113. Choose a value for y and graph $\log_a x/\log_a y$ and $\log_a x - \log_a y$. Notice that the graphs are different. To demonstrate the correct property, graph $\log_a(x/y)$ and $\log_a x - \log_a y$, choosing a value for y.

115. Let $\log_a u = x$ and $\log_a v = y$.

$a^x = u$ and $a^y = v$

$u \cdot v = a^x \cdot a^y = a^{x+y}$

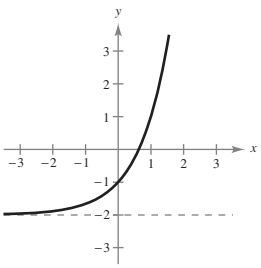
$\log_a a^{x+y} = x + y$

$\log_a uv = \log_a u + \log_a v$

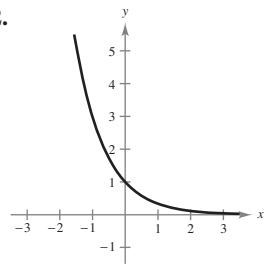
117. Answers will vary.

MID-CHAPTER QUIZ *(page 372)*

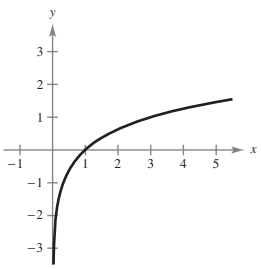
1.



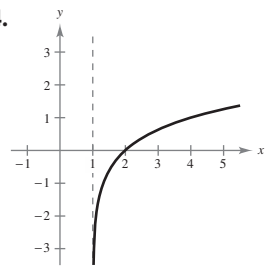
2.



3.



4.



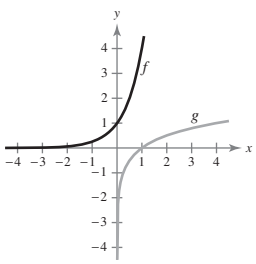
5. (a) Monthly: \$5,08504
(b) Continuously: \$5,09834

6. (a) 2000: 779 million (b) 2009: 1108 million
2005: 947 million 2010: 1153 million

7. (a) 100 (b) $\approx$ 364 (c) $\approx$ 1326

8. \$0854 **9.** 2 **10.** 4 **11.** -2 **12.** 0

13.



Domain of $f(x)$: $(-\infty, \infty)$

Domain of $g(x)$: $(0, \infty)$

The graphs are reflections of each other about the line $y = x$.

14. $\frac{3}{2}$ **15.** $\frac{6}{5}$ **16.** $\frac{1}{3}\log_{10} x + \frac{1}{3}\log_{10} y - \frac{1}{3}\log_{10} z$

17. $\ln(x^2 + 3) - 3\ln x$ **18.** $\ln\frac{xy}{3}$ **19.** $\log_{10}\frac{1}{64x^3}$

20. $\ln y = \frac{1}{3}\ln x$

SECTION 4.4 *(page 380)*

Skills Review *(page 380)*

1. $\frac{\ln 3}{\ln 2}$ **2.** $1 + \frac{2}{\ln 4}$ **3.** $\frac{e}{2}$ **4.** $2e$

5. $2 \pm i$ **6.** $\frac{1}{2}, 1$ **7.** x **8.** $2x$

9. $2x$ **10.** $-x^2$

1. 3 **3.** -2 **5.** 2 **7.** 64 **9.** $\frac{1}{10}$ **11.** x^2

13. $x^2 + 1$ **15.** $x^3 - 7$ **17.** $x^3 - 8$ **19.** $x + 5$

21. x^2 **23.** $\ln 3 \approx 1099$ **25.** $\log_3 8 \approx 1893$

27. $\ln 28 \approx 3332$ **29.** $\frac{1}{2}\log_3 80 \approx 1994$ **31.** 2

33. $\log_3 28 + 1 \approx 4033$ **35.** $3 - \log_2 565 \approx -6142$

37. $\frac{1}{3}\log_{10}\frac{3}{2} \approx 0059$ **39.** $\log_5 7 + 1 \approx 2209$

41. $\frac{1}{3}\ln 12 \approx 0828$ **43.** $\ln\frac{5}{3} \approx 0511$

45. $\ln\frac{1}{2} \approx -0693$ **47.** $\frac{1}{3} + \frac{1}{3}\log_2\frac{8}{3} \approx 0805$

49. $\ln 6 \approx 1792$, $\ln 2 \approx 0693$ **51.** $\ln 4 \approx 1386$

53. $2\ln 75 \approx 8635$ **55.** $\frac{1}{2}\ln 1498 \approx 3656$

57. $\dfrac{\ln 4}{365\ln\left(1 + \dfrac{0065}{365}\right)} \approx 21330$

59. $\dfrac{\ln 2}{12\ln\left(1 + \dfrac{010}{12}\right)} \approx 6960$

61. 10,000 **63.** $e^{-3} \approx 0050$

65. $\dfrac{e^{24}}{2} \approx 5512$ **67.** 5,000,000

69. $-1 + 3^{12/5} \approx 12967$ **71.** $\frac{1}{5}e^{10/3} \approx 5606$

73. $e^2 - 2 \approx 5389$ **75.** $e^{-2/3} \approx 0513$

77. No solution **79.** $1 + \sqrt{1 + e} \approx 2928$

81. No solution **83.** 7 **85.** $\dfrac{-1 + \sqrt{17}}{2} \approx 1562$

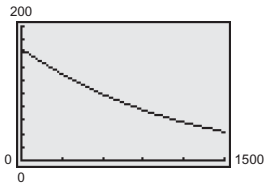
87. 2 **89.** $\dfrac{725 + 125\sqrt{33}}{8} \approx 180384$

91. $y = 2x + 1$ **93.** $y = \dfrac{(x-1)^2}{x+2}$

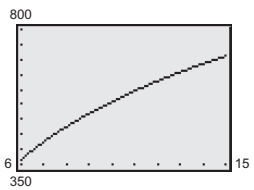
95. 2807 **97.** 20086 **99.** $\approx$ 1109 years

101. $\approx$ 1515 years **103.** 26 months

105. (a) ≈ 210 coin sets (b) ≈ 588 coin sets

(c) 

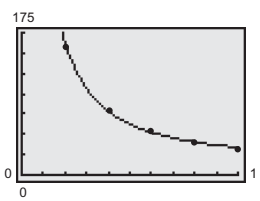
107. (a) ≈ 293 years (b) ≈ 398 years

109. (a) 

(b) and (c) 2001

111. (a)

02	1626
04	785
06	525
08	405
10	339

(b) 

The model is a good fit for the data.

(c) 1197 meters

(d) No. To reduce the g's to fewer than 23 requires a crumple zone of more than 227 meters, a length that exceeds the front width of most cars.

113. $\log_b uv = \log_b u + \log_b v$

True by the Product Rule in Section 43.

115. $\log_b(u - v) = \log_b u - \log_b v$

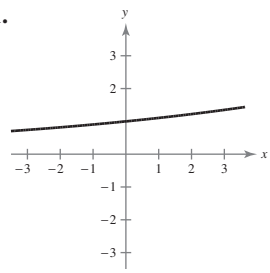
False.

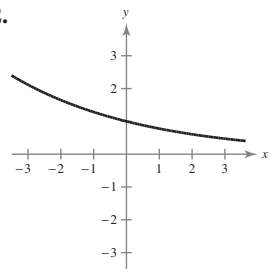
$195 \approx \log(100 - 10) \neq \log 100 - \log 10 = 1$

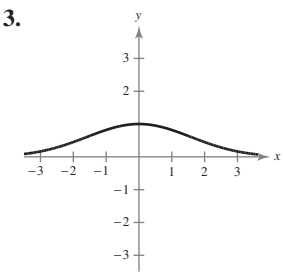
117. See Exercise 81.

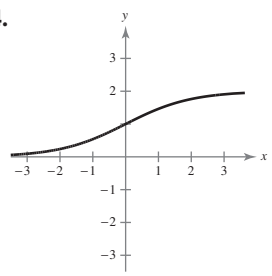
SECTION 4.5 *(page 391)*

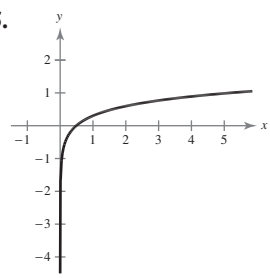
Skills Review *(page 391)*

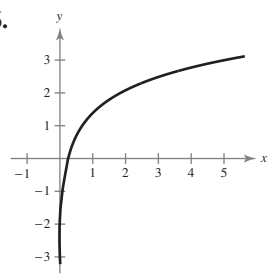
1. 

2. 

3. 

4. 

5. 

6. 

7. $\frac{1}{2}\ln\frac{7}{3}$ **8.** $\frac{1}{5}e^{7/2}$ **9.** ≈ 34539 **10.** ≈ 3695

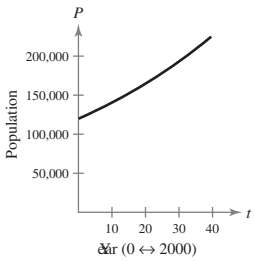
	Initial Investment	*Annual % Rate*	*Time to Double*	*Amount After 10 Years*
1.	$000	7%	990 years	$0,06876
3.	$00	693%	10 years	$00000
5.	$000	825%	840 years	$28188
7.	$39279	11%	630 years	$9,20500
9.	$000	8%	866 years	$1,12770

	Isotope	*Half-Life (Years)*	*Initial Quantity*	*Amount After 1000 Years*
11.	^{226}Ra	1599	4 g	259 g
13.	^{14}C	5715	395 g	35 g
15.	^{239}Pu	24,100	165 g	16 g

17. Exponential growth **19.** Exponential decay

21. $C = 1, k = \frac{1}{4}\ln 10$ **23.** $C = 1, k = \frac{1}{4}\ln\frac{1}{4}$

25.



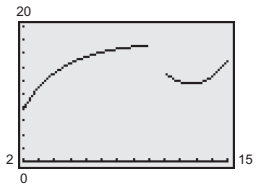
2025

27. ≈ 1236 hours to double
≈ 1959 hours to triple

29. 12,180 years **31.** ≈ 992%

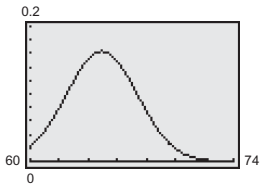
33. (a) $N = 40(1 - e^{-0049t})$ (b) ≈ 42 days

35. (a)



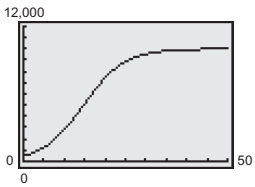
(b) S changed from a logarithmic function to a quadratic function.

37.



649 inches

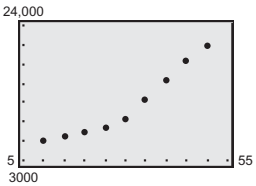
39. (a)



(b) ≈ 1252 fish

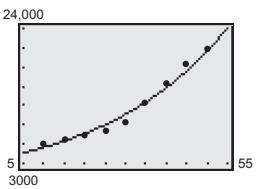
(c) ≈ 78 months

41. (a)



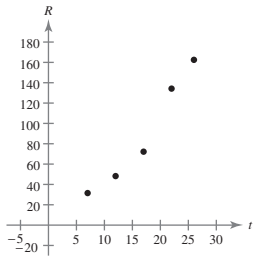
(b) $P = 397282e^{00328t}$

(c)



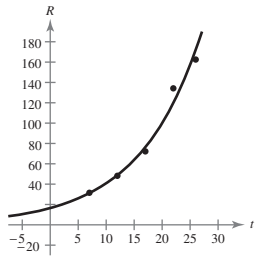
(d) 2022: ≈ 8,174,963 people
2042: ≈ 15,753,714 people

43. (a)



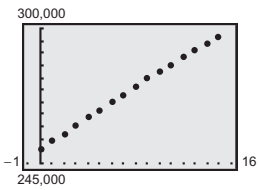
(b) $R = 1658e^{00899t}$

(c)



(d) 2009: ≈ $248 million
2010: ≈ $460 million

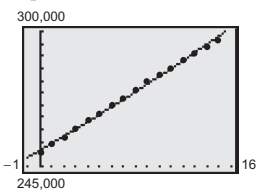
45. (a)



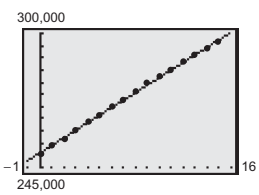
(b) $P = 251{,}37202e^{00114t}$

(c) Linear model: $P = 31109t + 250{,}815$
Quadratic model:
$P = -2055t^2 + 34192t + 250{,}096$

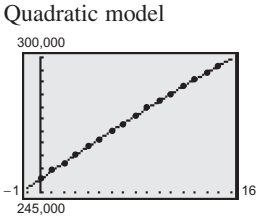
(d) Exponential model



Linear model



Quadratic model



(e) Exponential model

2008: 308,627,174 people

2009: 312,165,655 people

2010: 315,744,706 people

Linear model

2008: 306,811,200 people

2009: 309,922,100 people

2010: 313,033,000 people

Quadratic model

2008: 304,983,400 people

2009: 307,642,250 people

2010: 310,260,000 people

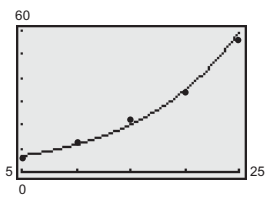
Answers will vary. Sample answer: The predictions given by all three models are relatively close to each other and seem reasonable.

47. (a) ≈ 7906 (b) ≈ 7684

49. (a) 20 decibels (b) 70 decibels

51. $\approx 1585 \times 10^{-6}$ **53.** $\approx 31{,}623$ **55.** 300 A.M.

57.



The data fit an exponential model.

Exponential model: $n = 39405e^{.01086t}$

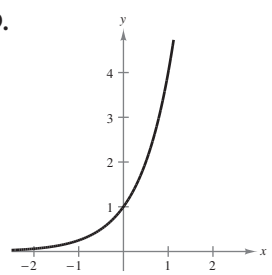
A logistic growth model would be more appropriate for this data because after the initial rapid growth in productivity, the worker's production rate will eventually level off.

REVIEW EXERCISES *(page 398)*

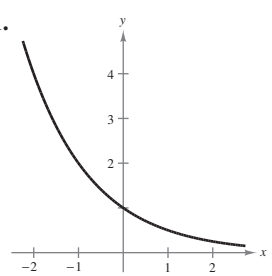
1. d **2.** f **3.** a **4.** b

5. c **6.** e **7.** h **8.** g

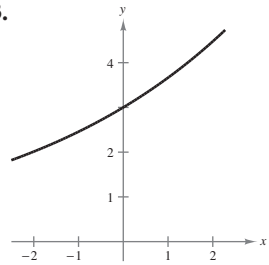
9.



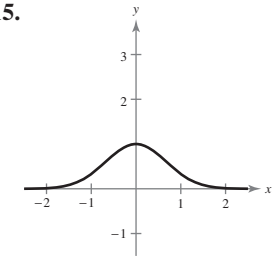
11.



13.



15.



17.

n	1	2	4
A	\$13,681.11	\$13,972.87	\$14,127.43

n	12	365	Continuous
A	\$14,233.93	\$14,286.46	\$14,288.26

19.

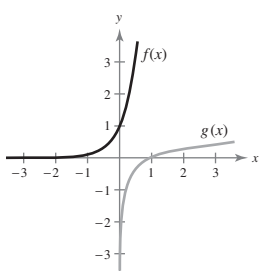
t	1	10	20
P	\$185,548.70	\$94,473.31	\$44,626.03

t	30	40	50
P	\$21,079.84	\$9957.41	\$4703.55

21. \$47194 **23.** $\log_4 64 = 3$ **25.** $\ln 73890 \ldots = 2$

27. $3^4 = 81$ **29.** $e^0 = 1$ **31.** 5 **33.** 7 **35.** $-\frac{1}{2}$

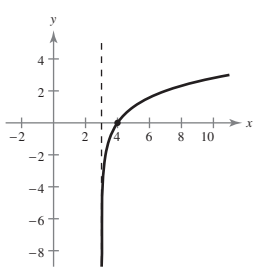
37.



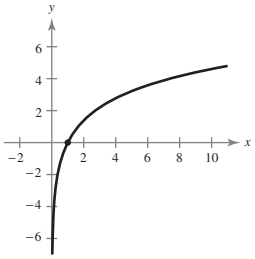
39. Domain: $(3, \infty)$

Vertical asymptote: $x = 3$

x-intercept: $(4, 0)$

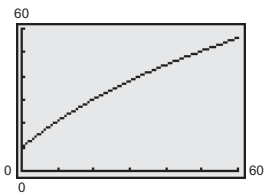


41. Domain: $(0, \infty)$
Vertical asymptote: $x = 0$
x-intercept: $(1, 0)$



43. The average score decreased from 82 to about 68.

45. (a) 5342 inches

(b) 

47. 2096 **49.** 2132 **51.** 09208 **53.** 02823

55. 2 **57.** 32 **59.** $\log_{10} x - \log_{10} y$

61. $\ln x + \frac{1}{2}\ln(x - 3)$ **63.** $4 \log_5(y - 3)$ **65.** $\log_4 6$

67. $\ln \sqrt{x}$ **69.** $\ln \dfrac{x}{(x - 3)(x + 1)}$

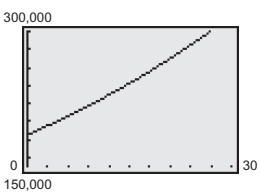
71. $\ln y = \frac{4}{3} \ln x$ **73.** $\ln 8 \approx 2079$ **75.** 1

77. $\ln 4 \approx 1386$ **79.** $\dfrac{1}{3}e^{82} \approx 1213650$

81. $\frac{1}{5}e^2 \approx 1478$ **83.** $3e^2 \approx 22167$ **85.** 7

87. (a) $\approx$ 197 desks (b) $\approx$ 257 desks

89. 1063 g

91. (a)  (b) 2016 ($t \approx 1673$)

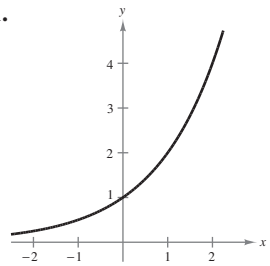
93. $\approx$ 993 hours to double
$\approx$ 1574 hours to triple

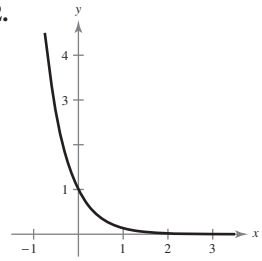
95. (a) $N = 50(1 - e^{-004838t})$ (b) 48 days

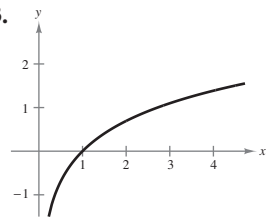
97. (a) $k \approx 02121$
(b) $\approx$ 196 deer; $\approx$ 294deer; $\approx$ 383deer

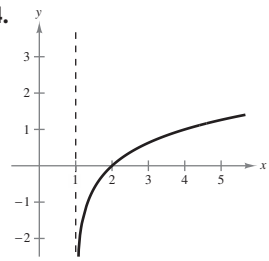
99. [illegible]

CHAPTER TEST *(page 402)*

1. 

2. 

3. 

4. 

5. 87; $\approx$ 798; $\approx$ 765

6. After 12 months: $\approx$ 703
After 18 months: $\approx$ 678
Human memory diminishes slowly over time.

7. $2 \ln x + 3 \ln y - \ln z$

8. $\log_{10} 3 + \log_{10} x + \log_{10} y + 2 \log_{10} z$

9. $\log_2 x + \frac{1}{3}\log_2(x - 2)$ **10.** $\frac{1}{5}\log_8(x^2 + 1)$

11. $\ln \dfrac{x^2y^3}{z}$ **12.** $\log_{10} \sqrt[3]{x^2y^2}$ **13.** $\dfrac{\log_2 21}{4}$

14. $\ln 6 \approx 1792$
$\ln 2 \approx 0693$

15. 127 **16.** $e^6 - 2 \approx 401429$ **17.** $\approx$ 163 years

18. About 2015 ($t \approx 154$). Because the exponent on e is positive, the population of the city is growing.

19. $\approx$ 99 hours **20.** $\approx$ 297 grams; $\approx$ 088 gram

21. Answers will vary.Sample answer:
The growth of the bear population will slow down as the population approaches the carrying capacity of the island. So, the population will grow logistically.

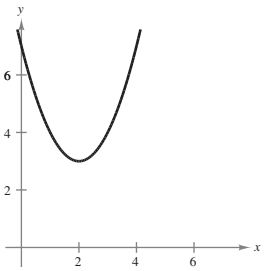
CUMULATIVE TEST: CHAPTERS 2–4
(page 403)

1. $x^2 + 3x - 4$ **2.** $x^2 - 3x + 6$

3. $3x^3 - 5x^2 + 3x - 5$ **4.** $\dfrac{x^2 + 1}{3x - 5}, x \neq \dfrac{5}{3}$

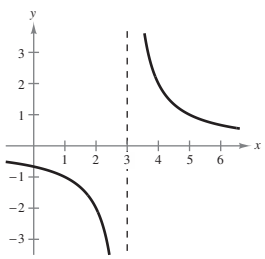
5. $9x^2 - 30x + 26$ **6.** $3x^2 - 2$

7.



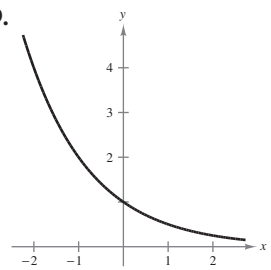
Domain: $(-\infty, \infty)$
Range: $[3, \infty)$

8.



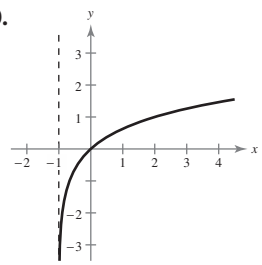
Domain: $(-\infty, 3) \cup (3, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

9.



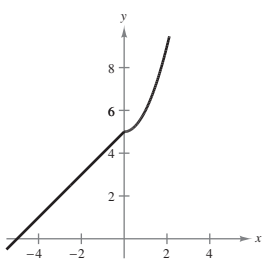
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$

10.



Domain: $(-1, \infty)$
Range: $(-\infty, \infty)$

11.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

12. 75,000 units **13.** $45 - 7i$ **14.** $-9 + 40i$

15. $-\frac{1}{29} - \frac{17}{29}i$ **16.** $\frac{5 \pm \sqrt{59}\, i}{6}$

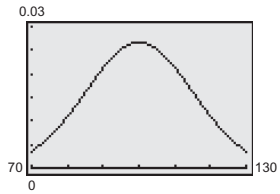
17. $-4i, 4i, i, -i$. Because a given zero is $4i$ and $f(x)$ is a polynomial with real coefficients, $-4i$ is also a zero.Using these two zeros, you can form the factors $(x + 4i)$ and $(x - 4i)$. Multiplying these two factors produces $x^2 + 16$. Using long division to divide $x^2 + 16$ into f produces $x^2 + 1$. Then, factoring $x^2 + 1$ gives the zeros i and $-i$.

18. (a) $3x - 2 + \frac{2 - 3x}{2x^2 + 1}$

(b) $2x^3 - x^2 + 2x - 10 + \frac{25}{x + 2}$

19. $\ln 6 \approx 1792$ **20.** $3 + e^{12} \approx 162{,}757791$

21.



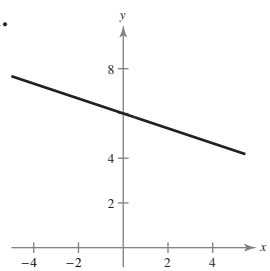
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CHAPTER 5

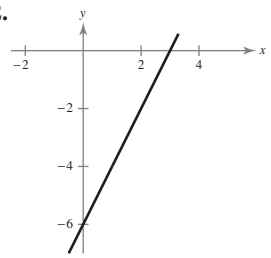
SECTION 5.1 *(page 412)*

Skills Review *(page 412)*

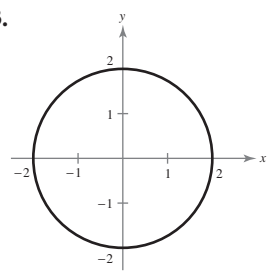
1.



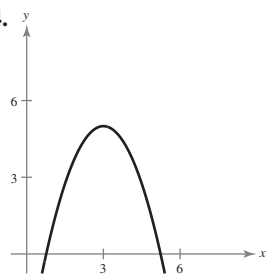
2.



3.



4.



5. x **6.** $-37v$ **7.** $2x^2 + 9$ **8.** -1

9. $x = 6$ **10.** $y = 1$

1. (a) No (b) Yes **3.** (a) Yes (b) No

5. (a) No (b) No **7.** $(2, -3)$ **9.** $(-1, 2), (2, 5)$

11. $(-2, -2), (0, 0), (2, 2)$ **13.** $(0, 4), \left(\frac{12}{5}, -\frac{16}{5}\right)$

15. $(-1, 0), (1, 0)$ **17.** $(-1, 1)$ **19.** $\left(\frac{1}{2}, 3\right)$

21. $(10, 3)$ **23.** $(15, 03\)$ **25.** $\left(\frac{20}{3}, \frac{40}{3}\right)$

27. No solution

29. $\left(1 + \sqrt{2}, 2 + 2\sqrt{2}\right), \left(1 - \sqrt{2}, 2 - 2\sqrt{2}\right)$

31. $\left(\frac{29}{10}, \frac{21}{10}\right), (-2, 0)$ **33.** No solution **35.** $(0, 1), (\pm 1, 0)$

37. $(2, 1)$ **39.** $(4, 3)$ **41.** $\left(\frac{5}{2}, \frac{3}{2}\right)$ **43.** $(2, 2), (4, 0)$

45. $(1, 4), (4, 7)$ **47.** $\left(4, -\frac{1}{2}\right)$ **49.** No solution

51. One solution **53.** Two solutions **55.** No solution

57. $\left(\frac{1}{2}, \frac{3}{4}\right), (-3, -1)$ **59.** (1, 4), (4, 7) **61.** (0, 1)

63. No points of intersection **65.** 192 units

67. 233,333 units **69.** 1500 CDs **71.** 1996 ($t \approx 570$)

73. 8%, at age 15. **75.** \$5,000 at 85%; \$30,000 at 12%

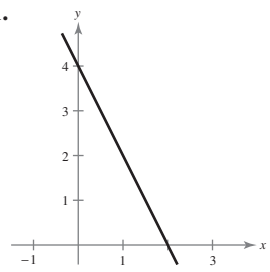
77. \$50,000

79. According to the graphs, Federal Perkins Loan awards will exceed Federal Pell Grant awards. Both models eventually decrease and become negative. So, it is unlikely that these models will continue to be accurate.

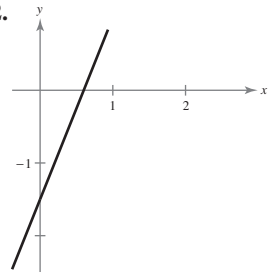
SECTION 5.2 *(page 423)*

Skills Review *(page 423)*

1.



2.



3. $x - y + 4 = 0$ **4.** $5x + 3y - 28 = 0$

5. $-\frac{1}{2}$ **6.** $\frac{7}{4}$ **7.** Perpendicular

8. Parallel **9.** Neither parallel nor perpendicular

10. Perpendicular

1. (2, 2) **3.** (2, 0) **5.** Inconsistent

7. $(2a, 3a - 3)$, where a is any real number

9. $\left(-\frac{1}{3}, -\frac{2}{3}\right)$ **11.** $\left(2, \frac{1}{2}\right)$; consistent

13. $(-7, -13)$; consistent **15.** $(4, -5)$; consistent

17. $(4, -1)$; consistent **19.** (40, 40); consistent

21. $\left(-\frac{6}{35}, \frac{43}{35}\right)$; consistent **23.** No solution; inconsistent

25. $\left(\frac{18}{5}, \frac{3}{5}\right)$; consistent **27.** $\left(\frac{19}{7}, -\frac{2}{7}\right)$; consistent

29. $\left(a, \frac{5}{6}a - \frac{1}{2}\right)$, where a is any real number; consistent

31. $\left(\frac{90}{31}, -\frac{67}{31}\right)$; consistent

33. No. The solution of the system is (79,400, 398).

35. $\begin{cases} x + y = 13 \\ x - y = 3 \end{cases}$; (8, 5) **37.** $\begin{cases} 2r + s = 8 \\ r - s = 7 \end{cases}$; (5, −2)

39. 550 miles per hour; 50 miles per hour

41. $6\frac{2}{3}$ gallons of 20% **43.** \$0,000 at 95%

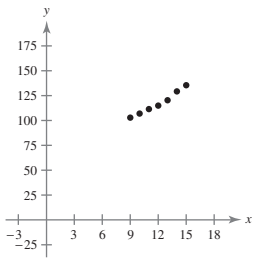
$3\frac{1}{3}$ gallons of 50% \$5,000 at 14%

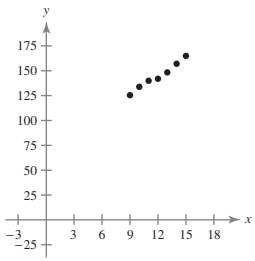
45. Let the number of adult tickets $= a$ and the number of children's tickets $= c$. By solving the system of equations $a + c = 740$, $4688 = 85a + 4c$, you obtain $a = 384$, $c = 356$.

47. $x \approx 309{,}091$ units; $p \approx \$3509$

49. $x = 2{,}000{,}000$ units; $p = \$0000$

51. (a) Fast-food Full-service





$y = 540x + 526$ $y = 619x + 702$

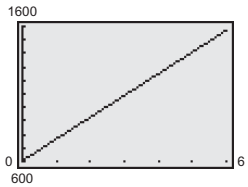
(b) No

53. 1380 units at \$1060 **55.** $y = 097x + 21$

57. $y = 0318x + 4061$ **59.** $y = -2x + 4$

61. (a) $y = 161t + 605$

(b)



1571 cars

(c) You obtain the same model: $y = 161t + 605$.

63. Answers will vary. Sample answer:

$\begin{cases} 3x - y = 6 \\ 6x - 2y = 12 \end{cases}$; $(a, 3a - 6)$, a is any real number.

Using the method of elimination, you obtain the statement $0 = 0$, which is true for all values of the variables. So, the system has infinitely many solutions.

65. (a) Any value of $k \neq 3$ (b) $k = 3$

For the system to be inconsistent, the lines must have the same slope and different y-intercepts. So, any value of k except for $k = 3$ will produce an inconsistent system. For the system to be consistent (dependent), the lines must have the same slope and the same y-intercept. So, $k = 3$ will produce a consistent (dependent) system.

SECTION 5.3 *(page 435)*

Skills Review *(page 435)*

1. $(15, 10)$ **2.** $\left(-2, -\frac{8}{3}\right)$ **3.** $(28, 4)$
4. $(4, 3)$ **5.** Not a solution **6.** Not a solution
7. Solution **8.** Solution **9.** $5a + 2$
10. $a + 13$

1. c **2.** a **3.** b **4.** d

5. Yes. The system has a "stair-step" pattern with leading coefficients of 1.

7. No. The system has a "stair-step" pattern, but not all of its leading coefficients are 1.

9. $(4, -2, -2)$ **11.** $(2, -3, -2)$ **13.** $(-1, -6, 8)$
15. Inconsistent **17.** $\left(1, -\frac{3}{2}, \frac{1}{2}\right)$
19. $(-3a + 10, 5a - 7, a)$ **21.** $\left(-4a + 13, -\frac{15}{2}a + \frac{45}{2}, a\right)$
23. Inconsistent **25.** $(-3, 4, 2)$ **27.** $(3, -1, 2)$
29. $\left(\frac{3}{4}a, -2a, a\right)$ **31.** $(-5a + 3, -a - 5, a)$
33. Inconsistent **35.** $(1, 1, 1, 1)$

37. Answers will vary. Sample answer:

$$\begin{cases} 2x - y + z = 9 \\ y + z = 1, \\ z = 2 \end{cases} \quad \begin{cases} -x + 2y - 4z = -13 \\ x + y + z = 4 \\ x + z = 5 \end{cases}$$

39. Answers will vary. Sample answer:

$$\begin{cases} x - y + z = 3 \\ y - z = -2, \\ z = -3 \end{cases} \quad \begin{cases} x + 3y + 4z = -26 \\ 4x - y - 5z = 24 \\ x + 2y = -9 \end{cases}$$

41. Answers will vary. Sample answer:
$a = 3$: $(3, -2, 3)$
$a = 6$: $(6, 1, 5)$
$a = -3$: $(-3, -8, -1)$

43. Answers will vary. Sample answer:
$a = 2$: $(1, 6, 5)$
$a = 4$: $(2, 12, 5)$
$a = 0$: $(0, 0, 5)$

45. $y = 2x^2 + 3x - 4$ **47.** $y = -4x^2 + 2x + 1$
49. $x^2 + y^2 - 4x = 0$ **51.** $x^2 + y^2 - 6x + 6y + 9 = 0$

53. \$00,000 at 7%
\$00,000 at 8%
\$00,000 at 10%

55. 15,000 units of \$5 candles
30,000 units of \$0 candles
5000 units of \$ candles

57. 18 gallons of spray X
1 gallon of spray Y
6 gallons of spray Z

59. Invest $\$3{,}33333 + 08a$ in certificates of deposit, $\$41{,}66667 - 08a$ in municipal bonds, $\$25{,}00000 - a$ in blue-chip stocks, and a in growth or speculative stocks, where $0 \le a \le 125{,}000$.

61. $y = 0079x^2 + 063x + 29$
63. $y = -0207x^2 - 089x + 51$
65. (a) $y = 0421x^2 - 099x + 148$ (b) Same model
67. (a) $y = -0371x^2 + 252x + 601$
(b) Same model (c) $\approx 634\%$
69. (a) $y = 0125x^2 - 255x + 18$
(b) $y = 0125x^2 - 255x + 18$ (c) No

71. Yes. A system of linear equations can have three possible types of solutions: exactly one solution, infinitely many solutions, or no solution.

73. The solution $(-a, 2a - 1, a)$ is generated by solving for x and y in terms of z. The solution $(b, -2b - 1, -b)$ is generated by solving for y and z in terms of x. Both methods yield equivalent solutions.

75. Answers will vary.

MID-CHAPTER QUIZ *(page 440)*

1. $(2, 5)$ **2.** No solution **3.** $(1, 3)$
4. $\left(-\frac{2}{5} \pm \frac{2\sqrt{11}}{5}, \frac{1}{5} \pm \frac{4\sqrt{11}}{5}\right)$ **5.** 1500 units
6. 500,000 units **7.** $(2, -1)$ **8.** $\left(1, \frac{3}{2}\right)$
9. $x = 5000$ units
$p = \$0$
10. $y = 062x + 400$ **11.** $(1, -2, 3)$
12. Answers will vary.
Sample answer: $(a + 6, a + 6, a)$, a is any real number.
13. Inconsistent **14.** $y = -05829x^2 + 3782x + 5985$

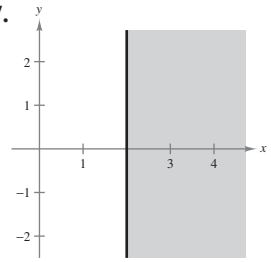
SECTION 5.4 *(page 448)*

Skills Review *(page 448)*

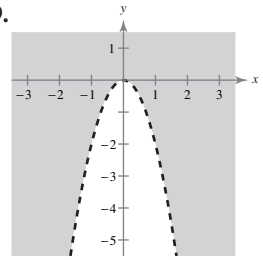
1. Line **2.** Parabola **3.** Circle **4.** Parabola
5. Line **6.** Circle **7.** $(1, 1)$ **8.** $(2, 0)$
9. $(2, 1), \left(-\frac{5}{2}, -\frac{5}{4}\right)$ **10.** $(2, 3), (3, 2)$

1. d **2.** b **3.** a **4.** c **5.** f **6.** e

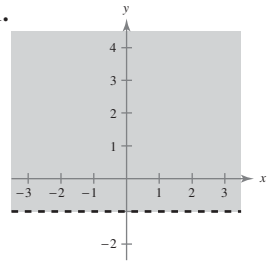
7.



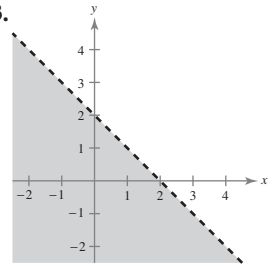
9.



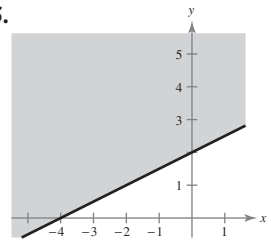
11.



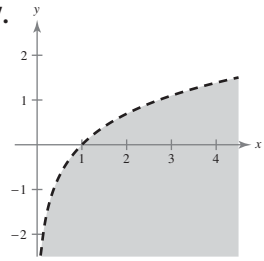
13.



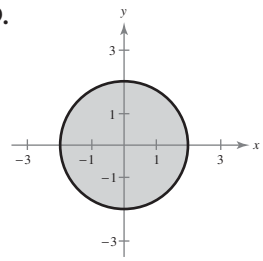
15.



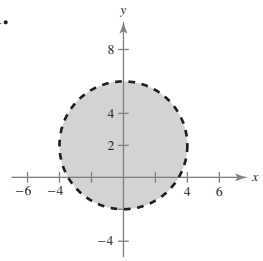
17.



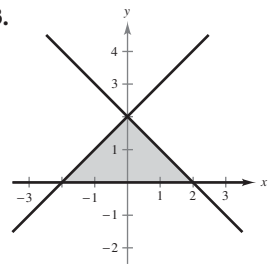
19.



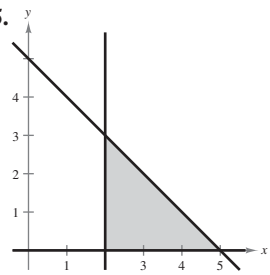
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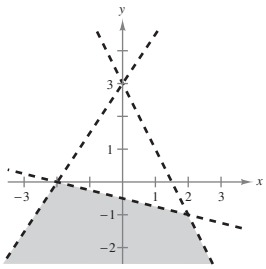
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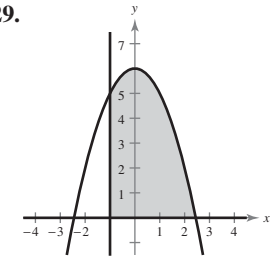
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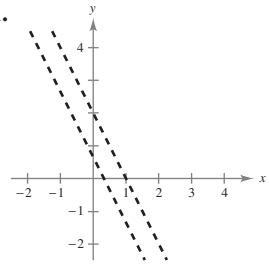
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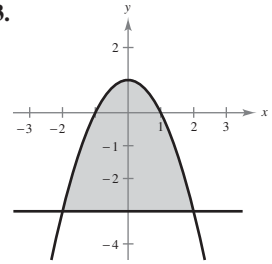
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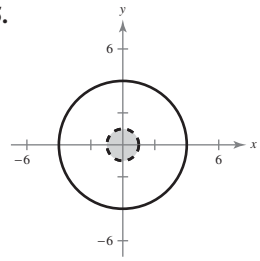
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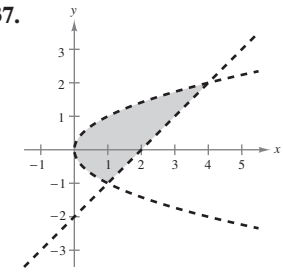
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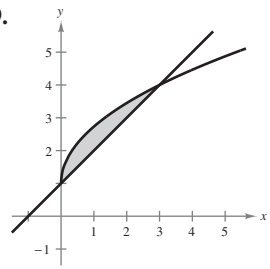
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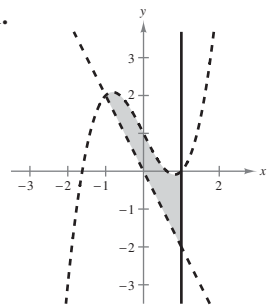
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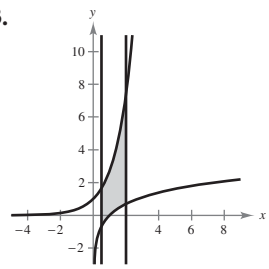
39.



41.



43.



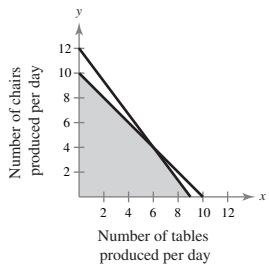
45. $0 \le x \le 8, 0 \le y \le 6$

47. $y \le \frac{4}{3}x, y \ge 0, y \le -4x + 16$

49. $x^2 + y^2 \le 16, x \ge 0, y \ge 0$

51. (a) $2x + \frac{3}{2}y \le 18, \frac{3}{2}x + \frac{3}{2}y \le 15, x \ge 0, y \ge 0$

(b)



53. Consumer surplus: $3,777,001.41

Producer surplus: $77,545.60

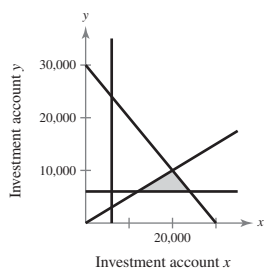
55. Consumer surplus: $40,000,000

Producer surplus: $20,000,000

57. The consumer surplus and producer surplus are equal when the slope of the demand equation is the negative of the slope of the supply equation.

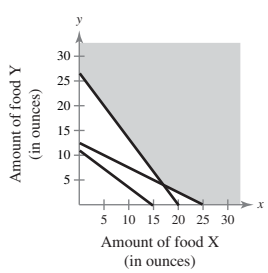
59. (a) $x + y \le 30{,}000$
$x \ge 6000$
$y \ge 6000$
$x - 2y \ge 0$

(b)



61. (a) $20x + 15y \ge 400$
$10x + 20y \ge 250$
$15x + 20y \ge 220$
$x \ge 0$
$y \ge 0$

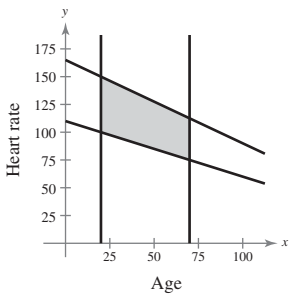
(b)



(c) Answers will vary. Sample answer: The nutritionist could give 10 ounces of food X and 15 ounces of food Y.

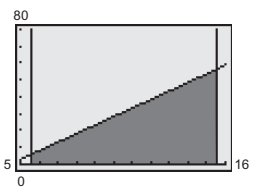
63. (a) $y \ge 0.5(220 - x)$
$y \le 0.75(220 - x)$
$x \ge 20$
$x \le 70$

(b)



(c) Answers will vary.

65. (a)



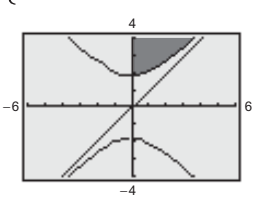
(b) $0.835 billion

67. Answers will vary. Sample answer:

$$\begin{cases} y \le \frac{1}{3}x + 4 \\ y \le -\frac{1}{3}x + 4 \\ y \ge 0 \end{cases}$$

69. (a) $\begin{cases} \pi y^2 - \pi x^2 \ge 10 \\ y > x \\ x > 0 \end{cases}$

(b)

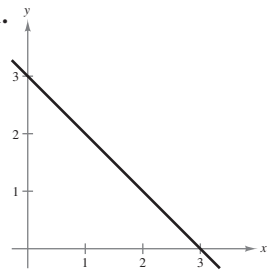


(c) The line is an asymptote to the boundary. The larger the circles, the closer the radii can be and the constraint will still be satisfied.

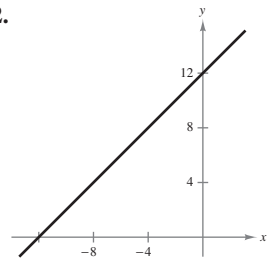
SECTION 5.5 *(page 457)*

Skills Review *(page 457)*

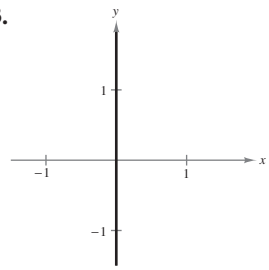
1.



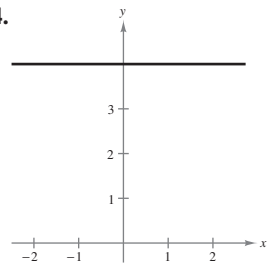
2.



3.

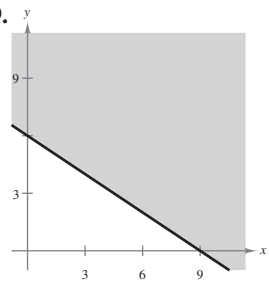


4.

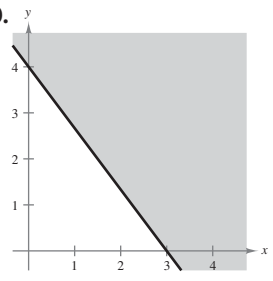


5. (0, 4) **6.** (12, 0) **7.** (3, 1) **8.** (2, 5)

9.



10.



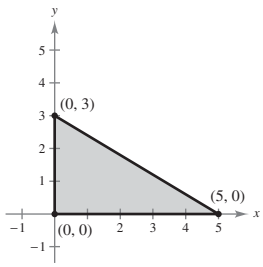
1. Minimum value at (0, 0):0
Maximum value at (6, 0):36

3. Minimum value at (0, 0):0
Maximum value at (6, 0):48

5. Minimum value at (0, 0):0
Maximum value at (3, 4):17

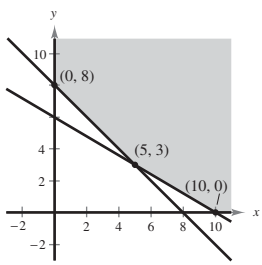
7. Minimum value at (0, 0):0
Maximum value at (4, 0):20

9. Minimum value at (0, 0):0
Maximum value at any point on the line segment connecting the points (5, 0) and (0, 3):30



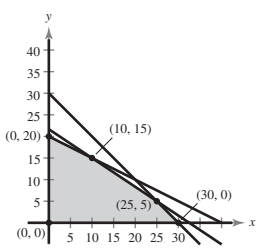
11. Minimum value at (0, 0):0
Maximum value at (5, 0):45
Same graph as in Answer 9

13. Minimum value at (5, 3):35
No maximum value



15. Minimum value at (10, 0):20
No maximum value
Same graph as in Answer 13

17. Minimum value at (0, 0):0
Maximum value at any point on the line segment connecting the points (0, 20) and (10, 15):40



19. Minimum value at (0, 0):0
Maximum value at any point on the line segment connecting the points (25, 5) and (30, 0):30
Same graph as in Answer 17

CHAPTER 5

21. Maximum value at $(3, 6)$: 12

23. Maximum value at any point on the line segment connecting the points $(0, 10)$ and $(3, 6)$: 30

25. Maximum value at $(4, 4)$: 28

27. Maximum value at $(7, 0)$: 84

29. Answers will vary. Sample answer: $z = 2x + 11y$

31. Answers will vary. Sample answer: $z = -x$

33. Answers will vary. Sample answer: $z = 2x + 5y$

35. Answers will vary. Sample answer: $z = 5x + 3y$

37. Crop A: 60 acres
Crop B: 90 acres
\$3,150

39. Brand X: 3 bags
Brand Y: 6 bags
\$40

41. Model A: 0 bicycles
Model B: 1600 bicycles
\$20,000

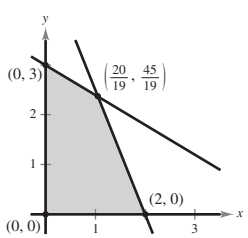
43. 12 audits and 0 tax returns

45. Television: None
Newspaper: \$1,000,000
250 million people

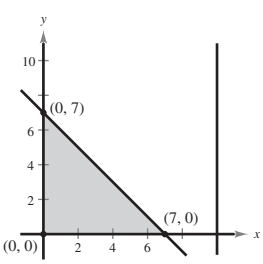
47. Type A: \$2,500
Type B: \$87,500
\$6,875

49. Model A: 929 units
Model B: 77 units
\$99,445

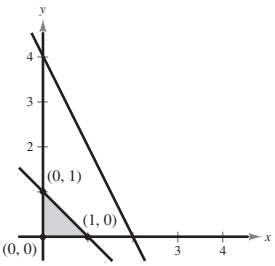
51. z is maximum at any point on the line segment connecting the vertices $(2, 0)$ and $\left(\frac{20}{19}, \frac{45}{19}\right)$.



53. The constraint $x \leq 10$ is extraneous. The maximum value of z occurs at $(0, 7)$.



55. The constraint $2x + y \leq 4$ is extraneous. The maximum value of z occurs at $(0, 1)$.



57. (a) Yes. The point $(1, 12)$ lies on the line segment connecting $(0, 14)$ and $(3, 8)$.

(b) No. The point $(4, 6)$ lies outside the line segment connecting $(0, 14)$ and $(3, 8)$.

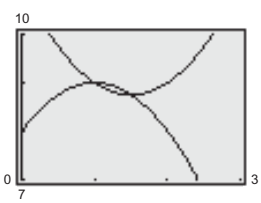
(c) $(2, 10)$

59. Yes. The objective function also has the maximum value at any point on the line segment connecting the two vertices, so there are an infinite number of points that produce the maximum value.

REVIEW EXERCISES *(page 462)*

1. $(-2, 4)$ **3.** $(8, -10)$ **5.** $(8, 6), (0, 10)$

7. $(1, 9), (15, 875)$



9. 800 plants

11. During the fourth month of the new format

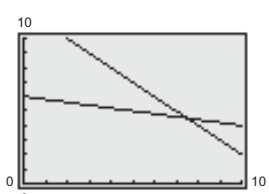
13. $(3, -5)$

15. $\left(a, \frac{4}{3}a - \frac{10}{3}\right)$, where a is any real number

17. $\left(a, \frac{5}{8}a - \frac{7}{4}\right)$, where a is any real number

19. $(8, 9)$ **21.** The graph is a point. Solution: $(-1, 1)$

23. (a) $\begin{cases} 01x + 05y = 025 & (12) \\ x + y = 12 \end{cases}$



(b) 75 gallons of 10% solution
45 gallons of 50% solution

25. $x = 71{,}429$ units

$p = \$271$

27. $(3, 5, 2)$ **29.** $(2, -1, 3)$

31. $\left(\frac{1}{5}a + \frac{8}{5}, -\frac{6}{5}a + \frac{42}{5}, a\right)$, a is any real number

33. Inconsistent **35.** $y = 2x^2 + x - 6$

37. $x^2 + y^2 - 4x + 2y - 4 = 0$

39. \$00,000 in certificates of deposit

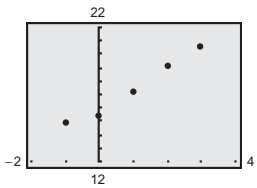
\$00,000 in municipal bonds

\$5,000 in blue-chip stocks

\$85,000 in growth or speculative stocks

41. $y = 101x + 154$

43. (a)



(b) $y = 149t + 159$

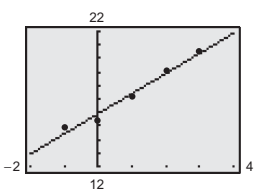
(c) $y = 0150t^2 + 119t + 158$

(d) $y = 149t + 159$

$y = 0150t^2 + 119t + 158$

They are the same as the regression models found in parts (b) and (c).

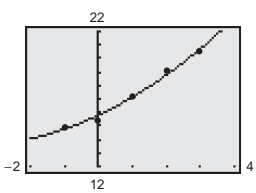
(e) Linear



2006: \$186 billion

2007: \$335 billion

Quadratic

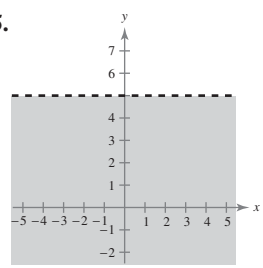


2006: \$296 billion

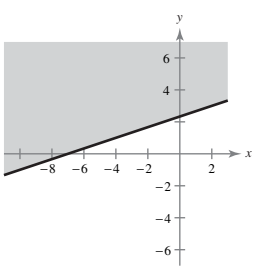
2007: \$550 billion

The predictions given by the quadratic model are greater than the predictions given by the linear model.

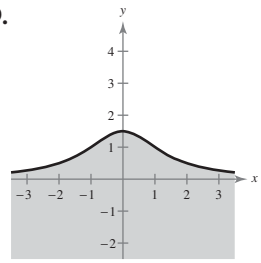
45.



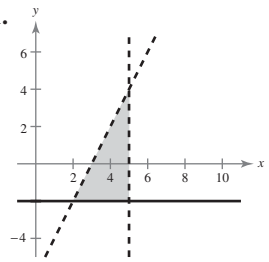
47.



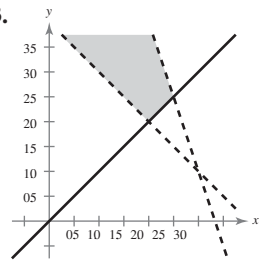
49.



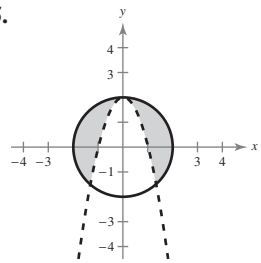
51.



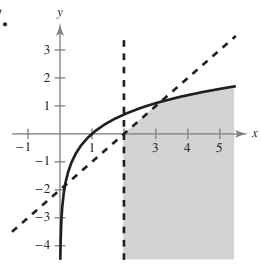
53.



55.



57.



59.

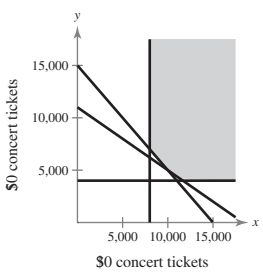
$$\begin{cases} y \le 3x - 5 \\ y \le -x + 11 \\ y \ge \frac{1}{3}x + \frac{1}{3} \end{cases}$$

61. Consumer surplus: \$8,000,000

Producer surplus: \$6,000,000

63.

$$\begin{cases} 30x + 50y \ge 550{,}000 \\ x + y \ge 15{,}000 \\ x \ge 8000 \\ y \ge 4000 \end{cases}$$



65. Minimum value at $(0, 0)$:0

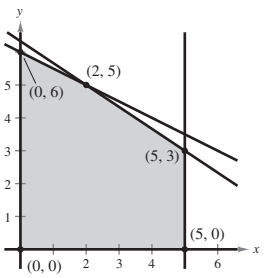
Maximum value at $(3, 3)$:81

67. Minimum value at $(0, 300)$:18,000

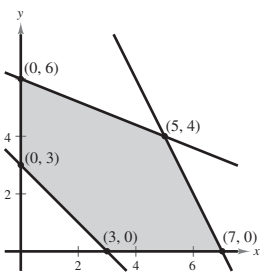
Maximum value at any point on the line segment connecting $(0, 500)$ and $(600, 0)$:30,000

CHAPTER 5

69. Minimum value at (0, 0): 0
Maximum value at (2, 5): 50



71. Minimum value at (3, 0): 30
Maximum value at (5, 4): 94



73. 700 units of the basic model
300 units of the deluxe model
$124,000

75. $270 model: 50 units
$455 model: 50 units
$3750

CHAPTER TEST *(page 466)*

1. (2, 4) **2.** (−2, 5), (3, 0)

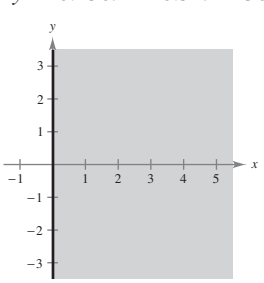
3. (1.68, 2.38), (−4.18, −26.88) **4.** (3.36, −1.32)

5. (2, −1, 3) **6.** (2, −3, 4)

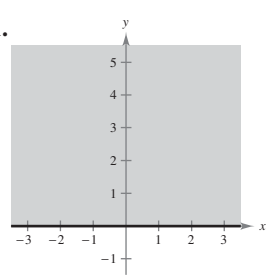
7. $60,000 at 9% **8.** (50,000, 34)
$20,000 at 9.5%

9. $y = 0.450t^2 + 0.51t + 36.8$; 42.38 million

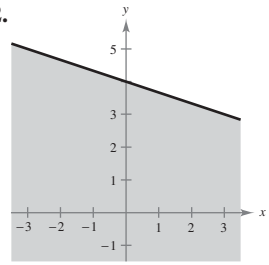
10.



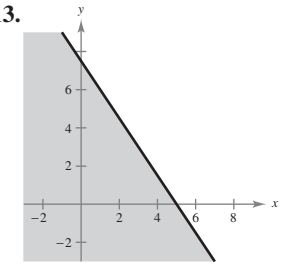
11.



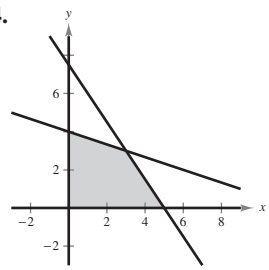
12.



13.



14.



15. Minimum value at (0, 0): 0
Maximum value at (3, 3): 39

16. Model A: 297 units; model B: 570 units
$216,150; The profit model is $P = 200x + 275y$ with the constraints

$$\begin{cases} 3.5x + 8y \le 5600 \\ 2.5x + 2y \le 2000 \\ 1.3x + 0.9y \le 900 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

and the maximum profit occurs at (297, 570).

CHAPTER 6

SECTION 6.1 *(page 478)*

Skills Review *(page 478)*

1. −3 **2.** 30 **3.** 6 **4.** $-\frac{1}{9}$ **5.** Solution
6. Not a solution **7.** (5, 2) **8.** $\left(\frac{12}{5}, -3\right)$
9. (40, 14, 2) **10.** $\left(\frac{15}{2}, 4, 1\right)$

1. 2 × 3 **3.** 4 × 3 **5.** 4 × 2 **7.** 2 × 4

9. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -7 & -1 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 & 0 & 14 & -11 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

13. Add 5 times R_2 to R_1.

15. Interchange R_1 and R_2. Add 4 times R_1 to R_3.

17. Reduced row-echelon form

19. Not in row-echelon form **21.** Not in row-echelon form

23. (a) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -9 \\ 2 & -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -9 \\ 0 & -3 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -9 \\ 0 & 0 & -30 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ **27.** $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **31.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

35. $\begin{cases} 2x + 4y = 6 \\ -x + 3y = -8 \end{cases}$ **37.** $\begin{cases} x + 2z = -10 \\ 3y - z = 5 \\ 4x + 2y = 3 \end{cases}$

39. $\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 5 & 7 & 12 \end{array}\right]$ **41.** $\left[\begin{array}{ccc|c} 1 & 10 & -3 & 2 \\ 5 & -3 & 4 & 0 \\ 2 & 4 & 0 & 6 \end{array}\right]$

43. $\left[\begin{array}{cccc|c} 9 & -3 & 20 & 1 & 13 \\ 12 & 0 & -8 & 0 & 5 \\ 1 & 2 & 3 & -4 & -2 \\ -1 & -1 & 1 & 1 & 1 \end{array}\right]$

45. $\begin{cases} x - 5y = 6 \\ y = -2 \end{cases}$ $(-4, -2)$

47. $\begin{cases} x + 3y - z = 15 \\ y + 4z = -12 \\ z = -5 \end{cases}$ $(-14, 8, -5)$

49. $(-4, 6)$ **51.** $(-4, -8, 2)$

53. $(-2a - 4, -a + 6, a)$ **55.** $(3, 2)$ **57.** $(4, -2)$

59. $\left(\frac{1}{2}, -\frac{3}{4}\right)$ **61.** Inconsistent **63.** $(-6, 8, 2)$

65. $\left(-\frac{3}{2}a + \frac{3}{2}, \frac{1}{3}a + \frac{1}{3}, a\right)$ **67.** Inconsistent

69. $(5a + 4, -3a + 2, a)$ **71.** $(0, 2 - 4a, a)$

73. $(-3b + 96a + 100, b, 52a + 54, a)$

75. $(0, 0)$ **77.** $(-2a, a, a)$ **79.** Yes; $(-1, 1, -3)$

81. No

83. \$1,200,000 was borrowed at 8%, \$200,000 was borrowed at 9%, and \$600,000 was borrowed at 12%.

85. Both are correct. Because there are infinitely many ordered triples that are solutions to this system, a solution can be written in many different ways. If $a = 3$, the ordered triple is $(3, -3, 5)$. You obtain the same triple when $b = 5$.

87. $y = 7.5t + 28$

133 new cases; Because the data values increased in a linear pattern, this estimate seems reasonable.

SECTION 6.2 *(page 492)*

Skills Review *(page 492)*

1. -5 **2.** -7

3. Not in reduced row-echelon form

4. Not in reduced row-echelon form

5. $\left[\begin{array}{cc|c} -5 & 10 & 12 \\ 7 & -3 & 0 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 10 & 15 & -9 & 42 \\ 6 & -5 & 0 & 0 \end{array}\right]$

7. $(0, 2)$ **8.** $(2 + a, 3 - a, a)$

9. $(1 - 2a, a, -1)$ **10.** $(2, -1, -1)$

1. $x = -3, y = 2$ **3.** $x = -2, y = 5$

5. (a) $\begin{bmatrix} 8 & -1 \\ 1 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 5 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 15 & -6 \\ 9 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 9 & -8 \\ 13 & -9 \end{bmatrix}$

7. (a) $\begin{bmatrix} 7 & 3 \\ 1 & 9 \\ -2 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -5 \\ 3 & -1 \\ -4 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 18 & -3 \\ 6 & 12 \\ -9 & 15 \end{bmatrix}$ (d) $\begin{bmatrix} 16 & -11 \\ 8 & 2 \\ -11 & -5 \end{bmatrix}$

9. (a) $\begin{bmatrix} 3 & 3 & -2 \\ -2 & 5 & 7 \\ 1 & -8 & 11 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 4 & -3 & -11 \\ 1 & 6 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & 6 & -3 \\ 3 & 3 & -6 \\ 3 & -3 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 4 & -1 \\ 9 & -5 & -24 \\ 3 & 11 & -7 \end{bmatrix}$

11. $\begin{bmatrix} -8 & -7 \\ 15 & -1 \end{bmatrix}$ **13.** $\begin{bmatrix} -24 & -4 & 12 \\ -12 & 32 & 12 \end{bmatrix}$

15. $\begin{bmatrix} 10 & 8 \\ -59 & 9 \end{bmatrix}$ **17.** $\begin{bmatrix} -17.143 & 2.143 \\ 11.571 & 10.286 \end{bmatrix}$

19. $\begin{bmatrix} -1.581 & -3.739 \\ -4.252 & -13.249 \\ 9.713 & -0.362 \end{bmatrix}$ **21.** $\begin{bmatrix} -6 & -9 \\ -1 & 0 \\ 17 & -10 \end{bmatrix}$

23. $\begin{bmatrix} 3 & 3 \\ -\frac{1}{2} & 0 \\ -\frac{13}{2} & \frac{11}{2} \end{bmatrix}$ **25.** $\begin{bmatrix} -7 & 10 & -12 & 17 \\ 6 & 21 & 7 & 15 \\ -3 & 3 & -5 & 6 \end{bmatrix}$

27. $\begin{bmatrix} -1 & 19 \\ 4 & -27 \\ 0 & 14 \end{bmatrix}$ **29.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix}$

31. Not possible

33. $\begin{bmatrix} 41 & 7 & 7 \\ 42 & 5 & 25 \\ -10 & -25 & 45 \end{bmatrix}$ **35.** $\begin{bmatrix} 151 & 25 & 48 \\ 516 & 279 & 387 \\ 47 & -20 & 87 \end{bmatrix}$

37. Not possible

39. (a) $\begin{bmatrix} 0 & 15 \\ 6 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 2 \\ 31 & 14 \end{bmatrix}$ (c) $\begin{bmatrix} 9 & 6 \\ 12 & 12 \end{bmatrix}$

41. (a) $\begin{bmatrix} 3 & 5 \\ 1 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 11 & 5 & 0 \\ -7 & -4 & -1 \\ 14 & 8 & 2 \end{bmatrix}$

(c) Not possible

43. (a) $[11]$ (b) $\begin{bmatrix} -4 & 2 & 3 \\ 0 & 0 & 0 \\ -20 & 10 & 15 \end{bmatrix}$ (c) Not possible

45. (a) $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

47. (a) $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

49. (a) $\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & -1 \\ 2 & -5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 17 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

51. $\begin{bmatrix} 110 & 132 & 66 & 44 \\ 154 & 176 & 220 & 88 \end{bmatrix}$

53.

Hotel x	Hotel y	Hotel z	
97.75	105.80	126.50	Single
115.00	138.00	149.50	Double
126.50	149.50	161.00	Triple
126.50	161.00	178.25	Quadruple

Occupancy

55. (a) \$19,550 (b) \$21,450

(c)

	Wholesale	Retail	
$ST =$	\$15,850	\$19,550	1
	\$26,350	\$30,975	2
	\$21,450	\$25,850	3

Outlet

ST represents the wholesale and retail prices of the computer inventories at the three outlets.

57. (a) $B = [2 \quad 0.5 \quad 3]$

(b)

	Calories burned	
	120-lb person	150-lb person

$BA = [473.5 \quad 588.5]$

BA represents the total calories burned by each person.

59. Cannot perform operation.

61. Cannot perform operation. **63.** 2×2 **65.** 2×3

67. (a)

	X	Y	Z	
$D =$	3	1	2	Sacks
	0	2	3	Interceptions
	4	5	3	Key tackles

Each entry d_{ij} represents the number of each type of defensive play made by each player.

(b)

	Sacks	Interceptions	Tackles
$B =$	\$2000	\$1000	\$800

Each entry b_{ij} represents the bonus received for each type of play.

(c)

	X	Y	Z
$BD =$	\$9200	\$8000	\$9400

Each entry represents the bonus each player will receive.

(d) Player Z

69. $\begin{bmatrix} 0.40 & 0.15 & 0.15 \\ 0.28 & 0.53 & 0.17 \\ 0.32 & 0.32 & 0.68 \end{bmatrix}$

71. $\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$

Matrix is unique.

73. $AC = \begin{bmatrix} 12 & -6 & 9 \\ 16 & -8 & 12 \\ 4 & -2 & 3 \end{bmatrix} = BC$, but $A \neq B$.

75. True. $\begin{bmatrix} 3 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -4 \end{bmatrix}$ and

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -4 \end{bmatrix}$

77. (a) Gold subscribers: 28,750

Galaxy subscribers: 35,750

Nonsubscribers: 35,500

Multiply the original matrix by the 3×1 matrix

$\begin{bmatrix} 25{,}000 \\ 30{,}000 \\ 45{,}000 \end{bmatrix}$

which represents the current numbers of subscribers for each company and the number of nonsubscribers.

(b) Gold subscribers: 30,813

Galaxy subscribers: 39,675

Nonsubscribers: 29,513

Multiply the original matrix by the 3×1 matrix

$$\begin{bmatrix} 28{,}750 \\ 35{,}750 \\ 35{,}500 \end{bmatrix}$$

which represents the numbers of subscribers for each company and the number of nonsubscribers after 1 year.

(c) Gold subscribers: 31,947

Galaxy subscribers: 42,330

Nonsubscribers: 25,724

Multiply the original matrix by the 3×1 matrix

$$\begin{bmatrix} 30{,}813 \\ 39{,}675 \\ 29{,}513 \end{bmatrix}$$

which represents the numbers of subscribers for each company and the number of nonsubscribers after 2 years.

(d) The number of subscribers to each company is increasing each year. The number of nonsubscribers is decreasing each year.

SECTION 6.3 *(page 503)*

Skills Review *(page 503)*

1. $\begin{bmatrix} 4 & 24 \\ 0 & -16 \\ 48 & 8 \end{bmatrix}$ **2.** $\begin{bmatrix} \frac{11}{2} & 5 & 24 \\ \frac{1}{2} & 0 & 8 \\ 0 & 1 & 4 \end{bmatrix}$

3. $\begin{bmatrix} -5 & -2 & -13 \\ 4 & -13 & -2 \end{bmatrix}$ **4.** $\begin{bmatrix} -13 & 11 \\ -19 & 21 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **6.** $\begin{bmatrix} 6 & 5 \\ 3 & -2 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **9.** $\left[\begin{array}{cc:cc} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & -3 \end{array}\right]$

10. $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -6 & -4 & 3 \\ 0 & 1 & 0 & 11 & 6 & -5 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right]$

1–9. $AB = I$ and $BA = I$ **11.** $\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 0 & -1 \\ 1 & 11 \end{bmatrix}$ **15.** $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -1 \end{bmatrix}$ **17.** $\begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{4} & 0 \end{bmatrix}$

19. Does not exist

21. $\begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}$ **23.** $\frac{1}{2}\begin{bmatrix} -3 & 3 & 2 \\ 9 & -7 & -6 \\ -2 & 2 & 2 \end{bmatrix}$

25. $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ **27.** $\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{4} & 0 \\ \frac{7}{20} & -\frac{1}{4} & \frac{1}{5} \end{bmatrix}$

29. Does not exist

31. $\begin{bmatrix} -175 & 37 & -13 \\ 95 & -20 & 7 \\ 14 & -3 & 1 \end{bmatrix}$ **33.** $\frac{1}{11}\begin{bmatrix} 0 & -20 & 10 \\ -110 & 55 & 55 \\ 110 & -30 & -40 \end{bmatrix}$

35. Does not exist

37. $\begin{bmatrix} -24 & 7 & 1 & -2 \\ -10 & 3 & 0 & -1 \\ -29 & 7 & 3 & -2 \\ 12 & -3 & -1 & 1 \end{bmatrix}$ **39.** $\frac{1}{19}\begin{bmatrix} 3 & 2 \\ -2 & 5 \end{bmatrix}$

41. Does not exist

43. $\frac{1}{59}\begin{bmatrix} 16 & 15 \\ -4 & 70 \end{bmatrix}$ **45.** $(-2, 1)$ **47.** $(4, 2)$

49. $(-2, 3)$ **51.** $(2, 0)$ **53.** $(3, 8, -11)$

55. $(2, 1, 0, 0)$ **57.** $(2, -2)$ **59.** Inconsistent

61. $(-4, -8)$ **63.** $(-1, 3, 2)$ **65.** $(5, 0, -2, 3)$

67. $\begin{cases} 2x + y + 3z = 16 \\ 4x - 2z = -2 \\ 3y + 2z = 1 \end{cases}$

69. AAA bonds: \$20,000
A bonds: \$5000
B bonds: \$10,000

71. AAA bonds: \$21,000
A bonds: \$5000
B bonds: \$10,000

73. $I_1 = 4$ amperes, $I_2 = 1$ ampere, $I_3 = 5$ amperes

75. 100 bags of potting soil for seedlings
100 bags of potting soil for general potting
100 bags of potting soil for hardwood plants

77. 5 bags of potting soil for seedlings
100 bags of potting soil for general potting
120 bags of potting soil for hardwood plants

79. (a) $\begin{bmatrix} 99.28 & -17.66 & 0.76 \\ -17.66 & 3.17 & -0.14 \\ 0.76 & -0.14 & 0.006 \end{bmatrix}$

$y = -0.08t^2 + 3.1t - 5$

(b) 24.58 billion

(c) The estimates are close and both seem reasonable.

81. $AB = \begin{bmatrix} 13 & 4 \\ 1 & 8 \end{bmatrix}$, $BA = \begin{bmatrix} 8 & 1 \\ 4 & 13 \end{bmatrix}$

Row 1 of AB is Row 2 of BA with reversed entries.

Row 2 of AB is Row 1 of BA with reversed entries.

83. Answers will vary. Sample answer:

If $k = 3$, then

$$\begin{bmatrix} 4 & 3 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix}.$$

If $k = -\frac{3}{2}$, the matrix is singular.

85. True. The inverse of I_n is I_n.

87. Answers will vary.

MID-CHAPTER QUIZ *(page 507)*

1. Any matrix with four rows and three columns

2. Any matrix with three rows and one column

3. $\left[\begin{array}{ccc|c} 3 & 2 & \vdots & -2 \\ 5 & -1 & \vdots & 19 \end{array}\right]$ **4.** $\left[\begin{array}{ccc|c} 1 & 0 & 3 & -5 \\ 1 & 2 & -1 & 3 \\ 3 & 0 & 4 & 0 \end{array}\right]$

5. $(2.769, -5.154)$ **6.** $(4, -2, -3)$ **7.** $\begin{bmatrix} -2 & 2 \\ 5 & 15 \end{bmatrix}$

8. $\begin{bmatrix} 9 & -2 & 23 \\ 14 & -4 & 36 \end{bmatrix}$ **9.** $\begin{bmatrix} -1 & 10 \\ -11 & 3 \end{bmatrix}$

10. $\begin{bmatrix} -6 & -2 \\ 3 & -5 \end{bmatrix}$ **11.** $\begin{bmatrix} 3 & -2 \\ 1 & -\frac{1}{2} \end{bmatrix}$ **12.** Not possible

13. $\begin{bmatrix} -3 & 16 \\ -12 & 16 \end{bmatrix}$ **14.** $\begin{bmatrix} 2 & -10 \\ 7 & -11 \end{bmatrix}$

15. \$22.80 **16.** \$41.40 **17.** \$59.60

18.

	Plant 1	Plant 2	
$LW =$	\$22.80	\$20.20	A
	\$47.80	\$41.40	B } Model
	\$66.50	\$59.60	C

LW represents the total labor costs for each model at each plant.

19. $(4, -2)$ **20.** $(4, 2, -3)$

SECTION 6.4 *(page 515)*

Skills Review *(page 515)*

1. $\begin{bmatrix} 3 & 5 \\ 4 & 0 \end{bmatrix}$ **2.** $\begin{bmatrix} -2 & 8 \\ 2 & -4 \end{bmatrix}$

3. $\begin{bmatrix} 9 & -12 & 6 \\ 3 & 0 & -3 \\ 0 & 3 & -6 \end{bmatrix}$ **4.** $\begin{bmatrix} 0 & 8 & 12 \\ -4 & 8 & 12 \\ -8 & 4 & -8 \end{bmatrix}$

5. -22 **6.** 35 **7.** -15 **8.** $-\frac{1}{8}$

9. -45 **10.** -16

1. -5 **3.** 1 **5.** 3 **7.** 0 **9.** 5 **11.** 4

13. $\frac{11}{6}$ **15.** 0.14 **17.** -0.838 **19.** 248

21. (a) $M_{11} = -5, M_{12} = 2, M_{21} = 4, M_{22} = 3$

(b) $C_{11} = -5, C_{12} = -2, C_{21} = -4, C_{22} = 3$

23. (a) $M_{11} = -4, M_{12} = -2, M_{21} = 1, M_{22} = 3$

(b) $C_{11} = -4, C_{12} = 2, C_{21} = -1, C_{22} = 3$

25. (a) $M_{11} = 3, M_{12} = -4, M_{13} = 1, M_{21} = 2, M_{22} = 2, M_{23} = -4, M_{31} = -4, M_{32} = 10, M_{33} = 8$

(b) $C_{11} = 3, C_{12} = 4, C_{13} = 1, C_{21} = -2, C_{22} = 2, C_{23} = 4, C_{31} = -4, C_{32} = -10, C_{33} = 8$

27. (a) $M_{11} = 30, M_{12} = 12, M_{13} = 11, M_{21} = -36, M_{22} = 26, M_{23} = 7, M_{31} = -4, M_{32} = -42, M_{33} = 12$

(b) $C_{11} = 30, C_{12} = -12, C_{13} = 11, C_{21} = 36, C_{22} = 26, C_{23} = -7, C_{31} = -4, C_{32} = 42, C_{33} = 12$

29. (a) -99 (b) -99 **31.** (a) -145 (b) -145

33. (a) 170 (b) 170 **35.** -58 **37.** -30 **39.** 0

41. 0 **43.** -0.002 **45.** 0 **47.** -108 **49.** 0

51. 412 **53.** -126 **55.** 0 **57.** -336 **59.** 410

61. -6 **63.** -16 **65.** 120 **67.** 40 **69.** -168

71. -20 **73.** -18

75. (a) -3 (b) -2 (c) $\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ (d) 6

77. (a) -8 (b) 0 (c) $\begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}$ (d) 0

79. (a) -21 (b) -19 (c) $\begin{bmatrix} 7 & 1 & 4 \\ -8 & 9 & -3 \\ 7 & -3 & 9 \end{bmatrix}$ (d) 399

81. (a) 2 (b) -6 (c) $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ (d) -12

83. Matrices will vary. The determinant of each matrix is the product of the entries on the main diagonal, which in this case equals -18.

85. Matrices will vary. The determinant of each matrix is the product of the entries on the main diagonal, which in this case equals 28.

87. Rows 2 and 4 are identical.

89. Row 4 is a multiple of Row 2.

91. True. If an entire row is zeros, then each cofactor in the expansion is multiplied by zero.

93–95. Answers will vary.

SECTION 6.5 *(page 524)*

Skills Review *(page 524)*

1. 1 **2.** 0 **3.** -8 **4.** x^2 **5.** 8 **6.** 60

7. $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 2 \\ 2 & 5 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 0.14 \\ 0.31 \end{bmatrix}$ **10.** $[7 \;\; 14]$

1. 11 **3.** 28 **5.** $\frac{33}{8}$ **7.** $\frac{5}{2}$ **9.** 28

11. $y = \frac{16}{5}$ or $y = 0$ **13.** $y = -3$ or $y = -11$

15. 307.5 square miles **17.** Collinear

19. Not collinear **21.** Collinear **23.** Not collinear

25. $y = -4$ **27.** $x = 3$ **29.** $x - 6y + 13 = 0$

31. $x + 3y - 5 = 0$ **33.** $x = -4$

35. $2x + 3y - 8 = 0$

37. $[3 \;\; 15][13 \;\; 5][0 \;\; 8][15 \;\; 13][5 \;\; 0][19 \;\; 15][15 \;\; 14]$
48, 81, 28, 51, 24, 40, 54, 95, 5, 10, 64, 113, 57, 100

39. $[3 \;\; 1 \;\; 12][12 \;\; 0 \;\; 13][5 \;\; 0 \;\; 20][15 \;\; 13 \;\; 15]$
$[18 \;\; 18 \;\; 15][23 \;\; 0 \;\; 0]$
$-68, 21, 35, -66, 14, 39, -115, 35, 60, -62, 15, 32,$
$-54, 12, 27, 23, -23, 0$

41. $1, -25, -65, 17, 15, -9, -12, -62, -119,$
$27, 51, 48, 43, 67, 48, 57, 111, 117$

43. $-5, -41, -87, 91, 207, 257, 11, -5, -41,$
$40, 80, 84, 76, 177, 227$

45. 34, 55, 43, 20, 35, 28, 19, 36, 33, 16, 24, 12, 56, 107, 111

47. HAPPY NEW YEAR **49.** SOUND ALL CLEAR

51. SEND MORE MONEY

53. MEET ME TONIGHT RON

55. Because $[45 \;\; -35]\begin{bmatrix} w & x \\ y & z \end{bmatrix} = [10 \;\; 15]$
and $[38 \;\; -30]\begin{bmatrix} w & x \\ y & z \end{bmatrix} = [8 \;\; 14]$,
you can solve $\begin{bmatrix} 45 & -35 \\ 38 & -30 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 8 & 14 \end{bmatrix}$;
JOHN RETURN TO BASE

57. Answers will vary.

REVIEW EXERCISES *(page 529)*

1. 2×4 **3.** $\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. $(3, -2)$ **9.** $(3, 2, -1)$ **11.** $(10 - 4a, a, a)$

13. Inconsistent

15. \$40,000 was borrowed at 8%, \$120,000 was borrowed at 10%, and \$40,000 was borrowed at 12%.

17. (a) $\begin{bmatrix} 3 & 7 \\ -4 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & 3 \\ 8 & -2 \end{bmatrix}$
(c) $\begin{bmatrix} -4 & 20 \\ 8 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -16 & 14 \\ 26 & -5 \end{bmatrix}$

19. (a) $\begin{bmatrix} 3 & 4 & 2 & 1 \\ 3 & -5 & 6 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 & -6 & 11 \\ -3 & 7 & 0 & 4 \end{bmatrix}$
(c) $\begin{bmatrix} 4 & 12 & -8 & 24 \\ 0 & 4 & 12 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 9 & -20 & 39 \\ -9 & 22 & 3 & 14 \end{bmatrix}$

21. $\begin{bmatrix} 4 & -11 \\ -3 & 1 \\ -1 & -3 \end{bmatrix}$ **23.** $\frac{1}{2}\begin{bmatrix} 3 & -5 \\ 1 & 4 \\ 9 & 14 \end{bmatrix}$ **25.** $\begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **29.** Not possible

31. (a) $[4]$ (b) $\begin{bmatrix} 2 & -6 & 8 \\ -2 & 6 & -8 \\ -1 & 3 & -4 \end{bmatrix}$ (c) Not possible

33. $\begin{bmatrix} 96 & 144 & 24 & 48 \\ 48 & 72 & 96 & 24 \\ 168 & 72 & 120 & 96 \end{bmatrix}$

35. (a) \$8325 (b) \$5200

(c)
$$ST = \begin{array}{c} \\ \text{Outlet} \end{array}\begin{array}{c} \begin{matrix} \text{Wholesale} & \text{Retail} \end{matrix} \\ \begin{bmatrix} \$5200 & \$8265 \\ \$5075 & \$7985 \\ \$5125 & \$8325 \end{bmatrix} \end{array}\begin{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{matrix}$$

ST represents the wholesale and retail values of the car sound system inventory at each outlet.

37. $AB = I$ and $BA = I$

39. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ **41.** $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ **43.** $(3, 4)$

45. $\left(2, \frac{1}{2}, 3\right)$ **47.** $(-6, -1)$ **49.** $(2, -1, -2)$

51. 10 units of fluid X
8 units of fluid Y
5 units of fluid Z

CHAPTER 6

53. (a) $\begin{bmatrix} \frac{23}{5} & -\frac{33}{10} & \frac{1}{2} \\ -\frac{33}{10} & \frac{187}{70} & -\frac{3}{7} \\ \frac{1}{2} & -\frac{3}{7} & \frac{1}{14} \end{bmatrix}$

$y = 0.129t^2 - 1.29t + 4.4$

(b) 4.4% (c) The estimate is too high.

55. 4 **57.** 0

59. (a) $M_{11} = 4, M_{12} = 7, M_{21} = -1, M_{22} = 2$

(b) $C_{11} = 4, C_{12} = -7, C_{21} = 1, C_{22} = 2$

61. (a) $M_{11} = 30, M_{12} = -12, M_{13} = -21,$
$M_{21} = 20, M_{22} = 19, M_{23} = 22,$
$M_{31} = 5, M_{32} = -2, M_{33} = 19$

(b) $C_{11} = 30, C_{12} = 12, C_{13} = -21,$
$C_{21} = -20, C_{22} = 19, C_{23} = -22,$
$C_{31} = 5, C_{32} = 2, C_{33} = 19$

63. 44; Answers will vary **65.** -12; Answers will vary.

67. -39; Answers will vary. **69.** 10 **71.** 10

73. Not collinear **75.** Collinear

77. $x + 15y - 38 = 0$ **79.** $x = 2$

81. [20 18][1 14][19 13][9 20][0 14][15 23]

94, 132, 44, 59, 77, 109, 78, 107, 42, 56, 99, 137

83. SEIZE THE DAY

85. (a) Because $\begin{bmatrix} -57 & -13 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 23 & 5 \end{bmatrix}$ and

$\begin{bmatrix} 91 & 26 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 13 \end{bmatrix}$, you can solve

$\begin{bmatrix} -57 & -13 \\ 91 & 26 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 23 & 5 \\ 0 & 13 \end{bmatrix}$.

(b) WE MISS YOU BIG BOB

CHAPTER TEST *(page 533)*

1. $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 2 \\ 1 & 4 & -1 & 0 \\ -1 & 3 & 3 & -1 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 3 & 4 & 2 & 4 \\ 2 & 3 & 0 & -2 \\ 0 & 2 & -3 & -13 \end{array}\right]$

3. $\left(\frac{7}{3}a - \frac{10}{3}, -\frac{8}{3}a + \frac{29}{3}, a\right)$ **4.** $(-12, -16, -6)$

5. $(2, -3, 1)$ **6.** $\begin{bmatrix} 2 & 4 \\ 7 & 13 \end{bmatrix}$ **7.** $\begin{bmatrix} -4 & -8 \\ 13 & 29 \end{bmatrix}$

8. $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$ **9.** $\begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix}$ **10.** $\frac{1}{5}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **12.** $\frac{1}{5}\begin{bmatrix} -9 & 16 & -6 \\ 6 & -9 & 4 \\ 4 & -6 & 1 \end{bmatrix}$ **13.** -17

14. -23 **15.** -20 **16.** $(2, -2, 3)$

17. Matrices will vary. Sample answer:

$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

18. $\frac{21}{2}$ square units **19.** Collinear

20. $-x + y + 3 = 0$

21. $BA = [\$384{,}000 \quad \$631{,}000]$

BA represents the total value of each product at each warehouse.

CHAPTER 7

SECTION 7.1 *(page 542)*

Skills Review *(page 542)*

1. $\frac{4}{5}$ **2.** $\frac{1}{3}$ **3.** $(2n + 1)(2n - 1)$

4. $(2n - 1)(2n - 3)$ **5.** $(n - 1)(n - 2)$

6. $(n + 1)(n + 2)$ **7.** $\frac{1}{3}$ **8.** 24

9. $\frac{13}{24}$ **10.** $\frac{3}{4}$

1. $-5, -3, -1, 1, 3$ **3.** 2, 4, 8, 16, 32

5. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ **7.** $\frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}$ **9.** $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}$

11. $\frac{5}{3}, \frac{17}{9}, \frac{53}{27}, \frac{161}{81}, \frac{485}{243}$ **13.** $3, \frac{9}{2}, \frac{9}{2}, \frac{27}{8}, \frac{81}{40}$ **15.** 0, 1, 2, 3, 4

17. $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}$ **19.** 3, 6, 9, 12, 15

21. $5, \frac{19}{4}, \frac{43}{9}, \frac{77}{16}, \frac{121}{25}$ **23.** -73 **25.** $\frac{4}{14{,}175}$

27. $\frac{44}{239}$ **29.** a **30.** d **31.** c **32.** b

33. 56 **35.** 19,656 **37.** $n + 1$; 1, 2

39. $\frac{1}{2n(2n + 1)}$; undefined, $\frac{1}{6}$ **41.** $a_n = 3n - 2$

43. $a_n = (-1)^{n+1}(2n)$ **45.** $a_n = \frac{2}{n^2}$ **47.** $a_n = \frac{n + 1}{n + 2}$

49. $a_n = \frac{(-1)^{n+1}}{2^n}$ **51.** $a_n = 1 + \frac{1}{n}$ **53.** $a_n = \frac{1}{n!}$

55. 72 **57.** -84 **59.** 48 **61.** 30 **63.** $\frac{29}{40}$

65. 14 **67.** 4 **69.** $\frac{65}{24}$ **71.** $\frac{205{,}771}{33{,}264}$

73. $\sum_{i=1}^{9} \frac{1}{3i}$ **75.** $\sum_{i=1}^{8} \left[2\left(\frac{i}{8}\right) + 3\right]$ **77.** $\sum_{i=1}^{6} (-1)^{i+1}3^i$

79. $\sum_{i=1}^{20} \frac{(-1)^{i+1}}{i^2}$ **81.** $\sum_{i=1}^{5} \frac{2^i - 1}{2^{i+1}}$

83. $\sum_{j=0}^{4} j^2 = 30 = \sum_{k=2}^{6} (k-2)^2$; No

85. (a) $A_1 = \$25{,}145.83$

$A_2 = \$25{,}292.52$

$A_3 = \$25{,}440.06$

$A_4 = \$25{,}588.46$

$A_5 = \$25{,}737.72$

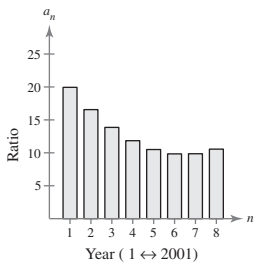
$A_6 = \$25{,}887.86$

(b) \$35,440.63 (c) No; $A_{120} = \$50{,}241.53$

87. (a) $a_1 = 19.94$, $a_2 = 16.56$, $a_3 = 13.86$, $a_4 = 11.84$, $a_5 = 10.50$, $a_6 = 9.84$, $a_7 = 9.86$, $a_8 = 10.56$

Each term represents the number of alligators divided by the number of pythons in the marshland for the corresponding year.

(b)



The ratio decreased until 2006, and increased thereafter.

(c) 731 alligators, 69 pythons

89. \$11.318 billion **91.** True by the properties of sums

93. $x, \frac{x^2}{2}, \frac{x^3}{6}, \frac{x^4}{24}, \frac{x^5}{120}$

95. $-\frac{x^2}{2}, \frac{x^4}{24}, -\frac{x^6}{720}, \frac{x^8}{40{,}320}, -\frac{x^{10}}{3{,}628{,}800}$

SECTION 7.2 *(page 551)*

Skills Review *(page 551)*

1. 36 **2.** 240 **3.** $\frac{11}{2}$ **4.** $\frac{10}{3}$ **5.** 18

6. 4 **7.** 143 **8.** 160 **9.** 430 **10.** 256

1. Arithmetic sequence; $d = 4$

3. Not an arithmetic sequence

5. Arithmetic sequence; $d = -\frac{1}{2}$

7. Not an arithmetic sequence

9. Arithmetic sequence; $d = 1$

11. Arithmetic sequence; $d = 0.4$

13. 8, 11, 14, 17, 20; Arithmetic sequence, $d = 3$

15. 2, 8, 24, 64, 160; Not an arithmetic sequence

17. 1, 1, 1, 1, 1; Arithmetic sequence, $d = 0$

19. Undefined, 2, 1, $\frac{2}{3}$, $\frac{1}{2}$; Not an arithmetic sequence

21. $-1, 1, -1, 1, -1$; Not an arithmetic sequence

23. $a_n = 5n - 1$ **25.** $a_n = -12n + 62$

27. $a_n = 5n + 2$ **29.** $a_n = 4n - 4$ **31.** $a_n = 6n - 4$

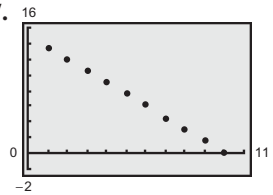
33. $a_n = \frac{3}{4}n + \frac{5}{4}$ **35.** 10, 6, 2, −2, −6

37. 5, 14, 23, 32, 41 **39.** 2, 6, 10, 14, 18

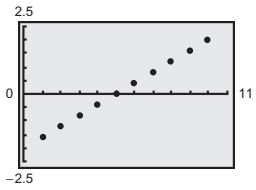
41. 54, 50, 46, 42, 38

43. d **44.** a **45.** b **46.** c

47.



49.



51. 216 **53.** 12 **55.** 23 **57.** 4000

59. 10,000 **61.** 1275 **63.** 22,770

65. 44,625 **67.** 670 **69.** 2725

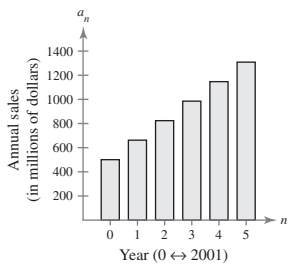
71. Answers will vary. Sample answer: $\sum_{n=1}^{100} 3n$

73. Answers will vary. Sample answer: $\sum_{n=1}^{10} \frac{n+1}{4}$

75. (a) \$43,750 (b) \$236,250

77. 2625 seats **79.** 405 bricks

81. (a)



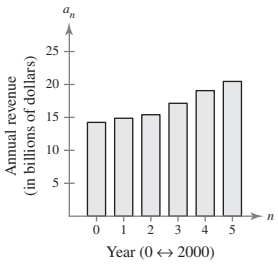
(b) \$5427 million

83. 1024 feet **85.** \$350,000; Answers will vary.

87. Yes; $a_n = a_1 - d + dn$

CHAPTER 7

89. (a)



(b) $a_n = 1.2973n + 13.621$, $n = 0, 1, \ldots, 5$

(c) $\sum_{n=0}^{5} (1.2973n + 13.621)$

Total revenue: \$101.1855 billion

SECTION 7.3 *(page 560)*

Skills Review *(page 560)*

1. $\frac{64}{125}$ **2.** $\frac{9}{16}$ **3.** $\frac{1}{16}$ **4.** 324

5. $6n^3$ **6.** $27n^4$ **7.** $4n^3$ **8.** n^2

9. $\sum_{n=1}^{4} 2(3)^{n-1}$ **10.** $\sum_{n=1}^{4} 3(2)^{n-1}$

1. 5 **3.** $\frac{1}{6}$ **5.** Geometric sequence; $r = 4$; $a_n = 4^{n-1}$

7. Geometric sequence; $r = -3$; $a_n = 4(-3)^{n-1}$

9. Geometric sequence; $r = -\frac{1}{2}$; $a_n = \left(-\frac{1}{2}\right)^{n-1}$

11. Not a geometric sequence

13. Not a geometric sequence

15. 4, 8, 16, 32, 64 **17.** $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

19. $\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}, -\frac{1}{54}, \frac{1}{162}$ **21.** $1, e, e^2, e^3, e^4$

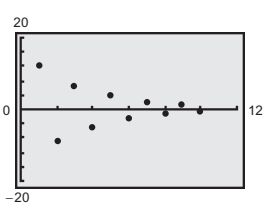
23. $16\left(\frac{1}{4}\right)^4 = \frac{1}{16}$ **25.** $6\left(-\frac{1}{3}\right)^{11} = -\frac{2}{3^{10}}$

27. $100e^8$ **29.** $500(1.02)^{39} \approx 1082.37$

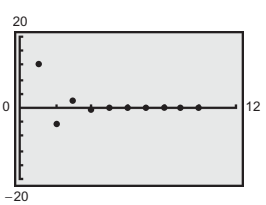
31. $-40\left(\frac{1}{2}\right) = -20$ or $-40\left(-\frac{1}{2}\right) = 20$ **33.** $54\left(-\frac{1}{3}\right)^5 = -\frac{2}{9}$

35. a **36.** c **37.** b **38.** d

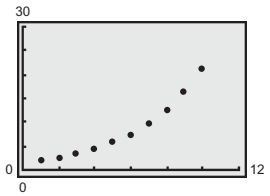
39.



41.



43.



45. 16,368 **47.** 511 **49.** $\frac{349{,}525}{32{,}768} \approx 10.67$

51. 43 **53.** $\frac{8}{5}$ **55.** $\approx 29{,}921.31$

57. $\approx -9.5 \times 10^{15}$ **59.** $\sum_{n=1}^{7} 5(3)^{n-1}$

61. $\sum_{n=1}^{7} 2\left(-\frac{1}{4}\right)^{n-1}$ **63.** $\sum_{n=1}^{6} 0.1(4)^{n-1}$

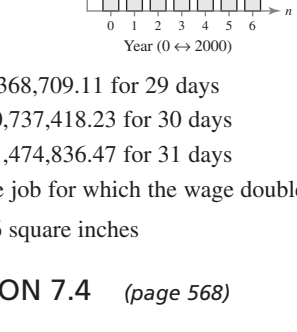
65. 2 **67.** $\frac{2}{3}$ **69.** Sum does not exist; $|r| \geq 1$

71. $\frac{10}{3}$ **73.** $\frac{4}{11}$ **75.** $\frac{7}{22}$

77. (a) No, $S_{48} = \$16{,}310.50$. (b) Yes, $S_{48} = \$17{,}385.63$.

79. \$18,416.57; \$18,444.46 **81.** \$3984.05 million

83. (a) (b) \$26.050 billion



85. \$5,368,709.11 for 29 days
\$10,737,418.23 for 30 days
\$21,474,836.47 for 31 days
The job for which the wage doubles daily pays more.

87. 126 square inches

SECTION 7.4 *(page 568)*

Skills Review *(page 568)*

1. $5x^5 + 15x^2$ **2.** $x^3 + 5x^2 - 3x - 15$

3. $x^2 + 8x + 16$ **4.** $4x^2 - 12x + 9$ **5.** $\frac{3x^3}{y}$

6. $-32z^5$ **7.** 120 **8.** 336 **9.** 720 **10.** 20

1. 15 **3.** 210 **5.** 792 **7.** 15,504 **9.** 1

11. 4950 **13.** 6 **15.** 35 **17.** 56

19. $x^4 + 4x^3 + 6x^2 + 4x + 1$ **21.** $x^3 + 6x^2 + 12x + 8$

23. $x^4 - 8x^3 + 24x^2 - 32x + 16$

25. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

27. $81s^4 + 216s^3 + 216s^2 + 96s + 16$

29. $64s^6 - 576s^5 + 2160s^4 - 4320s^3 + 4860s^2 - 2916s + 729$

31. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

33. $1 - 6x + 12x^2 - 8x^3$

35. $x^{10} - 5x^8y^2 + 10x^6y^4 - 10x^4y^6 + 5x^2y^8 - y^{10}$

37. $\frac{1}{x^5} + \frac{5y}{x^4} + \frac{10y^2}{x^3} + \frac{10y^3}{x^2} + \frac{5y^4}{x} + y^5$

39. The absolute values of corresponding numerical coefficients are the same. The terms of $(x - y)^n$ alternate in sign.

41. -4 **43.** $-2035 + 828i$ **45.** 1

47. $32t^5 - 80t^4 + 80t^3 - 40t^2 + 10t - 1$

49. $81 + 216z + 216z^2 + 96z^3 + 16z^4$

51. $120x^7y^3$ **53.** $360x^3y^2$ **55.** $1{,}259{,}712x^2y^7$

57. $\frac{12!}{4!8!}(10x)^4(-3y)^8 \approx -3.248 \times 10^{10}x^4y^8$

59. 8064 **61.** 180 **63.** 489,888 **65.** -56

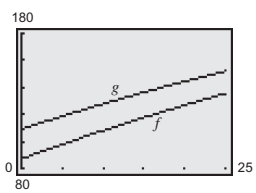
67. $\frac{1}{128} + \frac{7}{128} + \frac{21}{128} + \frac{35}{128} + \frac{35}{128} + \frac{21}{128} + \frac{7}{128} + \frac{1}{128}$

69. $0.07776 + 0.2592 + 0.3456 + 0.2304 + 0.0768 + 0.01024$

71. Fibonacci sequence

73. (a) $g(t) = -0.011t^2 + 1.99t + 108$

(b)



75. Answers will vary.

MID-CHAPTER QUIZ *(page 570)*

1. $-1, 1, 3, 5, 7$ **2.** $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}$ **3.** 20

4. 24,360 **5.** $a_n = \frac{3}{n^2}$ **6.** $a_n = \frac{1}{n+1}$

7. $a_n = 5n - 3$ **8.** $a_n = 3n + 4$

9. $a_n = 10(3)^{n-1}$ **10.** $a_n = 4\left(-\frac{1}{2}\right)^{n-1}$

11. Arithmetic; Common difference is 3.

12. Neither; No common ratio, no common difference

13. Geometric; Common ratio is $\frac{1}{2}$.

14. Geometric; Common ratio is 2.

15. 45 **16.** 60 **17.** 1225 **18.** 126

19. $\frac{3}{4}$ **20.** $\frac{25}{4}$ **21.** (a) 28 (b) 1

22. $x^4 - 20x^3 + 150x^2 - 500x + 625$

23. $32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$

24. $45x^4y^8$ **25.** (a) \$38,750 (b) \$213,750

26. \$59,294.72

SECTION 7.5 *(page 578)*

Skills Review *(page 578)*

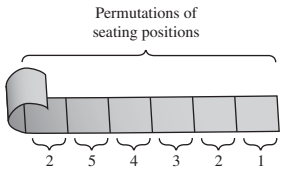
1. 6656 **2.** 291,600 **3.** 7920 **4.** 13,800

5. 792 **6.** 2300 **7.** $n(n-1)(n-2)(n-3)$

8. $n(n-1)(2n-1)$ **9.** $n!$ **10.** $n!$

1. 8 **3.** 6 **5.** 3 **7.** 11 **9.** 30 **11.** 15

13. 240



15. 6,760,000 **17.** 1024

19. (a) 900 (b) 648 (c) 180

21. A: 1800; B: 1575; A **23.** 64,000 **25.** 120

27. (a) 40,320 (b) 384 **29.** 24 **31.** 4 **33.** 336

35. 56 **37.** 11,880 **39.** 495 **41.** $n(n-1)(n-2)$

43. $n!$ **45.** 32,760 **47.** 142,506 **49.** 7,059,052

51. 120 **53.** 1260 **55.** 2520 **57.** 3,921,225

59. 70 **61.** 3744

63. Because order matters, use permutations.

65. Because order does not matter, use combinations.

67. 292,600 **69.** 5 **71.** 20

73. ${}_{10}P_6 > {}_{10}C_6$. Changing the order of any of the six elements selected results in a different permutation but the same combination.

75. Answers will vary.

SECTION 7.6 *(page 589)*

Skills Review *(page 589)*

1. $\frac{9}{16}$ **2.** $\frac{8}{15}$ **3.** $\frac{1}{6}$ **4.** $\frac{1}{80{,}730}$

5. $\frac{1}{804{,}722{,}688{,}000}$ **6.** $\frac{1}{24}$ **7.** $\frac{1}{12}$ **8.** $\frac{135}{323}$

9. 0.366 **10.** 0.997

1. $S = \{H1, T1, H2, T2, H3, T3, H4, T4, H5, T5, H6, T6\}$

3. $S = \{ABC, ACB, BAC, BCA, CAB, CBA\}$

5. $S = \{AB, AC, AD, AE, BC, BD, BE, CD, CE, DE\}$

7. $\frac{1}{2}$ **9.** $\frac{3}{10}$ **11.** $\frac{3}{8}$ **13.** $\frac{7}{8}$ **15.** $\frac{1}{9}$ **17.** $\frac{7}{12}$

19. $\frac{1}{3}$ **21.** $\frac{1}{2}$ **23.** $\frac{10}{13}$ **25.** $\frac{6}{13}$ **27.** $\frac{5}{9}$ **29.** $\frac{1}{36}$

31. $\frac{5}{12}$ **33.** 0.7 **35.** 0.9 **37.** 0.22

39. (a) $\frac{139}{198}$ (b) $\frac{59}{198}$ (c) $\frac{1}{33}$

41. (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{9}{100}$ (d) $\frac{1}{30}$ **43.** $\frac{3}{28}$

45. (a) $\frac{5}{13}$ (b) $\frac{1}{2}$ (c) $\frac{4}{13}$ **47.** (a) $\frac{1}{120}$ (b) $\frac{1}{24}$ (c) $\frac{1}{6}$

49. (a) $\frac{3}{8}$ (b) $\frac{19}{30}$ **51.** $\frac{6}{4165}$

53. (a) ≈ 0.922 (b) ≈ 0.078 **55.** 0.82

57. (a) $\frac{1}{81}$ (b) $\frac{16}{81}$ (c) $\frac{65}{81}$ **59.** (a) $\frac{1}{64}$ (b) $\frac{1}{32}$ (c) $\frac{63}{64}$

61. $\frac{7}{32}$ **63.** $\frac{76{,}545}{524{,}288} \approx 0.1459980011$

65. Answers will vary. **67.** 0.0121

69. 0.6561 **71.** 0.226981

73. No. ${}_nC_r \le {}_nP_r \le n!$ but $\frac{n!}{r!}$ may be greater than ${}_nP_r$.

SECTION 7.7 *(page 602)*

Skills Review *(page 602)*

1. 24 **2.** 40 **3.** $\frac{77}{60}$ **4.** $\frac{7}{2}$ **5.** $\frac{2k+5}{5}$

6. $\frac{3k+1}{6}$ **7.** $8 \cdot 2^{2k} = 2^{2k+3}$ **8.** $\frac{1}{9}$

9. $\frac{1}{k}$ **10.** $\frac{4}{5}$

1. $\frac{5}{(k+1)(k+2)}$ **3.** $\frac{(k+1)^2 k^2}{4}$

5–17. Answers will vary. **19.** 210

21. 91 **23.** 979 **25.** 70 **27.** -3402

29. $S_n = n(2n-1)$ **31.** $S_n = 10 - 10\left(\frac{9}{10}\right)^n$

33. $S_n = \frac{n}{2(n+1)}$

35. $n = 1$: No, $n = 2$: Yes, $n = 3$: Yes, $n = 4$: Yes

37. $n = 1$: No, $n = 2$: No, $n = 3$: Yes, $n = 4$: Yes

39. $n = 1$: No, $n = 2$: Yes, $n = 3$: Yes, $n = 4$: Yes

41–51. Answers will vary.

53. Answers will vary. See page 594.

55. Linear model: $a_n = 8n - 3$

57. Quadratic model: $a_n = 3n^2 + 3$

59. Quadratic model: $a_n = n^2 - 3$

61. 0, 3, 6, 9, 12

First differences: 3, 3, 3, 3

Second differences: 0, 0, 0

Linear model: $a_n = 3n - 3$

63. 3, 1, -2, -6, -11

First differences: -2, -3, -4, -5

Second differences: -1, -1, -1

Quadratic model: $a_n = -\frac{1}{2}n^2 - \frac{1}{2}n + 4$

65. 0, 1, 3, 6, 10

First differences: 1, 2, 3, 4

Second differences: 1, 1, 1

Quadratic model: $a_n = \frac{1}{2}n^2 + \frac{1}{2}n$

67. 2, 4, 6, 8, 10

First differences: 2, 2, 2, 2

Second differences: 0, 0, 0

Linear model: $a_n = 2n$

69. 1, 2, 6, 15, 31

First differences: 1, 4, 9, 16

Second differences: 3, 5, 7

Neither linear nor quadratic

71. $a_n = n^2 - n + 3$

73. $a_n = n^2 + 5n - 6$

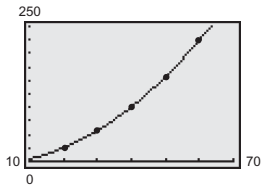
75. (a) First differences: 32, 44, 57, 69

Second differences: 12, 13, 12

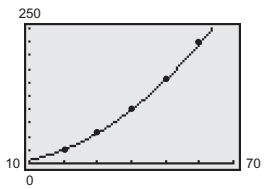
(b) Quadratic model: $d_n = \frac{3}{50}n^2 + \frac{1}{5}n - 3$

(c) $d_n = 0.06n^2 + 0.1n - 1$

(d) Model from part (b)



Model from part (c)



Both models are good fits for the actual data.

(e) No. For there to be an equal risk for each 10 mile-per-hour increase in speed, the data would have to follow a linear pattern.

REVIEW EXERCISES *(page 606)*

1. 5, 9, 13, 17, 21 **3.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$ **5.** $1, -\frac{4}{3}, 2, -\frac{16}{5}, \frac{16}{3}$

7. $a_n = 2^{n-1}$ **9.** $a_n = \dfrac{1}{n}$ **11.** $a_n = (-1)^{n+1}3^n$

13. 43,680 **15.** $\frac{1}{14}$ **17.** 51 **19.** 100

21. $\displaystyle\sum_{k=1}^{12} \frac{3}{1+k}$ **23.** $\displaystyle\sum_{k=1}^{13} (-1)^{k-1}2^k$

25. (a) $A_1 = \$4060.00, A_2 = \$4120.90, A_3 = \$4182.71,$
$A_4 = \$4245.45, A_5 = \$4309.14, A_6 = \$4373.77,$
$A_7 = \$4439.38, A_8 = \$4505.97$

(b) $A_{40} = \$7256.07$

(c) $A_{120} = \$23,877.29$

27. Arithmetic; $d = 5$ **29.** Arithmetic; $d = -\frac{1}{2}$

31. 9, 14, 19, 24, 29 **33.** $-6, -1, 4, 9, 14$

35. 24, 17, 10, 3, -4 **37.** $a_n = 3n - 1$

39. $a_n = 5n - 3$ **41.** 192 **43.** 765

45. (a) Company A: \$40,500 (b) \$219,000 (c) \$205,500
Company B: \$40,500

47. Geometric; $r = -3$; $a_n = (-3)^{n-1}$

49. Geometric; $r = \frac{1}{2}$; $a_n = 16\left(\frac{1}{2}\right)^{n-1}$

51. 3, 6, 12, 24, 48 **53.** $10, -2, \frac{2}{5}, -\frac{2}{25}, \frac{2}{125}$

55. $a_{40} = \dfrac{1}{2^{36}}$ **57.** $a_{10} = -\dfrac{25}{64}$ **59.** 8184

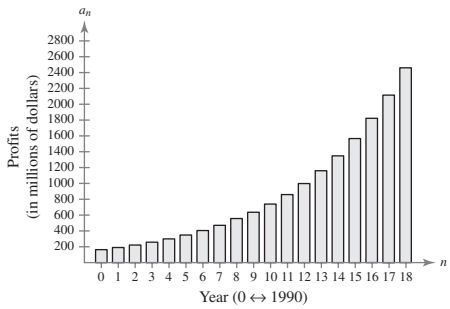
61. ≈ 22.2 **63.** 1.5

65. The sum does not exist; $\left|\frac{5}{4}\right| > 1$

67. (a) \$16,469.87

(b) No; it will be more because the interest is compounded.

69. (a)



(b) \$16,636.59 million

71. 35 **73.** 1 **75.** 792

77. $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

79. $243x^5 + 405x^4y + 270x^3y^2 + 90x^2y^3 + 15xy^4 + y^5$

81. $x^8 - 16x^7y + 112x^6y^2 - 448x^5y^3 + 1120x^4y^4$
$- 1792x^3y^5 + 1792x^2y^6 - 1024xy^7 + 256y^8$

83. $90a^3b^2$ **85.** 25,344 **87.** 210 **89.** 120

91. (a) 7000 (b) 3528 (c) 4500

93. 504 **95.** 40,320 **97.** 1,663,200

99. (a) 42,504 (b) 1,961,256 (c) 134,596

101. $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **103.** $\frac{3}{13}$

105. (a) $\frac{5}{12}$ (b) $\frac{7}{12}$ (c) $\frac{5}{36}$

107. (a) $\frac{1}{120}$ (b) $\frac{1}{6}$ **109.** $\frac{396}{4165}$

111. (a) $\frac{1}{128}$ (b) $\frac{1}{64}$ (c) $\frac{127}{128}$ **113.** 0.43

115–117. Answers will vary. **119.** 465 **121.** 1296

123. $S_n = \frac{5}{2}\left[1 - \left(\frac{3}{5}\right)^n\right]$ **125.** Neither

127. 2, 3, 5, 8, 12

n:	0	1	2	3	4
a_n:	2	3	5	8	12
1st differences:		1	2	3	4
2nd differences:			1	1	1

Quadratic model: $a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 2$

CHAPTER TEST *(page 610)*

1. 3, 5, 7, 9, 11 **2.** $-1, 4, -9, 16, -25$

3. 2, 6, 24, 120, 720 **4.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

5. Arithmetic; $d = 5$ **6.** Neither

7. Geometric; $r = 2$ **8.** Neither **9.** 2600

10. $\approx 29{,}918.311$ **11.** $\frac{1}{2}$ **12.** \$13,176.08

13. $32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$

14. $-192{,}456$ **15.** (a) 72 (b) 328,440

16. (a) 330 (b) 8,936,928 **17.** 120

18. 16,777,216 **19.** 45,000 **20.** 40,320

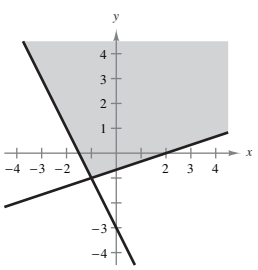
21. 220 **22.** $\frac{1}{216}$ **23.** 45%

24. Answers will vary.

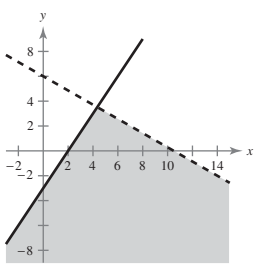
CUMULATIVE TEST: CHAPTERS 5–7 *(page 611)*

1. $x = 2, y = -3, z = 5$ **2.** $x = -2, y = 4, z = -5$

3.



4.



5. $\begin{bmatrix} 1 & 4 & 8 \\ -1 & 4 & 3 \\ 0 & 1 & 5 \end{bmatrix}$ **6.** $\begin{bmatrix} 16 & 9 \\ 8 & 7 \\ 7 & 2 \end{bmatrix}$ **7.** $\begin{bmatrix} 2 & 12 \\ -5 & -3 \\ 1 & 9 \end{bmatrix}$

8. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -1 \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$

9. $|A| = 4$, $|B|$ does not exist, $|C| = 17$, $|D| = 6$

10. $x = 3000$
$p = 68$

11. $BA = [170{,}000 \quad 230{,}000 \quad 237{,}000]$
\$170,000 is the value of the inventory in warehouse 1.
\$230,000 is the value of the inventory in warehouse 2.
\$237,000 is the value of the inventory in warehouse 3.

12. 4, 11, 18, 25, 32 **13.** $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}$ **14.** $\frac{130}{9}$

15. $64x^6 + 576x^5y + 2160x^4y^2 + 4320x^3y^3 + 4860x^2y^4 + 2961xy^5 + 729y^6$

16. 720 **17.** ≈ 0.9762 **18.** $\frac{1}{4}$

19. 2, 4, 7, 11, 16

n:	1	2	3	4	5
a_n:	2	4	7	11	16
1st differences:		2	3	4	5
2nd differences:			1	1	1

Quadratic model $a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1$

CHECKPOINTS

CHAPTER 0

SECTION 0.1 *(page 5)*

1. $x \geq 7$ denotes all real numbers greater than or equal to 7.

2. (a) $x \geq 5$ (b) $4 < y \leq 11$

3. 12 **4.** (a) $-|-6| = -|6|$ (b) $-|5| < |-5|$

5. 8 **6.** No

SECTION 0.2 *(page 10)*

1. (a) $8, -15x$ (b) $4x^2, -3y, -7$ **2.** 29 **3.** 10

4. (a) Multiplicative Identity Property
(b) Commutative Property of Addition

5. $\frac{11x}{12}$ **6.** $\frac{11}{12}$

7. Scientific calculator: 6 [×] [(] 8 [y^x] 3 [−] 481 [)] [=]
Graphing calculator: 6 [(] 8 [^] 3 [−] 481 [)] [ENTER]

8. 5.87

SECTION 0.3 *(page 21)*

1. 1024 **2.** $\frac{z^{12}}{27x^6}$ **3.** $\frac{7}{5}$ **4.** 3.45×10^{-3}

5. 428,000 **6.** 2000 **7.** 5 **8.** \$7021.36

9. 3.8%

SECTION 0.4 *(page 30)*

1. 2 **2.** $3x^2\sqrt{2x}$ **3.** $3x\sqrt[3]{2x}$ **4.** $\frac{\sqrt[3]{2}}{2}$

5. $3 + \sqrt{3}$ **6.** 9 **7.** 4 **8.** $6\sqrt{x}$ **9.** 2.621

10. 0.388 **11.** No

SECTION 0.5 *(page 40)*

1. $-9x^2 + 3x + 7$; degree 2 **2.** Not a polynomial

3. $2x^2 + 5x + 4$ **4.** $3x^2 - 2x - 1$

5. $x^4 - 2x^3 + 2x^2 - 11x + 4$ **6.** $-x^2 + 9$

7. $x^2 - 8x + 16$ **8.** $x^3 - 9x^2 + 27x - 27$

9. $x^2 + 10x - y^2 + 25$ **10.** \$12,282.98

11. Volume $= 4x^3 - 44x^2 + 120x$

$x = 2$ inches: $V = 96$ cubic inches

$x = 3$ inches: $V = 72$ cubic inches

SECTION 0.6 *(page 48)*

1. $(x + 1)(3x + 1)$ **2.** $x(x + 1)(x - 1)$

3. $4(5 + y)(5 - y)$ **4.** $(x - 6)^2$

5. $(y - 1)(y^2 + y + 1)$ **6.** $(x + 3)(x - 2)$

7. $(2x - 3)(x - 1)$ **8.** $(x + 1)(x^2 + 5)$

9. $(2x - 3)(x + 4)$

SECTION 0.7 *(page 55)*

1. All real numbers except $x = 5$ **2.** $\frac{2(x + 1)}{3}, x \neq 1$

3. $-\frac{3 + x}{2(x + 1)}, x \neq 1$ **4.** $\frac{1}{x + 1}, x \neq 2$

5. $\frac{8}{5}, x \neq -y$ **6.** $\frac{2(6 - x^2)}{3x}$ **7.** $\frac{9}{x}, x \neq 1$

8. $\frac{1}{3}, x \neq 3$

CHAPTER 1

SECTION 1.1 *(page 69)*

1. Identity **2.** 2 **3.** $\frac{11}{3}$

4. Infinitely many solutions **5.** No solution **6.** 4

7. No solution **8.** 6 **9.** 0.794 **10.** 2004 ($t = 4$)

SECTION 1.2 *(page 79)*

1. \$692.31 **2.** $S = 0.05(40{,}000) + 40{,}000$

3. 20% **4.** 1.5% **5.** 15 feet × 45 feet

6. 1.1 hours **7.** 32 feet

8. \$200 was invested at 4% and \$800 was invested at 5%

9. About 1.27 feet

SECTION 1.3 *(page 93)*

1. $-3, 4$ **2.** $0, 2$ **3.** -2 **4.** ± 2 **5.** $-3, 5$

6. 6 feet × 14 feet **7.** 3 seconds **8.** ≈ 8.5 feet

9. 2011 ($t \approx 10.82$)

SECTION 1.4 *(page 105)*

1. One real solution **2.** $-1 \pm \sqrt{3}$ **3.** $\frac{1}{3}$

4. $-0.831, 1.511$ **5.** 2:00 P.M. ($t \approx 1.91$)

6. ≈ 5.1 seconds

SECTION 1.5 *(page 115)*

1. $0, \pm 1$ **2.** $1, \pm\sqrt{2}$ **3.** $\pm 2, \pm 1$ **4.** 12 **5.** 81

6. 1, 3 **7.** 1, 5 **8.** 20 ski club members

9. ≈ 4.5% **10.** 4,010,025 copies

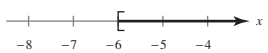
SECTION 1.6 *(page 127)*

1. (a) $2 \leq x < 7$; bounded

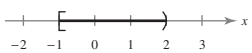
(b) $-\infty < x < 3$; unbounded

2. $x < 1$

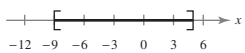
3. $x \geq -6$



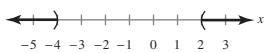
4. $-1 \leq x < 2$



5. $-9 \leq x \leq 5$



6. $x < -4$ or $x > 2$



7. More than 560 miles **8.** At most $12\frac{1}{2}$ weeks

9. Undercharged by as much as \$0.08 or overcharged by as much as \$0.08

SECTION 1.7 *(page 139)*

1. $(-2, 1)$ **2.** $(-\infty, 1) \cup (2, \infty)$

3. (a) The solution set is empty.

(b) The solution set consists of all real numbers except 1.

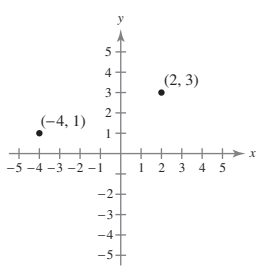
4. $(-\infty, 2] \cup [3, \infty)$ **5.** At least \$35 and at most \$75

6. (a) $[-2, 2]$ (b) $(-\infty, \infty)$

CHAPTER 2

SECTION 2.1 *(page 158)*

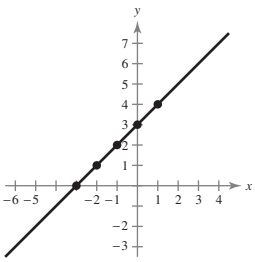
1.



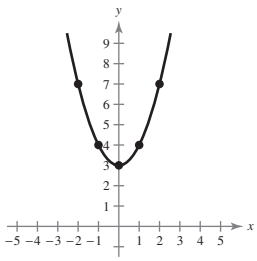
CHECKPOINTS

2. (a) 10 (b) $(2, -1)$ **3.** Yes

4.



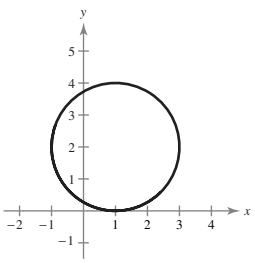
5.



6. $\left(-\frac{2}{3}, 0\right), (0, 2)$ **7.** y-axis symmetry

8. $x^2 + (y - 1)^2 = 25$

9.

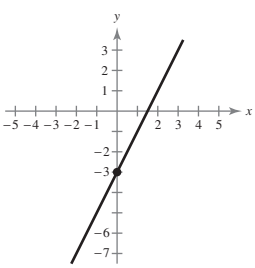


SECTION 2.2 *(page 172)*

1. $\frac{1}{2}$ **2.** (a) $y = -2x + 8$ (b) $y = \frac{3}{2}x + 9$

3. $y = 0.2x + 1$

4.



5. $y = 3x - 2$ **6.** $y = -4x + 4$

SECTION 2.3 *(page 182)*

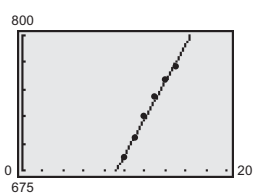
1. The model approximates the weight of the puppy best for $t = 2$ months and worst for $t = 10$ months.

2. $y = 0.06x$ **3.** $y = 33.84615x$ **4.** 5 hours

5. 235,826 people **6.** $V = -195t + 2300$

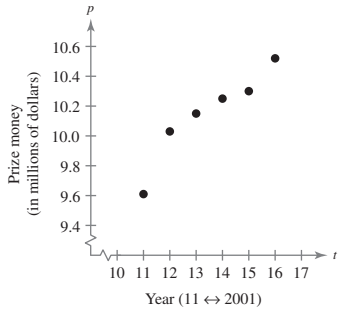
7. (a) $y = 17.11x + 518.2$

(b)



(c) $\approx$ 809,000 employees

8.



$p = 0.156t + 8.037$

SECTION 2.4 *(page 195)*

1. Yes **2.** Yes **3.** -3 **4.** 10, 2

5. All real numbers **6.** $V = 4\pi h^3$ **7.** No

8. 534 cat cadavers

SECTION 2.5 *(page 208)*

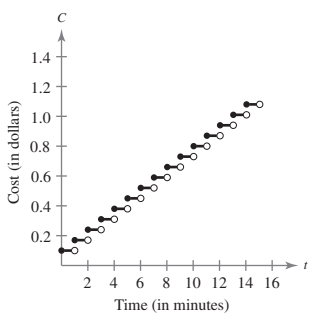
1. Domain: $(-\infty, \infty)$
Range: $[-3, \infty)$

2. Yes

3. Decreasing on $\left(-\infty, -\frac{3}{2}\right)$ and increasing on $\left(-\frac{3}{2}, \infty\right)$

4. (2, 2)

5.

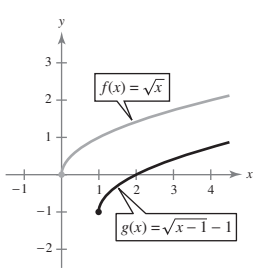


Less than 13 minutes

6. Neither

SECTION 2.6 *(page 220)*

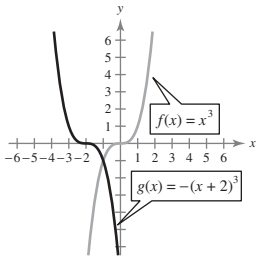
1.



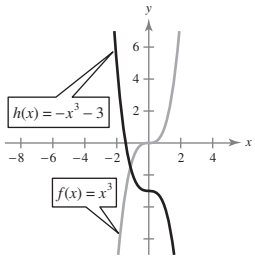
2. (a) The graph of g is a reflection of the graph of f in the x-axis.

(b) The graph of h is a reflection of the graph of f in the y-axis.

3. (a)



(b)

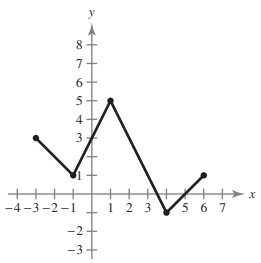


4. $h(x) = -(x - 2)^2 - 1$

5. (a) The graph of g is a vertical stretch of the graph of f by a factor of 4.

(b) The graph of h is a vertical shrink of the graph of f by a factor of $\frac{1}{4}$.

6.



SECTION 2.7 *(page 228)*

1. $2x^2 + x - 1$ **2.** $-x - 7; -10$

3. Domain of $\frac{f}{h}$: All real numbers except $x = 3$

Domain of $\frac{h}{f}$: All real numbers except $x = 1$

4. $x^2 + 2x - 1$ **5.** $\sqrt{3 - x^2}; [-\sqrt{3}, \sqrt{3}]$

6. Answers will vary.

Sample answer: $f(x) = x^2 + 2,\ g(x) = x - 1$

$h(x) = (x - 1)^2 + 2 = f(x - 1) = f(g(x))$

7. f represents the number of Independent senators.

8. About 1 hour and 18 minutes

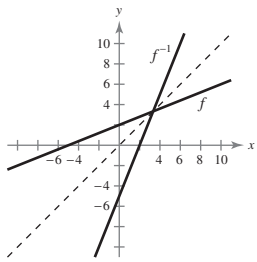
SECTION 2.8 *(page 238)*

1. $f^{-1}(x) = 6x$ **2.** $f^{-1}(x) = x - 10$

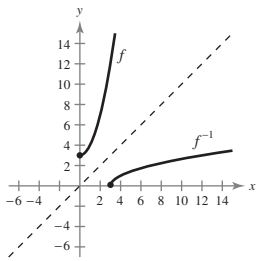
3. $\left(\sqrt[3]{x - 6}\right)^3 + 6 = x;\ \sqrt[3]{(x^3 + 6) - 6} = x$

4. $g(x)$ **5.** $f^{-1}(x) = \dfrac{x - 5}{4}$

6.



7.



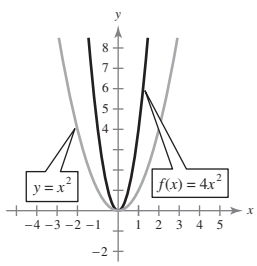
8. (a) f does not have an inverse function.

(b) f has an inverse function.

CHAPTER 3

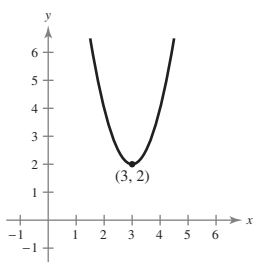
SECTION 3.1 *(page 259)*

1.



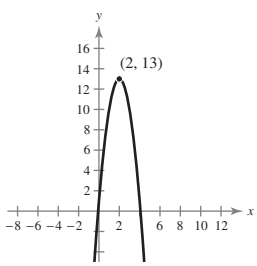
Compared with the graph of $y = x^2$, each output of $f(x) = 4x^2$ vertically stretches the graph by a factor of 4.

2.



Vertex: (3, 2)

3.



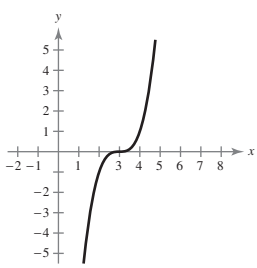
Vertex: (2, 13)

4. $f(x) = x^2 - 6x + 13$

5. $\approx$ 39.7 feet **6.** $\approx$ 342 sparrows **7.** 30 units

SECTION 3.2 *(page 270)*

1.

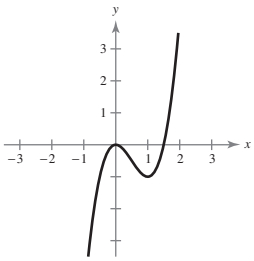


2. Falls to the left
Falls to the right

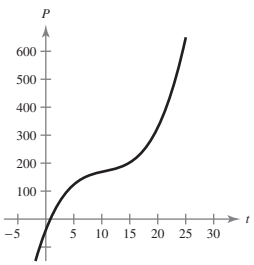
3. Rises to the left
Rises to the right

4. ± 2 **5.** $0, \pm 1$

6.



7.



The median price was about \$195,000 in 2004.

SECTION 3.3 *(page 279)*

1. $(x - 4)(x - 3)(x + 1)$ **2.** $x^2 - 2x + 4$

3. $5x^2 + 13 + \dfrac{2x + 13}{x^2 + 2x - 4}$ **4.** $2x^2 + 3x + 15 - \dfrac{5}{x - 5}$

5. 3

6.

$$\begin{array}{r|rrrrr} -2 & 1 & 6 & 7 & -6 & -8 \\ & & -2 & -8 & 2 & 8 \\ \hline & 1 & 4 & -1 & -4 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -4 & 1 & 4 & -1 & -4 \\ & & -4 & 0 & 4 \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

$f(x) = (x + 2)(x + 4)(x^2 - 1)$
$= (x + 2)(x + 4)(x + 1)(x - 1)$

7. $\approx 10.8\%$

SECTION 3.4 *(page 289)*

1. No rational zeros **2.** $-2, 1$ **3.** $-2, -1, \frac{1}{2}$

4. -2 **5.** The function has a zero between 1.3 and 1.4.

6. -1.290 **7.** $-0.247, 1.445, 2.802$ **8.** $x \approx 1.89$

9. $\approx$ \$289,000

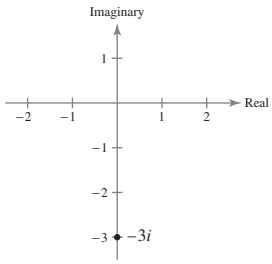
SECTION 3.5 *(page 305)*

1. (a) $5 + i$ (b) $-5i$

2. (a) $-8 + 12i$ (b) $16 - 30i$ **3.** $4 + i$

4. $4 + 3i$ **5.** $-\dfrac{3}{2} \pm \dfrac{\sqrt{7}}{2}i$

6.



7. The complex number -3 is not in the Mandelbrot Set because for $c = -3$, the corresponding Mandelbrot sequence is $-3, 6, 33, 1086, 1, 179, 393, \ldots$, which is unbounded.

SECTION 3.6 *(page 315)*

1. Four zeros: $\pm\sqrt{6}, \pm\sqrt{6}\,i$

2. (a) $\pm 1, \pm 3i$; $f(x) = (x - 1)(x + 1)(x - 3i)(x + 3i)$

(b) $3, 3, -1, i, -i$;
$g(x) = (x - 3)(x - 3)(x + 1)(x - i)(x + i)$

3. Answers will vary. Sample answer: $f(x) = x^4 - 5x^2 - 36$

4. (a) $(x^2 + 4)(x^2 - 3)$ (b) $(x^2 + 4)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$

(c) $(x + 2i)(x - 2i)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$

(d) Two irrational zeros and two imaginary zeros.

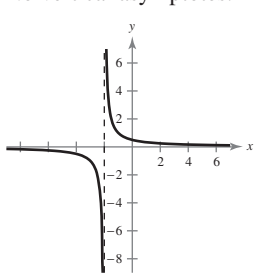
5. $\frac{5}{3}, \pm 4i$

SECTION 3.7 *(page 322)*

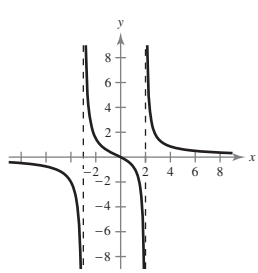
1. The domain of f is all real numbers except $x = 1$. As x approaches 1 from the left, $f(x)$ decreases without bound. As x approaches 1 from the right, $f(x)$ increases without bound.

2. Horizontal asymptote: $y = 1$
No vertical asymptotes.

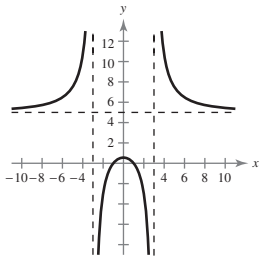
3.



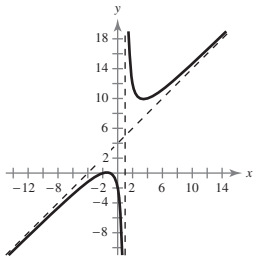
4.



5.



6.



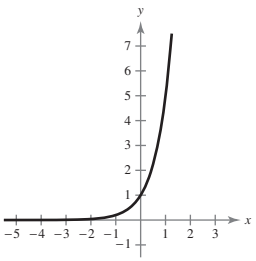
7. $1,066,667 **8.** ≈ 6.9 acres per person

CHAPTER 4

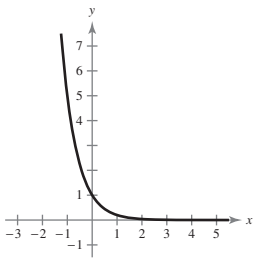
SECTION 4.1 *(page 342)*

1. 4.134

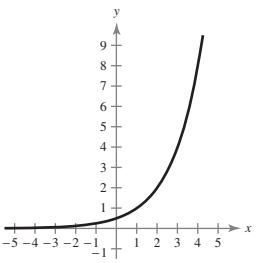
2.



3.

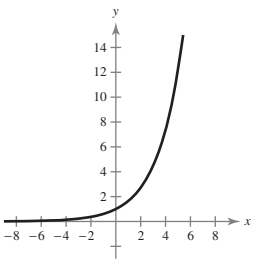


4.



5. 403.4287935

6.

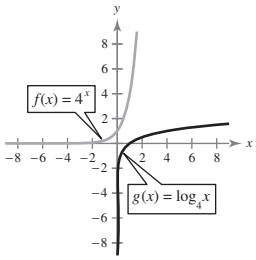


7. $7938.78 **8.** ≈ 9.971 pounds

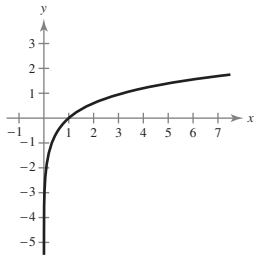
SECTION 4.2 *(page 354)*

1. −2 **2.** 2.301 **3.** 0

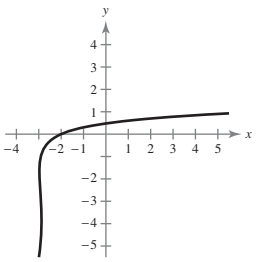
4.



5.



6.



7. (a) 7 (b) 0

8. −2.303 **9.** $(-5, \infty)$

10. 103.0 in., 114.0 in., 128.2 in., 132.8 in., 141.7 in.

SECTION 4.3 *(page 364)*

1. 1.936 **2.** 1.936 **3.** $2 \log_{10} 5 - \log_{10} 3$

4. $-\ln \dfrac{2}{e} = -(\ln 2 - \ln e)$

$= -\ln 2 + \ln e$

$= 1 - \ln 2$

5. $\ln 2 + \ln m + 2 \ln n$ **6.** $\log_{10} \dfrac{(x+1)^2}{(x-1)^3}$

7. $\ln y = \frac{1}{2} \ln x$ **8.** 30 decibels

SECTION 4.4 *(page 373)*

1. (a) 4 (b) 216 **2.** $\log_6 84 \approx 2.473$

3. $\log_{10} 38 \approx 1.580$ **4.** $\frac{1}{2}(7 + \log_4 24) \approx 4.646$

5. $\ln 3 \approx 1.099$, $\ln 4 \approx 1.386$ **6.** $\frac{81}{2}$ **7.** 3

8. $e^4 \approx 54.598$ **9.** 4 **10.** ≈ 2.70 years

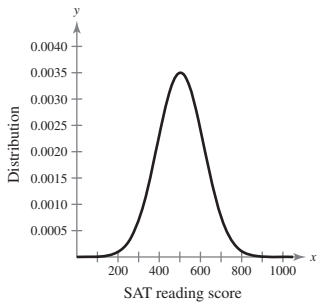
11. 2004 ($t \approx 14.03$)

SECTION 4.5 *(page 384)*

1. 2015 ($t \approx 24.83$) **2.** $y = 3e^{0.19617x}$

3. ≈ 7681 years

4.



Average score: 503

5. 9 days **6.** ≈ 2,511,886

7. 1×10^{-5} mole of hydrogen per liter

8. Logistic growth model

CHAPTER 5

SECTION 5.1 *(page 406)*

1. (5, 1)

2. $11,500 is invested at 9% and $3500 is invested at 11%.

3. $(-4, -7), (2, 5)$ **4.** No solution **5.** (4, 0)

6. ≈ 5455 pairs of shoes **7.** Plan A

SECTION 5.2 *(page 415)*

1. $\left(2, -\frac{1}{3}\right)$ **2.** (3, 2) **3.** $(-3, -3)$ **4.** $(1, -4)$

5. No solution **6.** $(a, -a - 5)$

7. Speed of plane: ≈ 471.18 miles per hour
Speed of wind: ≈ 16.63 miles per hour

8. (130,000, 22)

SECTION 5.3 *(page 427)*

1. (7, 2, 2) **2.** $\left(-\frac{1}{2}, \frac{1}{2}\right)$ **3.** (1, 2, 3)

4. No solution **5.** $\left(-\frac{3}{5}a + \frac{3}{10}, -\frac{9}{5}a + \frac{2}{5}, a\right)$

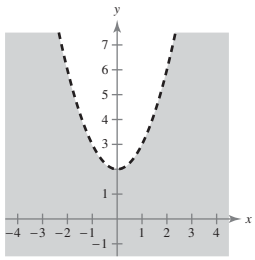
6. $\left(\frac{1}{4}a, \frac{11}{4}a - 1, a\right)$

7. Answers will vary. Sample answer: $55,000 in certificates of deposit, $185,000 in municipal bonds, $105,000 in blue-chip stocks, and $15,000 in growth or speculative stocks

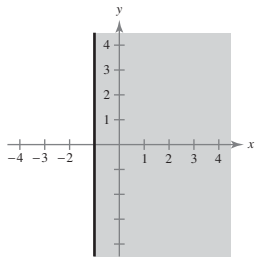
8. $y = 2x^2 - 2x + 3$

SECTION 5.4 *(page 441)*

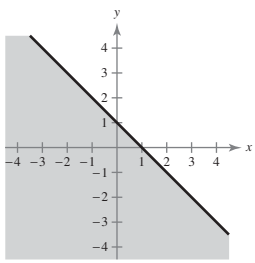
1.



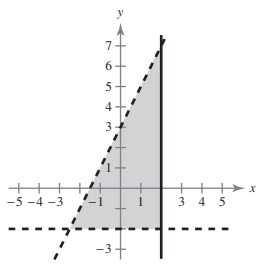
2.



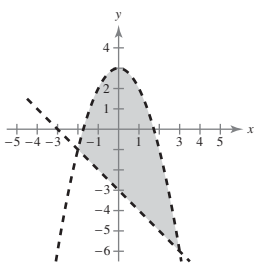
3.



4.



5.



6. Consumer surplus: $845,000
Producer surplus: $845,000

7. No. The combination of 4 cups of dietary drink X and 1 cup of dietary drink Y does not meet all the minimum daily requirements.

SECTION 5.5 *(page 452)*

1. Maximum value at (0, 3): 9

2. (a) Maximum value at (27, 0): 135

(b) Minimum value at (0, 0): 0

3. No maximum value

4. The maximum profit would be $2925, and it would occur at monthly production levels of 1050 units of product I and 150 units of product II.

5. The minimum cost would be $0.56 per day, and it would occur when 1 cup of drink X and 4 cups of drink Y were consumed each day.

CHAPTER 6

SECTION 6.1 *(page 468)*

1. 2×3 **2.** Multiply the second row by $\frac{1}{3}$.

3. $\begin{cases} x - 2y + 5z = 3 \\ y + 4z = -3 \\ z = 2 \end{cases}$

$(-29, -11, 2)$

4. Row-echelon form **5.** $(-1, 0, -1, 3)$

6. No solution **7.** $(6, 5, 5)$

8. It is not the same row-echelon form, but it does yield the same solution found in Example 8.

9. $(-9a - 10, -5a - 2, a)$

SECTION 6.2 *(page 482)*

1. $a_{11} = 5, a_{12} = 2, a_{21} = -1, a_{22} = 3$

2. $\begin{bmatrix} 6 & -2 \\ -2 & -3 \end{bmatrix}$

3. (a) $\begin{bmatrix} 4 & 8 & -2 \\ 0 & 2 & 6 \\ -6 & 4 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & -5 \\ -7 & 6 & 5 \\ -8 & 4 & 12 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 \\ 12 & -4 \end{bmatrix}$ **5.** $\begin{bmatrix} 5 & \frac{1}{2} \\ \frac{1}{2} & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} 8 & -8 \\ 4 & -16 \\ -10 & 4 \end{bmatrix}$

7. $\begin{bmatrix} -3 & -22 \\ 3 & 10 \\ -5 & 10 \end{bmatrix}$ **8.** Not possible

9. $AB = [6]$, $BA = \begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}$

10. $\begin{bmatrix} -2 & -3 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -36 \end{bmatrix}$

$X = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$

11. Company B

SECTION 6.3 *(page 497)*

1. $AB = I$ and $BA = I$

2. $\begin{bmatrix} 1 & 1 \\ -5 & -4 \end{bmatrix}$ **3.** $\begin{bmatrix} -4 & -2 & 5 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$ **5.** $(-1, 2, 1)$

SECTION 6.4 *(page 509)*

1. -7

2. Minors: $M_{11} = -9, M_{12} = -10, M_{13} = 2, M_{21} = 5, M_{22} = -2, M_{23} = -3, M_{31} = 13, M_{32} = 5, M_{33} = -1$

Cofactors: $C_{11} = -9, C_{12} = 10, C_{13} = 2, C_{21} = -5, C_{22} = -2, C_{23} = 3, C_{31} = 13, C_{32} = -5, C_{33} = -1$

3. -32 **4.** -133 **5.** 27 **6.** -30

SECTION 6.5 *(page 518)*

1. 14 **2.** Not collinear **3.** $x - y + 2 = 0$

4. $[15 \quad 23 \quad 12][19 \quad 0 \quad 1][18 \quad 5 \quad 0]$
$[14 \quad 15 \quad 3][20 \quad 21 \quad 18][14 \quad 1 \quad 12]$

5. 110, -39, -59, 25, -21, -3, 23, -18, -5, 47, -20, -24, 149, -56, -75, 87, -38, -37

6. OWLS ARE NOCTURNAL

CHAPTER 7

SECTION 7.1 *(page 535)*

1. 3, 5, 7, 9 **2.** $-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}$ **3.** 6, 3, 1, $\frac{1}{4}, \frac{1}{20}$

4. $\frac{1}{2}, \frac{3}{4}$, 2, $\frac{15}{2}$, 36 **5.** $4(n + 1)$ **6.** 44

7. $\sum_{i=1}^{4} \frac{i - 1}{2}$; Not unique

8. 340 million; The prediction is 4 million greater than the projection.

SECTION 7.2 *(page 545)*

1. (a) Arithmetic sequence; $d = 2$

(b) Not an arithmetic sequence

(c) Arithmetic sequence; $d = -\frac{1}{2}$

2. $a_n = 2n + 3$ **3.** 28 **4.** 43

5. 6, 9, 12, 15, 18, . . . **6.** 175 **7.** 6275

8. 30,100 **9.** 1470 seats **10.** \$147,500

SECTION 7.3 *(page 554)*

1. Geometric; $a_n = 2(4)^{n-1}$ **2.** 2, 6, 18, 54, 162

3. ≈ 104.021 **4.** 7,812,500 **5.** $\frac{2187}{32}$ **6.** 59,048

7. $\frac{25}{3}$ **8.** Yes, $S_{24} = \$1733.52$.

SECTION 7.4 *(page 564)*

1. (a) 10 (b) 1 **2.** (a) 36 (b) 15

3. 84, 126, 36

4. $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

5. $243x^5 + 405x^4 + 270x^3 + 90x^2 + 15x + 1$

6. (a) $12ab^5$ (b) $1500ab^3$

SECTION 7.5 *(page 571)*

1. 3 **2.** 2 **3.** 17,576 **4.** 2,600,000

5. 24 **6.** 120 **7.** 1260 **8.** 10

9. 22,100 **10.** Permutations

SECTION 7.6 *(page 581)*

1. $S = \{AA, AB, AC, BA, BB, BC, CA, CB, CC\}$

2. $\frac{13}{52} = \frac{1}{4}$ **3.** $\frac{1}{12}$ **4.** $\frac{1}{6}$ **5.** $\frac{1}{575{,}757}$

6. ≈ 0.493 **7.** $\frac{4}{13} \approx 0.308$ **8.** $\frac{66}{529} \approx 0.125$

9. $\frac{1}{4}$ **10.** 0.0000293 **11.** 0.139

SECTION 7.7 *(page 594)*

1. $P_{k+1} = S_{k+1} = (k + 1)(2k + 3)$

2. Answers will vary. **3.** Answers will vary.

4. Answers will vary. **5.** 385

6. $S_k = (n + 1)^2 - 1$

7. Quadratic model: $a_n = 2n^2 - n + 2$

8. Linear model: $a_n = 4n + 2$

APPENDIX B

APPENDIX B.1 *(page A17)*

Skills Review *(page A17)*

1. $9x^2 + 16y^2 = 144$ **2.** $x^2 + 4y^2 = 32$

3. $16x^2 - y^2 = 4$ **4.** $243x^2 + 4y^2 = 9$

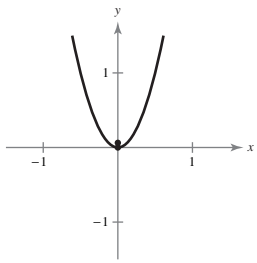
5. $c = 2\sqrt{2}$ **6.** $c = \sqrt{13}$ **7.** $c = 2\sqrt{3}$

8. $c = \sqrt{5}$ **9.** 4 **10.** 2

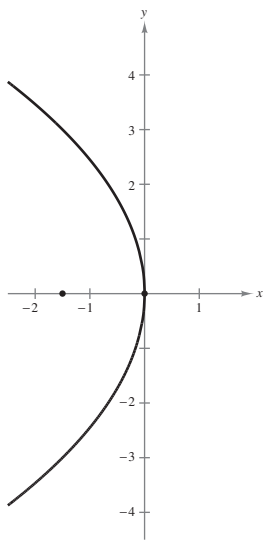
1. a **2.** c **3.** d **4.** h

5. g **6.** f **7.** e **8.** b

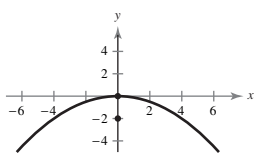
9. Vertex: $(0, 0)$
Focus: $\left(0, \frac{1}{16}\right)$



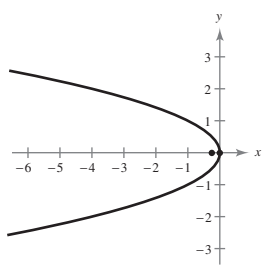
11. Vertex: $(0, 0)$
Focus: $\left(-\frac{3}{2}, 0\right)$



13. Vertex: $(0, 0)$
Focus: $(0, -2)$



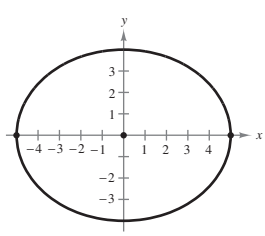
15. Vertex: $(0, 0)$
Focus: $\left(-\frac{1}{4}, 0\right)$



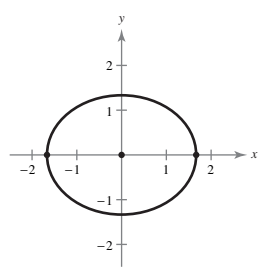
17. $x^2 = -6y$ **19.** $y^2 = -8x$ **21.** $x^2 = 4y$

23. $y^2 = -12x$ **25.** $y^2 = 9x$

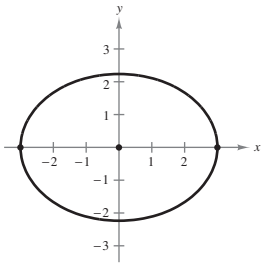
27. Center: $(0, 0)$
Vertices: $(\pm 5, 0)$



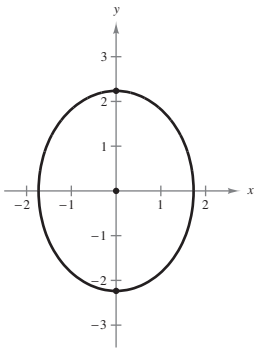
29. Center: $(0, 0)$
Vertices: $\left(\pm\frac{5}{3}, 0\right)$



31. Center: $(0, 0)$
Vertices: $(\pm 3, 0)$



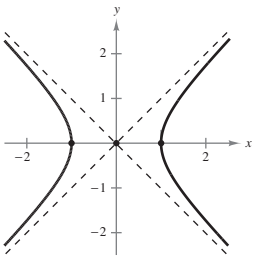
33. Center: $(0, 0)$
Vertices: $\left(0, \pm\sqrt{5}\right)$



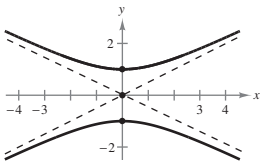
35. $\dfrac{x^2}{1} + \dfrac{y^2}{4} = 1$ **37.** $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$

39. $\dfrac{x^2}{36} + \dfrac{y^2}{11} = 1$ **41.** $\dfrac{21x^2}{400} + \dfrac{y^2}{25} = 1$

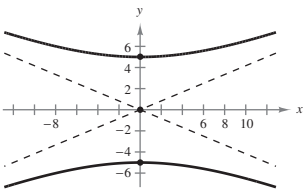
43. Center: $(0, 0)$
Vertices: $(\pm 1, 0)$



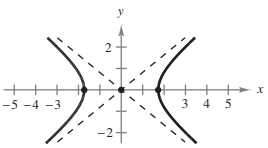
45. Center: $(0, 0)$
Vertices: $(0, \pm 1)$



47. Center: $(0, 0)$
Vertices: $(0, \pm 5)$



49. Center: $(0, 0)$
Vertices: $\left(\pm\sqrt{3}, 0\right)$



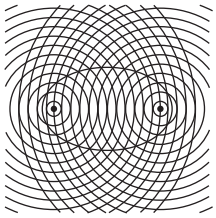
51. $\dfrac{y^2}{4} - \dfrac{x^2}{12} = 1$ **53.** $\dfrac{x^2}{1} - \dfrac{y^2}{9} = 1$

55. $\dfrac{y^2}{16/5} - \dfrac{x^2}{64/5} = 1$ **57.** $\dfrac{y^2}{9} - \dfrac{x^2}{9/4} = 1$

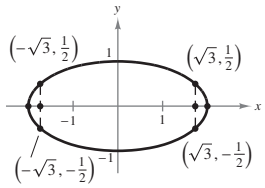
59. $x^2 = 12y$

61. The tacks should be placed $\frac{3}{2}$ feet to the left and right of the center along the base. The string should be 5 feet long.

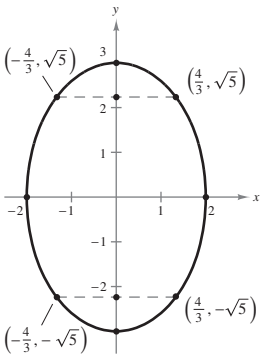
63.



65.



67.



69. $x \approx 110.3$

APPENDIX B.2 *(page A26)*

Skills Review *(page A26)*

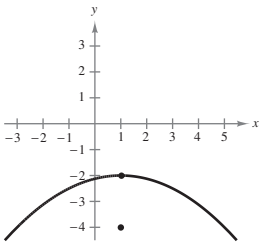
1. Hyperbola	**2.** Ellipse	**3.** Parabola
4. Hyperbola	**5.** Ellipse	**6.** Circle
7. Hyperbola	**8.** Parabola	**9.** Parabola
10. Ellipse	**11.** Hyperbola	**12.** Ellipse

1. Shifted one unit up and two units to the left

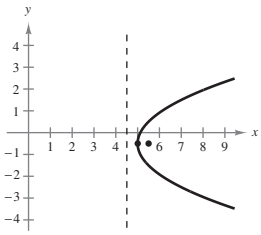
3. Shifted three units down and one unit to the right

5. Shifted two units down and one unit to the right

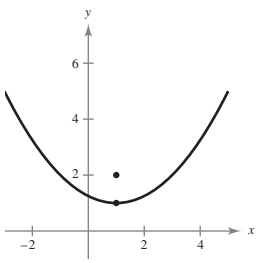
7. Vertex: $(1, -2)$
Focus: $(1, -4)$
Directrix: $y = 0$



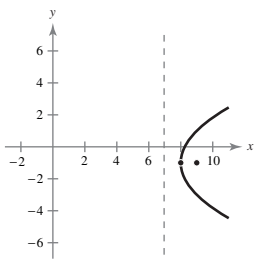
9. Vertex: $\left(5, -\frac{1}{2}\right)$
Focus: $\left(\frac{11}{2}, -\frac{1}{2}\right)$
Directrix: $x = \frac{9}{2}$



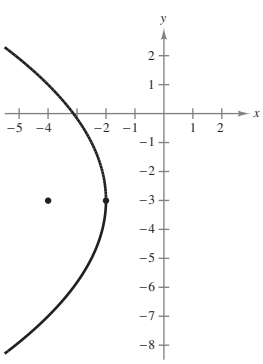
11. Vertex: $(1, 1)$
Focus: $(1, 2)$
Directrix: $y = 0$



13. Vertex: $(8, -1)$
Focus: $(9, -1)$
Directrix: $x = 7$



15. Vertex: $(-2, -3)$
Focus: $(-4, -3)$
Directrix: $x = 0$



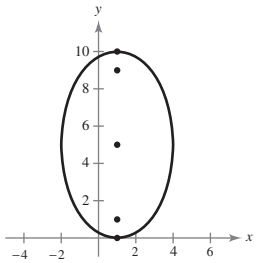
17. $(y - 2)^2 = 4(-2)(x - 3)$

19. $x^2 = 4(2)(y - 4)$

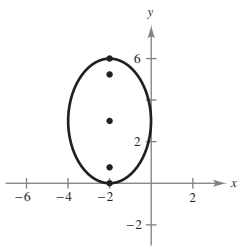
21. $(y - 2)^2 = 4(2)x$

23. $x^2 = 4\left(-\frac{1}{4}\right)(y - 4)$

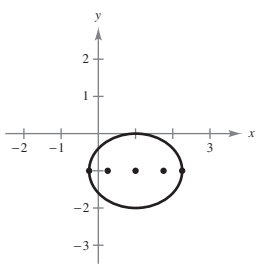
25. Center: $(1, 5)$
Foci: $(1, 9), (1, 1)$
Vertices: $(1, 10), (1, 0)$



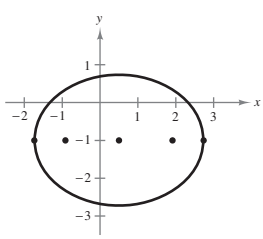
27. Center: $(-2, 3)$
Foci: $\left(-2, 3 - \sqrt{5}\right), \left(-2, 3 + \sqrt{5}\right)$
Vertices: $(-2, 0), (-2, 6)$



29. Center: $(1, -1)$
Foci: $\left(\frac{1}{4}, -1\right), \left(\frac{7}{4}, -1\right)$
Vertices: $\left(-\frac{1}{4}, -1\right), \left(\frac{9}{4}, -1\right)$



31. Center: $\left(\frac{1}{2}, -1\right)$
Foci: $\left(\frac{1}{2} - \sqrt{2}, -1\right), \left(\frac{1}{2} + \sqrt{2}, -1\right)$
Vertices: $\left(\frac{1}{2} - \sqrt{5}, -1\right), \left(\frac{1}{2} + \sqrt{5}, -1\right)$



33. $\dfrac{(x-2)^2}{4}+\dfrac{(y-2)^2}{1}=1$ **35.** $\dfrac{(x-2)^2}{16}+\dfrac{y^2}{12}=1$

37. $(x-2)^2+\dfrac{4(y+1)^2}{9}=1$

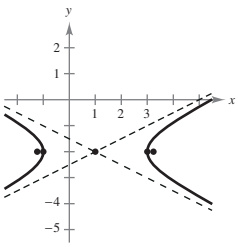
39. $\dfrac{(x-3)^2}{36}+\dfrac{(y-2)^2}{32}=1$

41. $\dfrac{(x-5)^2}{25}+\dfrac{(y-6)^2}{36}=1$

43. Center: $(1,-2)$

Vertices: $(-1,-2),(3,-2)$

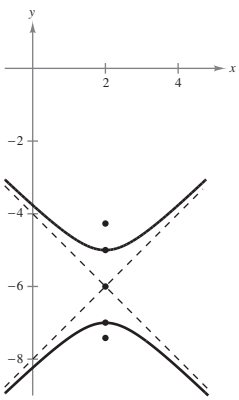
Foci: $\left(1-\sqrt{5},-2\right),\left(1+\sqrt{5},-2\right)$



45. Center: $(2,-6)$

Vertices: $(2,-7),(2,-5)$

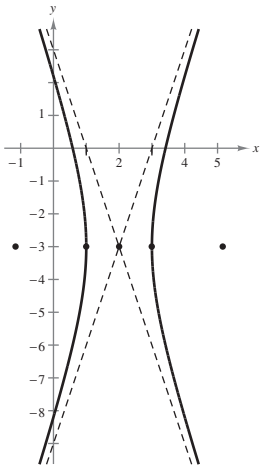
Foci: $\left(2,-6-\sqrt{2}\right),\left(2,-6+\sqrt{2}\right)$



47. Center: $(2,-3)$

Vertices: $(3,-3),(1,-3)$

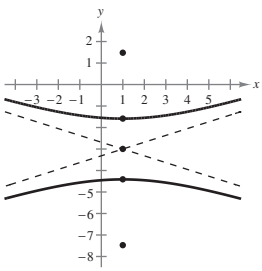
Foci: $\left(2-\sqrt{10},-3\right),\left(2+\sqrt{10},-3\right)$



49. Center: $(1,-3)$

Vertices: $\left(1,-3-\sqrt{2}\right),\left(1,-3+\sqrt{2}\right)$

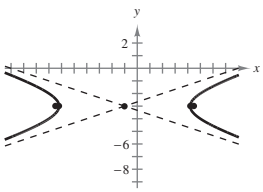
Foci: $\left(1,-3-2\sqrt{5}\right),\left(1,-3+2\sqrt{5}\right)$



51. Center: $(-1,-3)$

Vertices: $\left(-1-3\sqrt{3},-3\right),\left(-1+3\sqrt{3},-3\right)$

Foci: $\left(-1-\sqrt{30},-3\right),\left(-1+\sqrt{30},-3\right)$



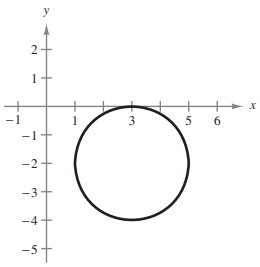
53. $\dfrac{(x-4)^2}{4}-\dfrac{y^2}{12}=1$ **55.** $\dfrac{(y-5)^2}{16}-\dfrac{(x-4)^2}{9}=1$

57. $\dfrac{y^2}{9}-\dfrac{(x-2)^2}{9/4}=1$ **59.** $\dfrac{(x-3)^2}{9}-\dfrac{(y-2)^2}{4}=1$

APPENDIX

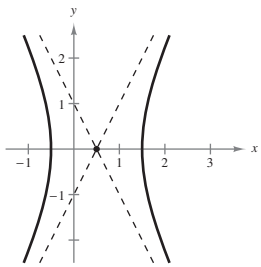
61. $\dfrac{(x-3)^2}{4}+\dfrac{(y+2)^2}{4}=1$

Circle



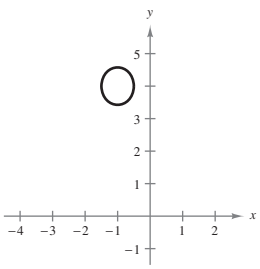
63. $\left(x-\dfrac{1}{2}\right)^2-\dfrac{y^2}{4}=1$

Hyperbola



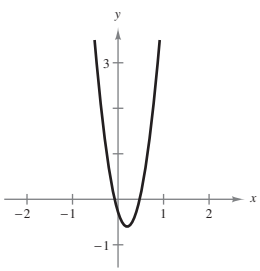
65. $\dfrac{(x+1)^2}{1/4}+\dfrac{(y-4)^2}{1/3}=1$

Ellipse



67. $\left(x-\dfrac{1}{5}\right)^2=8\left(y+\dfrac{3}{5}\right)$

Parabola



69. (a) $\approx$ 24,748.74 miles per hour

(b) $x^2=-16{,}400(y-4100)$

71. $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

73. Perihelion: $\approx 9.138\times10^7$ miles

Aphelion: $\approx 9.454\times10^7$ miles

75. $e\approx 5.414\times10^{-2}$

77. $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$

$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{a^2-c^2}=1$

$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{(a^2/a^2)(a^2-c^2)}=1$

$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{a^2(1-c^2/a^2)}=1$

$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{a^2[1-(c/a)^2]}=1$

$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{a^2(1-e^2)}=1$

79. $\dfrac{x^2}{4.88}+\dfrac{y^2}{1.39}=1$

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Algebra

Factors and Zeros of Polynomials

Given the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. If $p(b) = 0$, then b is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - b)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, $a \neq 0$ and $b^2 - 4ac \geq 0$, then the real zeros of p are $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$.

Example

If $p(x) = x^2 + 3x - 1$, then $p(x) = 0$ if

$$x = \frac{-3 \pm \sqrt{13}}{2}.$$

Special Factors

$x^2 - a^2 = (x - a)(x + a)$

$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

$x^4 - a^4 = (x - a)(x + a)(x^2 + a^2)$

$x^4 + a^4 = \left(x^2 + \sqrt{2}ax + a^2\right)\left(x^2 - \sqrt{2}ax + a^2\right)$

$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-1})$, for n odd

$x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + \cdots + a^{n-1})$, for n odd

$x^{2n} - a^{2n} = (x^n - a^n)(x^n + a^n)$

Examples

$x^2 - 9 = (x - 3)(x + 3)$

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

$x^3 + 4 = \left(x + \sqrt[3]{4}\right)\left(x^2 - \sqrt[3]{4}x + \sqrt[3]{16}\right)$

$x^4 - 4 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)(x^2 + 2)$

$x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$

$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$

$x^7 + 1 = (x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$

$x^6 - 1 = (x^3 - 1)(x^3 + 1)$

Binomial Theorem

$(x + a)^2 = x^2 + 2ax + a^2$

$(x - a)^2 = x^2 - 2ax + a^2$

$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$

$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$

$(x - a)^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$

$$(x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} + \cdots + na^{n-1}x + a^n$$

$$(x - a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} - \cdots \pm na^{n-1}x \mp a^n$$

Examples

$(x + 3)^2 = x^2 + 6x + 9$

$(x^2 - 5)^2 = x^4 - 10x^2 + 25$

$(x + 2)^3 = x^3 + 6x^2 + 12x + 8$

$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$

$\left(x + \sqrt{2}\right)^4 = x^4 + 4\sqrt{2}x^3 + 12x^2 + 8\sqrt{2}x + 4$

$(x - 4)^4 = x^4 - 16x^3 + 96x^2 - 256x + 256$

$(x + 1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

$(x - 1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

Rational Zero Test

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational* zero of $p(x) = 0$ is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Example

If $p(x) = 2x^4 - 7x^3 + 5x^2 - 7x + 3$, then the only possible *rational* zeros are $x = \pm 1, \pm\frac{1}{2}, \pm 3$, and $\pm\frac{3}{2}$. By testing, you find the two rational zeros to be $\frac{1}{2}$ and 3.

Factoring by Grouping

$$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d)$$
$$= (ax^2 + b)(cx + d)$$

Example

$$3x^3 - 2x^2 - 6x + 4 = x^2(3x - 2) - 2(3x - 2)$$
$$= (x^2 - 2)(3x - 2)$$

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Algebra

Arithmetic Operations

$$ab + ac = a(b + c)$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c}$$

$$\frac{a - b}{c - d} = \frac{b - a}{d - c}$$

$$\frac{ab + ac}{a} = b + c, \; a \neq 0$$

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$$

$$\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

Exponents and Radicals

$$a^0 = 1, \; a \neq 0$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$\sqrt[n]{a^m} = a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

$$a^{-x} = \frac{1}{a^x}$$

$$(a^x)^y = a^{xy}$$

$$\sqrt{a} = a^{1/2}$$

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

$$a^x a^y = a^{x+y}$$

$$(ab)^x = a^x b^x$$

$$\sqrt[n]{a} = a^{1/n}$$

$$\sqrt[n]{\left(\frac{a}{b}\right)} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Algebraic Errors to Avoid

$\frac{a}{x + b} \neq \frac{a}{x} + \frac{a}{b}$ (To see this error, let $a = b = x = 1$.)

$\sqrt{x^2 + a^2} \neq x + a$ (To see this error, let $x = 3$ and $a = 4$.)

$a - b(x - 1) \neq a - bx - b$ [Remember to distribute negative signs. The equation should be $a - b(x - 1) = a - bx + b$.]

$\frac{\left(\frac{x}{a}\right)}{b} \neq \frac{bx}{a}$ [To divide fractions, invert and multiply. The equation should be $\frac{\left(\frac{x}{a}\right)}{b} = \frac{\left(\frac{x}{a}\right)}{\left(\frac{b}{1}\right)} = \left(\frac{x}{a}\right)\left(\frac{1}{b}\right) = \frac{x}{ab}$.]

$\sqrt{-x^2 + a^2} \neq -\sqrt{x^2 - a^2}$ (The negative sign cannot be factored out of the square root.)

$\frac{a + bx}{a} \neq 1 + bx$ (This is one of many examples of incorrect cancellation. The equation should be $\frac{a + bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{bx}{a}$.)

$\frac{1}{x^{1/2} - x^{1/3}} \neq x^{-1/2} - x^{-1/3}$ (This error is a more complex version of the first error.)

$(x^2)^3 \neq x^5$ [This equation should be $(x^2)^3 = x^2x^2x^2 = x^6$.]

Conversion Table

1 centimeter ≈ 0.394 inch	1 joule ≈ 0.738 foot-pound	1 mile ≈ 1.609 kilometers
1 meter ≈ 39.370 inches	1 gram ≈ 0.035 ounce	1 gallon ≈ 3.785 liters
≈ 3.281 feet	1 kilogram ≈ 2.205 pounds	1 pound ≈ 4.448 newtons
1 kilometer ≈ 0.621 mile	1 inch ≈ 2.540 centimeters	1 foot-lb ≈ 1.356 joules
1 liter ≈ 0.264 gallon	1 foot ≈ 30.480 centimeters	1 ounce ≈ 28.350 grams
1 newton ≈ 0.225 pound	≈ 0.305 meter	1 pound ≈ 0.454 kilogram

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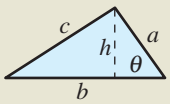
Formulas from Geometry

Triangle

$h = a \sin \theta$

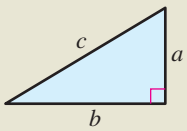
Area $= \frac{1}{2}bh$

(Law of Cosines)
$c^2 = a^2 + b^2 - 2ab \cos \theta$



Right Triangle

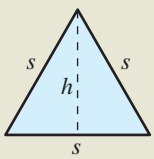
(Pythagorean Theorem)
$c^2 = a^2 + b^2$



Equilateral Triangle

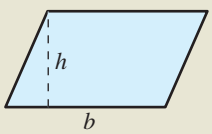
$h = \frac{\sqrt{3}s}{2}$

Area $= \frac{\sqrt{3}s^2}{4}$



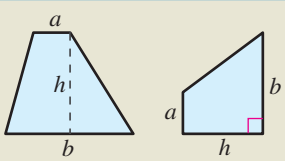
Parallelogram

Area $= bh$



Trapezoid

Area $= \frac{h}{2}(a + b)$



Circle

Area $= \pi r^2$

Circumference $= 2\pi r$

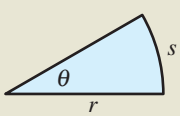


Sector of Circle

(θ in radians)

Area $= \frac{\theta r^2}{2}$

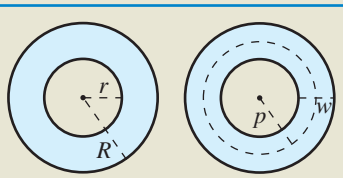
$s = r\theta$



Circular Ring

(p = average radius,
w = width of ring)

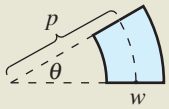
Area $= \pi(R^2 - r^2)$
$= 2\pi pw$



Sector of Circular Ring

(p = average radius,
w = width of ring,
θ in radians)

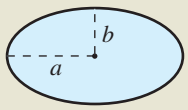
Area $= \theta pw$



Ellipse

Area $= \pi ab$

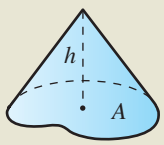
Circumference $\approx 2\pi\sqrt{\frac{a^2 + b^2}{2}}$



Cone

(A = area of base)

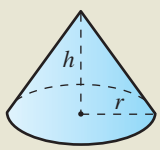
Volume $= \frac{Ah}{3}$



Right Circular Cone

Volume $= \frac{\pi r^2 h}{3}$

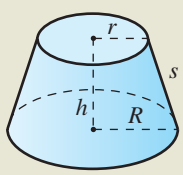
Lateral Surface Area $= \pi r\sqrt{r^2 + h^2}$



Frustum of Right Circular Cone

Volume $= \frac{\pi(r^2 + rR + R^2)h}{3}$

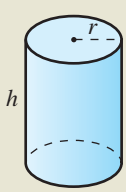
Lateral Surface Area $= \pi s(R + r)$



Right Circular Cylinder

Volume $= \pi r^2 h$

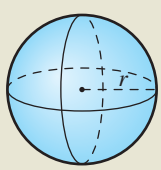
Lateral Surface Area $= 2\pi rh$



Sphere

Volume $= \frac{4}{3}\pi r^3$

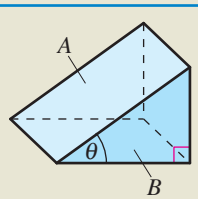
Surface Area $= 4\pi r^2$



Wedge

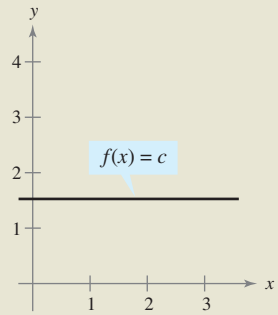
(A = area of upper face,
B = area of base)

$A = B \sec \theta$

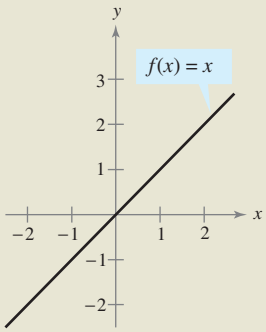


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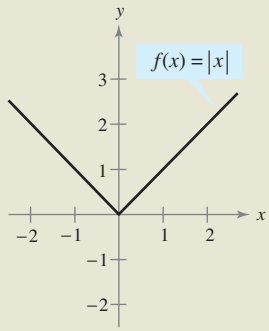
Graphs of Common Functions



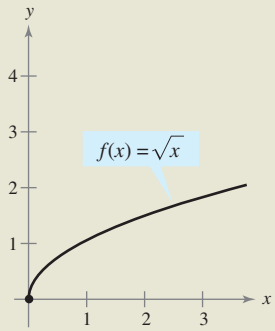
Constant Function



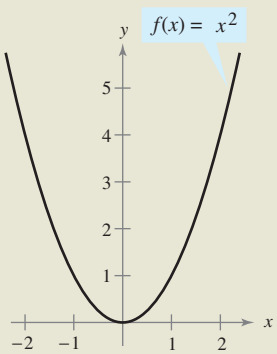
Identity Function



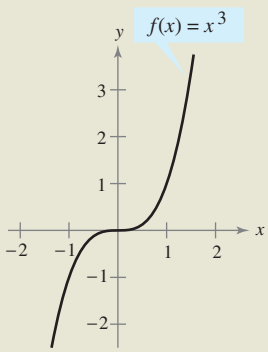
Absolute Value Function



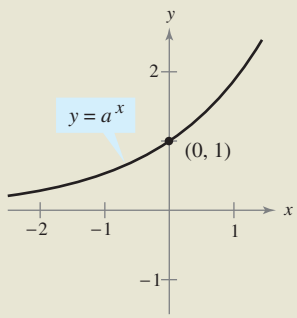
Square Root Function



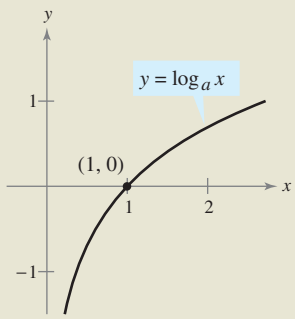
Squaring Function



Cubing Function

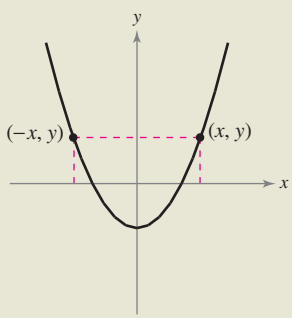


Exponential Function

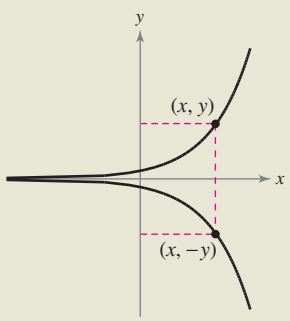


Logarithmic Function

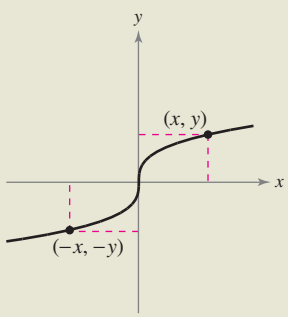
Symmetry



y-Axis Symmetry



x-Axis Symmetry



Origin Symmetry

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